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Modified Proximal point algorithms for finding a zero point of maximal monotone operators, generalized mixed equilibrium problems and variational inequalities

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Abstract

In this article, we prove strong and weak convergence theorems of modified proximal point algorithms for finding a common element of the zero point of maximal monotone operators, the set of solutions of generalized mixed equilibrium problems, the set of solutions of variational inequality problems and the fixed point set of relatively nonexpansive mappings in a Banach space under difference conditions. Our results modify and improve previous result of Li and Song.

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1. Introduction

Let E be a Banach space with norm $||\cdot||$, C be a nonempty closed convex subset of E and let E^* denote the dual of E. Let B be a *monotone* operator of C into E^*. The *variational inequality problem* is to find a point $x \in C$ such that

$$\langle Bx, y - x\rangle \geq 0 \quad \text{for all } y \in C. \tag{1.1}$$

The set of solutions of the variational inequality problem is denoted by $VI(C, B)$. Such a problem is connected with the convex minimization problem, the complementarity problem, the problem of finding a point $u \in E$ satisfying $0 = Bu$ and so on. An operator B of C into E^* is said to be *inverse-strongly monotone*, if there exists a positive real number α such that

$$\langle x - y, Bx - By\rangle \geq \alpha \|Bx - By\|^2 \tag{1.2}$$

for all $x, y \in C$. In such a case, B is said to be *α-inverse-strongly monotone*. If an operator B of C into E^* is α-inverse-strongly monotone, then B is *Lipschitz continuous*, that is $\|Bx - By\| \leq \frac{1}{\alpha}\|x - y\|$ for all $x, y \in C$.

A point $x \in C$ is a *fixed point* of a mapping S: $C \to C$ if $Sx = x$, by $F(S)$ denote the set of fixed points of S; that is, $F(S) = \{x \in C: Sx = x\}$. A point p in C is said to be an *asymptotic fixed point* of S (see [1]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} ||x_n - Sx_n|| = 0$. The set of asymptotic fixed points of S will be denoted by $\widehat{F}(S)$. A mapping S from C into itself is said to be *relatively nonexpansive* [2-4] if $\widehat{F}(S) = F(S)$ and $\varphi(p, Sx) \le \varphi(p, x)$ for all $x \in C$ and $p \in F(S)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [5,6]. A mapping S is said to be φ-nonexpansive, if $\varphi(Sx, Sy) \le \varphi(x, y)$ for $x, y \in C$. A mapping S is said to be *quasi φ-nonexpansive* if $F(S) \neq \emptyset$ and $\varphi(p, Sx) \le \varphi(p, x)$ for $x \in C$ and $p \in F(S)$.

Let E be a Banach space with norm $||\cdot||$, C be a nonempty closed convex subset of E and let E^* be the dual of E. Let Θ: $C \times C \to \mathbb{R}$ be a bifunction, ϕ: $C \to \mathbb{R}$ be a real-valued function, and B: $C \to E^*$ be a nonlinear mapping. The *generalized mixed equilibrium problem*, which is to find $x \in C$ such that

$$\Theta(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x) \ge 0, \quad \forall y \in C. \tag{1.3}$$

The solutions set to (1.3) is denoted by Ω, i.e.,

$$\Omega = \{x \in C : \Theta(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x) \ge 0, \quad \forall y \in C\}. \tag{1.4}$$

If $B = 0$, the problem (1.3) reduce into the *mixed equilibrium problem for* Θ, denoted by MEP(Θ, ϕ), which is to find $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \ge 0, \quad \forall y \in C. \tag{1.5}$$

If $\Theta \equiv 0$, the problem (1.3) reduce into the *mixed variational inequality* of Browder type, denoted by $VI(C, B, \phi)$, is to find $x \in C$ such that

$$\langle Bx, y - x\rangle + \varphi(y) - \varphi(x) \ge 0, \quad \forall y \in C. \tag{1.6}$$

If $B = 0$ and $\phi = 0$ the problem (1.3) reduce into the *equilibrium problem for* Θ, denoted by $EP(\Theta)$, is to find $x \in C$ such that

$$\Theta(x, y) \ge 0, \quad \forall y \in C. \tag{1.7}$$

The above formulation (1.7) was shown in [7] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems, Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an EP(Θ). In other words, the EP(Θ) is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many articles have appeared in the literature on the existence of solutions of EP(Θ); see, for example [7-10] and references therein. Some solution methods have been proposed to solve the EP(Θ) (see, for example, [8,10-15] and references therein). In 2005, Combettes and Hirstoaga [11] introduced an iterative scheme of finding the best approximation to the initial data when EP(Θ) is nonempty and they also proved a strong convergence theorem.

In 2004, in Hilbert space H, Iiduka et al. [16] proved that the sequence $\{x_n\}$ defined by: $x_1 = x \in C$ and

$$x_{n+1} = P_C(x_n - \lambda_n Bx_n), \tag{1.8}$$

where P_C is the metric projection of H onto C and $\{\lambda_n\}$ is a sequence of positive real numbers, converges weakly to some element of $VI(C, B)$.

In 2008, Iiduka and Takahashi [17] introduced the following iterative scheme for finding a solution of the variational inequality problem for an inverse-strongly monotone operator B that satisfies the following conditions in a 2-uniformly convex and uniformly smooth Banach space E:

(C1) B is inverse-strongly monotone,
(C2) $VI(C, B) \neq \emptyset$,
(C3) $||Ay|| \leq ||Ay - Au||$ for all $y \in C$ and $u \in VI(C, B)$.

Let $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Bx_n) \tag{1.9}$$

for every $n = 1, 2, 3, ...$, where Π_C is the generalized metric projection from E onto C, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.9) converges weakly to some element of $VI(C, B)$.

Consider the problem of finding:

$$v \in E \quad \text{such that } 0 \in A(v), \tag{1.10}$$

where A is an operator from E into E^*. Such $v \in E$ is called a *zero point* of A. When A is a maximal monotone operator, a well-know methods for solving (1.10) in a Hilbert space H is the *proximal point algorithm*: $x_1 = x \in H$ and,

$$x_{n+1} = J_{r_n} x_n, \quad n = 1, 2, 3, \ldots, \tag{1.11}$$

where $\{r_n\} \subset (0, \infty)$ and $J_{r_n} = (I + r_n A)^{-1}$, then Rockafellar [18] proved that the sequence $\{x_n\}$ converges weakly to an element of $A^{-1}(0)$. Such a problem contains numerous problems in economics, optimization, and physics and is connected with a variational inequality problem. It is well known that the variational inequalities are equivalent to the fixed point problems.

In 2000, Kamimura and Takahashi [19] proved the following strong convergence theorem in Hilbert spaces, by the following algorithm

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) J_{r_n} x_n, \quad n = 1, 2, 3, \ldots, \tag{1.12}$$

where $J_r = (I + rA)^{-1} J$, then the sequence $\{x_n\}$ converges strongly to $P_{A^{-1}0}(x)$, where $P_{A^{-1}0}$ is the projection from H onto $A^{-1}(0)$. These results were extended to more general Banach spaces see [20,21].

In 2003, Kohsaka and Takahashi [21] introduced the following iterative sequence for a maximal monotone operator A in a smooth and uniformly convex Banach space: $x_1 = x \in E$ and

$$x_{n+1} = J^{-1}(\alpha_n Jx + (1 - \alpha_n) J(J_{r_n} x_n)), \quad n = 1, 2, 3, \ldots, \tag{1.13}$$

where J is the duality mapping from E into E^* and $J_r = (I + rA)^{-1} J$.

In 2004, Kamimura et al. [22] considered the algorithm (1.14) in a uniformly smooth and uniformly convex Banach space E, namely

$$x_{n+1} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J(J_{r_n}x_n)), \quad n = 1, 2, 3, \ldots. \tag{1.14}$$

They proved that the algorithm (1.14) converges weakly to some element of $A^{-1}0$.

In 2008, Li and Song [23] proved a strong convergence theorem in a Banach space, by the following algorithm: $x_1 = x \in E$ and

$$\begin{aligned} y_n &= J^{-1}(\beta_n J(x_n) + (1-\beta_n)J(J_{r_n}x_n)), \\ x_{n+1} &= J^{-1}(\alpha_n Jx + (1-\alpha_n)J(y_n)), \end{aligned} \tag{1.15}$$

with the coefficient sequences $\{\alpha_n\}$, $\{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty}\beta_n = 0$, and $\lim_{n\to\infty} r_n = \infty$, where J is the duality mapping from E into E^* and $J_r = (I + rA)^{-1} J$. Then they proved that the sequence $\{x_n\}$ converges strongly to $\Pi_C x$, where Π_C is the generalized projection from E onto C.

In this article, motivated and inspired by Kamimura et al. [22], Li and Song [23], Iiduka and Takahashi [17], Zhang [24] and Inoue et al. [25], we introduce the new hybrid algorithm (3.1) below. Under appropriate difference conditions, we will prove that the sequence $\{x_n\}$ generated by algorithms (3.1) converges strongly to the point $\prod_{\Omega\cap VI(C,A)\cap A^{-1}(0)\cap F(S)} x_0$ and converges weakly to the point $\lim_{n\to\infty} \prod_{\Omega\cap VI(C,A)\cap A^{-1}(0)\cap F(S)} x_n$. The results presented in this article extend and improve the corresponding ones announced by Kamimura et al. [22], Li and Song [23] and some authors in the literature.

2. Preliminaries

A Banach space E is said to be *strictly convex* if $\left\|\frac{x+y}{2}\right\| < 1$ for all $x, y \in E$ with $||x|| = ||y|| = 1$ and $x \neq y$. Let $U = \{x \in E: ||x|| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be *smooth* provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in E$. The *modulus of convexity* of E is the function δ: $[0, 2] \to [0, 1]$ defined by

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\right\}. \tag{2.1}$$

A Banach space E is *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0, 2]$ (see [26,27] for more details). Observe that every p-uniform convex is uniformly convex. One should note that no a Banach space is p-uniform convex for $1 < p < 2$. It is well known that a Hilbert space is 2-uniformly convex and uniformly smooth. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ *is defined by*

$$J_p(x) = \{x^* \in E^* : \langle x, x^* \rangle = ||x||^p, ||x^*|| = ||x||^{p-1}\} \tag{2.2}$$

for all $x \in E$. In particular, $J = J_2$ is called *the normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

We know the following (see [28]):

(1) if E is smooth, then J is single-valued;

(2) if E is strictly convex, then J is one-to-one and $\langle x - y, x^* - y^* \rangle > 0$ holds for all $(x, x^*), (y, y^*) \in J$ with $x \neq y$;

(3) if E is reflexive, then J is surjective;

(4) if E is uniformly convex, then it is reflexive;

(5) if E^* is uniformly convex, then J is uniformly norm-to-norm continuous on each bounded subset of E.

The duality J from a smooth Banach space E into E^* is said to be *weakly sequentially continuous* [29] if $x_n \rightharpoonup x$ implies $Jx_n \rightharpoonup^* Jx$, where $\rightharpoonup^*$ implies the weak* convergence.

Lemma 2.1. [30,31]*If E be a 2-uniformly convex Banach space. Then, for all $x, y \in E$ we have*

$$\|x - y\| \leq \frac{2}{c^2} \|Jx - Jy\|,$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

The best constant $\frac{1}{c}$ in Lemma is called the 2-uniformly convex constant of E (see [26]).

Lemma 2.2. [30,32]*If E be a p-uniformly convex Banach space and let p be a given real number with $p \geq 2$. Then for all $x, y \in E$, $J_x \in J_p(x)$ and $J_y \in J_p(y)$*

$$\langle x - y, Jx - Jy \rangle \geq \frac{c^p}{2^{p-2}p} \|x - y\|^p,$$

where J_p is the generalized duality mapping of E and $\frac{1}{c}$ is the p-uniformly convexity constant of E.

Lemma 2.3. (Xu [31]) *Let E be a uniformly convex Banach space. Then for each $r > 0$, there exists a strictly increasing, continuous and convex function g*: $[0, \infty) \to [0, \infty)$ *such that* $g(0) = 0$ *and*

$$\|\lambda x + (1 - \lambda y)\|^2 \leq \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda(1 - \lambda) g(\|x - y\|) \tag{2.3}$$

for all $x, y \in \{z \in E: ||z|| \leq r\}$ and $\lambda \in [0, 1]$.

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Throughout this article, we denote by φ the function defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \text{for } x, y \in E. \tag{2.4}$$

Following Alber [33], the *generalized projection* Π_C: $E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\varphi(x, y)$, that is, $\prod_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x},\ x) = \inf_{y\in C}\phi(y,\ x) \tag{2.5}$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\varphi(x, y)$ and strict monotonicity of the mapping J. It is obvious from the definition of function φ that (see [33])

$$\left(\|y\| - \|x\|\right)^2 \leq \phi(y,\ x) \leq \left(\|y\| + \|x\|\right)^2, \quad \forall x, y \in E. \tag{2.6}$$

If E is a Hilbert space, then $\varphi(x, y) = ||x - y||^2$.

If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\varphi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\varphi(x, y) = 0$ then $x = y$. From (2.6), we have $||x|| = ||y||$. This implies that $\langle x, Jy\rangle = ||x||^2 = ||Jy||^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$ (see [28,34] for more details).

Lemma 2.4. (Kamimura and Takahashi [20]) *Let E be a uniformly convex and smooth real Banach space and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\varphi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $||x_n - y_n|| \to 0$.*

Lemma 2.5. (Alber [33]) *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y,\ Jx - Jx_0\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.6. (Alber [33]) *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y,\ \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y,\ x), \quad \forall y \in C.$$

Let E be a strictly convex, smooth and reflexive Banach space, let J be the duality mapping from E into E^*. Then J^{-1} is also single-valued, one-to-one, and surjective, and it is the duality mapping from E^* into E. Define a function $V: E \times E^* \to \mathbb{R}$ as follows (see [21]):

$$V(x,\ x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \left\|x^*\right\|^2 \tag{2.7}$$

for all $x \in Ex \in E$ and $x^* \in E^*$. Then, it is obvious that $V(x, x^*) = \varphi(x, J^{-1}(x^*))$ and $V(x, J(y)) = \varphi(x, y)$.

Lemma 2.7. (Kohsaka and Takahashi [[21], Lemma 3.2]) *Let E be a strictly convex, smooth and reflexive Banach space, and let V be as in* (2.7). *Then*

$$V(x,\ x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x,\ x^* + y^*) \tag{2.8}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Let E be a reflexive, strictly convex and smooth Banach space. Let C be a closed convex subset of E. Because $\varphi(x, y)$ is strictly convex and coercive in the first variable, we know that the minimization problem $\inf_{y\in C} \varphi(x, y)$ has a unique solution. The operator $\Pi_C x := \arg\min_{y\in C} \varphi(x, y)$ is said to be the generalized projection of x on C.

A set-valued mapping $A: E \to E^*$ with domain $D(A) = \{x \in E : A(x) \neq \emptyset\}$ and range $R(A) = \{x^* \in E^*: x^* \in A(x), x \in D(A)\}$ is said to be *monotone* if $\langle x - y, x^* - y^*\rangle \geq 0$ for all $x^* \in A(x)$, $y^* \in A(y)$. We denote the set $\{s \in E: 0 \in Ax\}$ by $A^{-1}0$. A is *maximal monotone* if its graph $G(A)$ is not properly contained in the graph of any other monotone operator. If A is maximal monotone, then the solution set $A^{-1}0$ is closed and convex.

Let E be a reflexive, strictly convex and smooth Banach space, it is knows that A is a maximal monotone if and only if $R(J + rA) = E^*$ for all $r > 0$.

Define the *resolvent* of A by $J_r x = x_r$. In other words, $J_r = (J + rA)^{-1}J$ for all $r > 0$. J_r is a single-valued mapping from E to $D(A)$. Also, $A^{-1}(0) = F(J_r)$ for all $r > 0$, where $F(J_r)$ is the set of all fixed points of J_r. Define, for $r > 0$, the *Yosida approximation* of A by $A_r = (J - JJ_r)/r$. We know that $A_r x \in A(J_r x)$ for all $r > 0$ and $x \in E$.

Lemma 2.8. (Kohsaka and Takahashi [[21], Lemma 3.1]) *Let E be a smooth, strictly convex and reflexive Banach space, let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$, let $r > 0$ and let $J_r = (J + rT)^{-1}J$. Then*

$$\phi(x, J_r y) + \phi(J_r y, y) \leq \phi(x, y)$$

for all $x \in A^{-1}0$ and $y \in E$.

Let B be an inverse-strongly monotone mapping of C into E^* which is said to be *hemicontinuous* if for all $x, y \in C$, the mapping F of $[0, 1]$ into E^*, defined by $F(t) = B(tx + (1 - t)y)$, is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$ the *normal cone* for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \ \forall y \in C\}. \tag{2.9}$$

Theorem 2.9. (Rockafellar [18]) *Let C be a nonempty, closed convex subset of a Banach space E and B a monotone, hemicontinuous operator of C into E^*. Let $T \subset E \times E^*$ be an operator defined as follows:*

$$Tv = \begin{cases} Bv + N_C(v), & v \in C; \\ \emptyset, & \text{otherwise.} \end{cases} \tag{2.10}$$

Then T is maximal monotone and $T^{-1}0 = VI(C, B)$.

Lemma 2.10. (Tan and Xu [35]) *Let $\{a_n\}$ and $\{b_n\}$ be two sequence of nonnegative real numbers satisfying the inequality*

$$a_{n+1} = a_n + b_n \quad \text{for all } n \geq 0.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.11. (Xu [36]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n t_n + r_n \, n \geq 1,$$

where $\{\alpha_n\}$, $\{t_n\}$, and $\{r_n\}$ satisfy $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\limsup_{n\to\infty} t_n \leq 0$ and $r_n \geq 0$, $\sum_{n=1}^{\infty} r_n < \infty$. Then $\lim_{n\to\infty} s_n = 0$.

For solving the mixed equilibrium problem, let us assume that the bifunction $\Theta: C \times C \to \mathbb{R}$ and $\phi: C \to \mathbb{R}$ is convex and lower semi-continuous satisfies the following conditions:

(A1) $\Theta(x, x) = 0$ for all $x \in C$;

(A2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t\downarrow 0} \Theta(tz + (1-t)x,\ y) \le \Theta(x,\ y);$$

(A4) for each $x \in C$, $y \mapsto \Theta(x, y)$ is convex and lower semi-continuous.

Lemma 2.12. (Blum and Oettli [7]) *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and let Θ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$\Theta(z,\ y) + \frac{1}{r}\langle y - z,\ z - x\rangle \ge 0 \quad \textit{for all } y \in C.$$

Lemma 2.13. (Takahashi and Zembayashi [37]) *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). For all $r > 0$ and $x \in E$, define a mapping T_r: $E \to C$ as follows:*

$$T_r x = \left\{ z \in C : \Theta(z,\ y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \forall y \in C \right\} \tag{2.11}$$

for all $x \in E$. Then, the followings hold:

(1) *T_r is single-valued;*
(2) *T_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle T_r x - T_r y,\ JT_r x - JT_r y\rangle \le \langle T_r x - T_r y,\ Jx - Jy\rangle;$$

(3) $F(T_r) = \mathrm{EP}(\Theta)$;
(4) $\mathrm{EP}(\Theta)$ *is closed and convex.*

Lemma 2.14. (Takahashi and Zembayashi [37]) *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q,\ T_r x) + \phi(T_r x,\ x) \le \phi(q,\ x).$$

Lemma 2.15. (Zhang [24]) *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let B: $C \to E^*$ be a continuous and monotone mapping, ϕ: $C \to \mathbb{R}$ is convex and lower semi-continuous and Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in E$, then there exists $u \in C$ such that*

$$\Theta(u,\ y) + \langle Bu,\ y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u,\ Ju - Jx\rangle \ge 0, \quad \forall y \in C.$$

Define a mapping K_r: $C \to C$ as follows:

$$K_r(x) = \left\{ u \in C : \Theta(u,\ y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \ge 0, \forall y \in C \right\} \tag{2.12}$$

for all $x \in E$. Then, the followings hold:

(i) *K_r is single-valued;*

(ii) K_r *is firmly nonexpansive, i.e., for all* $x, y \in E$, $\langle K_rx - K_ry, JK_rx - JK_ry\rangle \le \langle K_rx - K_ry, Jx - Jy\rangle$;

(iii) $F(K_r) = \Omega$;

(iv) Ω *is closed and convex;*

(v) $\varphi(p, K_rz) + \varphi(K_rz, z) \le \varphi(p, z)$ $\forall p \in F(K_r)$, $z \in E$.

Remark 2.16. (Zhang [24]) It follows from Lemma 2.13 that the mapping $K_r: C \to C$ defined by (2.12) is a relatively nonexpansive mapping. Thus, it is quasi-φ-nonexpansive.

Lemma 2.17. (Xu [31] and Zalinescu [32]) *Let E be a uniformly convex Banach space and let r >0. Then there exists a strictly increasing, continuous and convex function g:* $[0, \infty) \to [0, \infty)$ *such that* $g(0) = 0$ *and*

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x-y\|) \tag{2.13}$$

for all $x, y \in B_r(0)$ *and* $t \in [0, 1]$, *where* $B_r(0) = \{z \in E: ||z|| \le r\}$.

3. Strong convergence theorem

In this section, we prove a strong convergence theorem for finding a common element of the set of solutions of mixed equilibrium problems, the set of solution of the variational inequality problem, the fixed point set of relatively nonexpansive mappings and the zero point of a maximal monotone operators in a Banach space by using the shrinking hybrid projection method.

Theorem 3.1. *Let E be a 2-uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E. Let* Θ *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying (A1)-(A4) let* $\phi: C \to \mathbb{R}$ *be a proper lower semicontinuous and convex function, let* $T: E \to E^*$ *be a maximal monotone operator satisfying* $D(T) \subset C$. *Let* $J_r = (J + rT)^{-1}J$ *for r >0, let* $B: C \to E^*$ *be a continuous and monotone mappings and S be a relatively nonexpansive mappings from C into itself, with* $F := \Omega \cap VI(C, A) \cap T^{-1}(0) \cap F(S) \neq \emptyset$. *Assume that A an operator of C into E* that satisfies the conditions (C1)-(C3). Let* $\{x_n\}$ *be a sequence generated by* $x_1 = x \in C$ *and,*

$$\begin{cases} u_n = K_{r_n}x_n, \\ z_n = \Pi_C J^{-1}(J - \lambda_n A)u_n, \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JSJ_{r_n}z_n), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx + (1-\alpha_n)Jy_n), \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N}$, *where* Π_C *is the generalized projection from E onto C, J is the duality mapping on E. The coefficient sequences* $\{\lambda_n\} \subset [a, b]$ *for some a, b with* $0 < a < b < \frac{c^2\alpha}{2}$, $\frac{1}{c}$ *is the 2-uniformly convexity constant of E and* $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset (0, 1]$ $\{r_n\} \subset (0, \infty)$ *satisfying*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\limsup_{n\to\infty} \beta_n < 1$,

(iii) $\liminf_{n\to\infty} r_n > 0$.

Then the sequence $\{x_n\}$ *converges strongly to* $\Pi_F x_0$.

Proof. Let $H(u_n, y) = \Theta(u_n, y) + \langle Bu_n, y - u_n\rangle + \phi(y) - \phi(u_n)$, $y \in C$ and $K_{r_n} = \{u_n \in C : H(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \forall y \in C\}$. We first show that $\{x_n\}$ is bounded. Put $v_n = J^{-1}(J - \lambda_n A)u_n$ and $w_n = J_{r_n} z_n$ for all $n \geq 0$. Let $p \in F: = \Omega \cap VI(C, A) \cap T^{-1}(0) \cap F(S)$ and $u_n = K_{r_n} x_n$. Since S, J_{r_n} and K_{r_n} are relatively nonexpansive mappings, we get

$$\phi(p, u_n) = \phi(p, K_{r_n} x_n) \leq \phi(p, x_n) \tag{3.2}$$

and Lemma 2.7, the convexity of the function V in the second variable, we obtain

$$\begin{aligned}
\phi(p, z_n) &= \phi(p, \Pi_C v_n) \\
&\leq \phi(p, v_n) = \phi(p, J^{-1}(Ju_n - \lambda_n Au_n)) \\
&\leq V(p, Ju_n - \lambda_n Au_n + \lambda_n Au_n) - 2\langle J^{-1}(Ju_n - \lambda_n Au_n) - p, \lambda_n Au_n\rangle \\
&= V(p, Ju_n) - 2\lambda_n\langle v_n - p, Au_n\rangle \\
&= \phi(p, u_n) - 2\lambda_n\langle u_n - p, Au_n\rangle + 2\langle v_n - u_n, -\lambda_n Au_n\rangle.
\end{aligned} \tag{3.3}$$

Since $p \in VI(C, A)$ and A is α-inverse-strongly monotone, we have

$$\begin{aligned}
-2\lambda_n\langle u_n - p, Au_n\rangle &= -2\lambda_n\langle u_n - p, Au_n - Ap\rangle - 2\lambda_n\langle u_n - p, Ap\rangle \\
&\leq -2\alpha\lambda_n\|Au_n - Ap\|^2,
\end{aligned} \tag{3.4}$$

and by Lemma 2.1, we obtain

$$\begin{aligned}
2\langle v_n - u_n, -\lambda_n Au_n\rangle &= 2\langle J^{-1}(Ju_n - \lambda_n Au_n) - u_n, -\lambda_n Au_n\rangle \\
&\leq 2\|J^{-1}(Ju_n - \lambda_n Au_n) - u_n\| \, \|\lambda_n Au_n\| \\
&\leq \frac{4}{c^2}\|Ju_n - \lambda_n Au_n - Ju_n\| \, \|\lambda_n Au_n\| \\
&= \frac{4}{c^2}\lambda_n^2\|Au_n\|^2 \\
&\leq \frac{4}{c^2}\lambda_n^2\|Au_n - Ap\|^2.
\end{aligned} \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), we get

$$\begin{aligned}
\phi(p, z_n) &\leq \phi(p, u_n) - 2\alpha\lambda_n\|Au_n - Ap\|^2 + \frac{4}{c^2}\lambda_n^2\|Au_n - Ap\|^2 \\
&\leq \phi(p, u_n) + 2\lambda_n\left(\frac{2}{c^2}\lambda_n - \alpha\right)\|Au_n - Ap\|^2 \\
&\leq \phi(p, u_n) \\
&\leq \phi(p, x_n).
\end{aligned} \tag{3.6}$$

By Lemmas 2.7, 2.8 and (3.6), we have

$$\begin{aligned}
\phi(p, y_n) &= \phi(p, J^{-1}(\beta_n Jx_n + (1 - \beta_n)JSw_n)) \\
&= V(p, \beta_n Jx_n + (1 - \beta_n)JSw_n) \\
&\leq V(p, \beta_n Jx_n) + (1 - \beta_n)V(p, JSw_n) \\
&= \beta_n\phi(p, x_n) + (1 - \beta_n)\phi(p, Sw_n) \\
&\leq \beta_n\phi(p, x_n) + (1 - \beta_n)\phi(p, w_n) \\
&\leq \beta_n\phi(p, x_n) + (1 - \beta_n)(\phi(p, z_n) - \phi(w_n, z_n)) \\
&\leq \beta_n\phi(p, x_n) + (1 - \beta_n)\phi(p, z_n) \\
&\leq \beta_n\phi(p, x_n) + (1 - \beta_n)\phi(p, x_n) \\
&= \phi(p, x_n),
\end{aligned} \tag{3.7}$$

it follows that

$$\begin{aligned}\phi(p,\ x_{n+1}) &= \phi(p,\ \Pi_C J^{-1}(\alpha_n Jx_1 + (1-\alpha_n)Jy_n)) \\ &\le\ \phi(p,\ J^{-1}(\alpha_n Jx_1 + (1-\alpha_n)Jy_n)) \\ &=\ V(p,\ \alpha_n Jx_1 + (1-\alpha_n)Jy_n) \\ &\le\ \alpha_n V(p,\ Jx_1) + (1-\alpha_n)V(p,\ Jy_n) \\ &=\ \alpha_n\phi(p,\ x_1) + (1-\alpha_n)\phi(p,\ y_n) \\ &\le \alpha_n\phi(p,\ x_1) + (1-\alpha_n)\phi(p,\ x_n)\end{aligned} \tag{3.8}$$

for all $n \in \mathbb{N}$. Hence, by induction, we have that $\varphi(p, x_n) \le \varphi(p, x_1)$ for all $n \in \mathbb{N}$. Since $(||x_n|| - ||p||)^2 \le \varphi(p, x_n)$. It implies that $\{x_n\}$ is bounded and $\{y_n\}$, $\{z_n\}$, $\{w_n\}$ are also bounded.

From (3.6)-(3.8), we have

$$\begin{aligned}\phi(p,\ x_{n+1}) &\le \alpha_n\phi(p,\ x_1) + (1-\alpha_n)[\beta_n\phi(p,\ x_n) + (1-\beta_n)(\phi(p,\ x_n) - \phi(w_n,\ z_n))] \\ &\le\ \alpha_n\phi(p,\ x_1) + (1-\alpha_n)\phi(p,\ x_n) - (1-\alpha_n)(1-\beta_n)\phi(w_n,\ z_n)\end{aligned}$$

and then

$$(1-\alpha_n)(1-\beta_n)\phi(w_n,\ z_n) \le \alpha_n\phi(p,\ x_1) + (1-\alpha_n)\phi(p,\ x_n) - \phi(p,\ x_{n+1})$$

for all $n \in \mathbb{N}$. Since $\lim_{n \text{-} \infty} \alpha_n = 0$, $\limsup_{n\to\infty} \beta_n < 1$, it follows that $\lim_{n\to\infty} \varphi(w_n, z_n) = 0$. Applying Lemma 2.4, we have

$$\lim_{n\to\infty} \|w_n - z_n\| = \lim_{n\to\infty} \left\|J_{r_n} z_n - z_n\right\| = 0. \tag{3.9}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|Jw_n - Jz_n\| = \lim_{n\to\infty} \left\|JJ_{r_n} z_n - Jz_n\right\| = 0. \tag{3.10}$$

By (3.2), (3.6)-(3.8) again, we note that

$$\begin{aligned}\phi(p,\ x_{n+1}) &\le \alpha_n\phi(p,\ x_1) + \\ &\quad (1-\alpha_n)\left\{\beta_n\phi(p,\ x_n) + (1-\beta_n)\left[\phi(p,\ x_n) - 2\lambda_n\left(\alpha - \frac{2}{c^2}\lambda_n\right)\left\|Au_n - Ap\right\|^2\right]\right\} \\ &\le\ \alpha_n\phi(p,\ x_1) + (1-\alpha_n)\phi(p,\ x_n) - (1-\alpha_n)(1-\beta_n)2\lambda_n\left(\alpha - \frac{2}{c^2}\lambda_n\right)\left\|Au_n - Ap\right\|^2\end{aligned}$$

and hence

$$2\lambda_n\left(\alpha - \frac{2}{c^2}\lambda_n\right)\left\|Au_n - Ap\right\|^2 \le \frac{1}{(1-\alpha_n)(1-\beta_n)}(\alpha_n\phi(p,\ x_1) + (1-\alpha_n)\phi(p,\ x_n) - \phi(p,\ x_{n+1}))$$

for all $n \in \mathbb{N}$. Since $0 < a < b < \dfrac{c^2\alpha}{2}$, $\lim_{n\to\infty} \alpha_n = 0$, $\limsup_{n\to\infty} \beta_n < 1$, we have

$$\lim_{n\to\infty} \left\|Au_n - Ap\right\| = 0. \tag{3.11}$$

From Lemmas 2.6, 2.7 and (3.5), we get

$$\begin{aligned}\phi(u_n, z_n) = \phi(u_n, \Pi_C v_n) &\le \phi(u_n, v_n)\\ &= \phi(u_n, J^{-1}(Ju_n - \lambda_n A u_n))\\ &= V(u_n, Ju_n - \lambda_n A u_n)\\ &\le V(u_n, (Ju_n - \lambda_n A u_n) + \lambda_n A u_n)\\ &\quad - 2\langle J^{-1}(Ju_n - \lambda_n A u_n) - u_n, \lambda_n A u_n\rangle\\ &= \phi(u_n, u_n) + 2\langle v_n - u_n, -\lambda_n A u_n\rangle\\ &= 2\langle v_n - u_n, -\lambda_n A u_n\rangle\\ &\le \frac{4}{c^2}\lambda_n^2 \|Au_n - Ap\|^2.\end{aligned}$$

From Lemma 2.4 and (3.11), we have

$$\lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{3.12}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|Ju_n - Jz_n\| = 0. \tag{3.13}$$

From Lemmas 2.6, 2.7 and (3.5), we obtain

$$\begin{aligned}\phi(x_n, z_n) &= \phi(x_n, \Pi_C J^{-1}(Ju_n - \lambda_n A u_n))\\ &\le \phi(x_n, J^{-1}(Ju_n - \lambda_n A u_n))\\ &= V(x_n, Ju_n - \lambda_n A u_n)\\ &\le V(x_n, (Ju_n - \lambda_n A u_n) + \lambda_n A u_n) - 2\langle J^{-1}(Ju_n - \lambda_n A u_n) - u_n, \lambda_n A u_n\rangle\\ &= \phi(x_n, u_n) + 2\langle J^{-1}(Ju_n - \lambda_n A u_n) - u_n, -\lambda_n A u_n\rangle\\ &= \phi(x_n, x_n) + 2\langle J^{-1}(Ju_n - \lambda_n A u_n) - u_n, -\lambda_n A u_n\rangle\\ &= \frac{4}{c^2}\|Au_n - Ap\|^2\end{aligned}$$

for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty}||Au_n - Ap||^2 = 0$, we have $\lim_{n\to\infty} \varphi(x_n, z_n) = 0$. Applying Lemma 2.4, we get

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.14}$$

Since J is uniformly norm-to-norm continuous on bounded set, we obtain

$$\lim_{n\to\infty} \|Jx_n - Jz_n\| = 0. \tag{3.15}$$

So, by the triangle inequality, we get

$$\|x_n - u_n\| \le \|x_n - z_n\| + \|z_n - u_n\|.$$

By (3.12) and (3.14), we also have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.16}$$

From (3.1), we obtain

$$\begin{aligned}\phi(y_n, z_n) &= \phi(x_n, J^{-1}(\beta_n Jx_n + (1-\beta_n)JSw_n)) \\ &= V(x_n, \beta_n Jx_n + (1-\beta_n)JSw_n) \\ &\le \beta_n V(x_n, Jx_n) + (1-\beta_n)V(x_n, JSw_n) \\ &= \beta_n\phi(x_n, x_n) + (1-\beta_n)\phi(x_n, Sw_n) \\ &\le \beta_n\phi(x_n, x_n) + (1-\beta_n)\phi(x_n, w_n) \\ &= (1-\beta_n)\phi(x_n, z_n)\end{aligned}$$

for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \varphi(x_n, z_n) = 0$, we have $\lim_{n\to\infty} \varphi(y_n, z_n) = 0$. Applying Lemma 2.4, we get

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.17}$$

Since J is uniformly norm-to-norm continuous on bounded set, we obtain

$$\lim_{n\to\infty} \|Jy_n - Jz_n\| = 0. \tag{3.18}$$

From

$$\|x_n - y_n\| \le \|x_n - z_n\| + \|z_n - y_n\|,$$

we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.19}$$

Since J is uniformly norm-to-norm continuous on bounded set, we obtain

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{3.20}$$

From Lemma 2.17 and (3.7), we have

$$\begin{aligned}\phi(p, y_n) &= \phi(p, J^{-1}(\beta_n Jx_n + (1-\beta_n)JSw_n)) \\ &= \|p\|^2 - 2\langle p\beta_n Jx_n + (1-\beta_n)JSw_n\rangle + \|\beta Jx_n + (1-\beta_n)JSw_n\|^2 \\ &\le \|p\|^2 - 2\beta_n\langle p, Jx_n\rangle - 2(1-\beta_n)\langle p, JSw_n\rangle + \beta_n\|x_n\|^2 + (1-\beta_n)\|Sw_n\|^2 \\ &\quad - \beta_n(1-\beta_n)g(\|Jx_n - JSw_n\|) \\ &= \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, Sw_n) - \beta_n(1-\beta_n)g(\|Jx_n - JSw_n\|) \\ &\le \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JSw_n\|) \\ &= \phi(p, x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JSw_n\|).\end{aligned} \tag{3.21}$$

This implies that

$$\beta_n(1-\beta_n)g(\|Jx_n - JSw_n\|) \le \phi(p, x_n) - \phi(p, y_n). \tag{3.22}$$

On the other hand, we have

$$\begin{aligned}\phi(p, x_n) - \phi(p, y_n) &= \|x_n\|^2 - \|y_n\|^2 - 2\langle p, Jx_n - Jy_n\rangle \\ &= \|x_n - y_n\|(\|x_n\| + \|y_n\|) + 2\|p\|\,\|Jx_n - Jy_n\|.\end{aligned} \tag{3.23}$$

Noticing (3.19) and (3.20), we obtain

$$\phi(p, x_n) - \phi(p, y_n) \to 0, \quad \text{as } n \to \infty. \tag{3.24}$$

Since $\limsup_{n\to\infty} \beta_n < 1$ and (3.24), it follows from (3.22) that

$$g(\|Jx_n - JSw_n\|) \to 0, \quad \text{as } n \to \infty. \tag{3.25}$$

If follows from the property of g that

$$\lim_{n\to\infty} \|Jx_n - JSw_n\| = 0. \tag{3.26}$$

Since J is uniformly norm-to-norm continuous on bounded set, we see that

$$\lim_{n\to\infty} \|x_n - Sw_n\| = 0. \tag{3.27}$$

Since

$$\|z_n - Sw_n\| \le \|z_n - x_n\| + \|x_n - Sw_n\|,$$

from (3.14) and (3.27), we obtain that

$$\lim_{n\to\infty} \|z_n - Sw_n\| = 0. \tag{3.28}$$

By (3.9) and (3.14), we obtain

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{3.29}$$

Also, by (3.9) and (3.28), we obtain

$$\lim_{n\to\infty} \|w_n - Sw_n\| = 0. \tag{3.30}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u \in C$. It follows from (3.29), we have $w_{n_i} \rightharpoonup u$ as $i \to \infty$ and S be a relatively nonexpansive, we have that $u \in \widehat{F}(S) = F(S)$.

Next, we show that $u \in T^{-1}0$. Indeed, since $\liminf_{n\to\infty} r_n > 0$, it follows from (3.10) that

$$\lim_{n\to\infty} ||A_{r_n} z_n|| = \lim_{n\to\infty} \frac{1}{r_n} ||Jz_n - Jw_n|| = 0. \tag{3.31}$$

If $(z, z^*) \in T$, then it holds from the monotonicity of A that

$$\langle z - z_{n_i}, z^* - A_{r_{n_i}} z_{n_i} \rangle \ge 0$$

for all $i \in \mathbb{N}$. Letting $i \to \infty$, we get $\langle z - u, z^* \rangle \ge 0$. Then, the maximality of T implies $u \in T^{-1}0$.

Next, we show that $u \in VI(C, A)$. Let $B \subset E \times E^*$ be an operator as follows:

$$Bv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & \text{otherwise} \end{cases}$$

By Theorem 2.9, B is maximal monotone and $B^{-1}0 = VI(C, A)$. Let $(v, w) \in G(B)$. Since $w \in Bv = Av + N_C(v)$, we get $w - Av \in N_C(v)$. From $z_n \in C$, we have

$$\langle v - z_n, w - Av \rangle \ge 0. \tag{3.32}$$

On the other hand, since $z_n = \Pi_C J^{-1}(Ju_n - \lambda_n Au_n)$. Then by Lemma 2.5, we have

$$\langle v - z_n,\ Jz_n - (Ju_n - \lambda_n Au_n)\rangle \geq 0,$$

thus

$$\left\langle v - z_n, \frac{Ju_n - Jz_n}{\lambda_n} - Aux_n \right\rangle \leq 0. \tag{3.33}$$

It follows from (3.32) and (3.33) that

$$\begin{aligned}
\langle v - z_n, w\rangle \geq\ & \langle v - z_n, Av\rangle \\
\geq\ & \langle v - z_n,\ Av\rangle + \left\langle v - z_n,\ \frac{Ju_n - Jz_n}{\lambda_n} - Au_n \right\rangle \\
=\ & \langle v - z_n,\ Av - Au_n\rangle + \left\langle v - z_n,\ \frac{Ju_n - Jz_n}{\lambda_n} \right\rangle \\
=\ & \langle v - z_n,\ Av - Az_n\rangle + \langle v - z_n,\ Az_n - Au_n\rangle + \left\langle v - z_n,\ \frac{Ju_n - Jz_n}{\lambda_n} \right\rangle \\
\geq & -\| v - z_n\| \frac{\|z_n - u_n\|}{\alpha} - \|v - z_n\| \frac{\|Ju_n - Jz_n\|}{a} \\
\geq\ & -M\left(\frac{\|z_n - u_n\|}{\alpha} + \frac{\|Ju_n - Jz_n\|}{a}\right),
\end{aligned}$$

where $M = \sup_{n\geq 1}\{||v - z_n||\}$. From (3.12) and (3.13), we obtain $\langle v - u, w\rangle \geq 0$. By the maximality of B, we have $u \in B^{-1}0$ and hence $u \in VI(C, A)$.

Next, we show that $u \in \Omega$. From (3.16) and J is uniformly norm-to-norm continuous on bounded set, we obtain

$$\lim_{n\to\infty} \|Ju_n - Jx_n\| = 0. \tag{3.34}$$

From the assumption $\liminf_{n\to\infty} r_n > a$, we get

$$\lim_{n\to\infty} \frac{\|Ju_n - Jx_n\|}{r_n} = 0.$$

Noticing that $u_n = K_{r_n}x_n$, we have

$$H(u_n,\ y) + \frac{1}{r_n}\langle y - u_n,\ Ju_n - Jx_n\rangle \geq 0, \qquad \forall y \in C.$$

Hence,

$$H(u_{n_i},\ y) + \frac{1}{r_{n_i}}\langle y - u_{n_i},\ Ju_{n_i} - Jx_{n_i}\rangle \geq 0, \qquad \forall y \in C.$$

From the (A2), we note that

$$\|y - u_{n_i}\| \frac{\|Ju_{n_i} - Jx_{n_i}\|}{r_{n_i}} \geq \frac{1}{r_{n_i}}\langle y - u_{n_i}, Ju_{n_i} - Jx_{n_i}\rangle \geq -H(u_{n_i},\ y) \geq H(y,\ u_{n_i}), \qquad \forall y \in C.$$

Taking the limit as $n \to \infty$ in above inequality and from (A4) and $u_{n_i} \rightharpoonup u$, we have $H(y, u) \leq 0$, $\forall y \in C$. For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1 - t)u$. Noticing that y, $u \in C$, we obtains $y_t \in C$, which yields that $H(y_t, u) \leq 0$. It follows from (A1) that

$$0 = H(y_t, y_t) \le tH(y_t, y) + (1-t)H(y_t, \hat{x}) \le tH(y_t, y).$$

That is, $H(y_t, y) \geq 0$.

Let $t \downarrow 0$, from (A3), we obtain $H(u, y) \geq 0$, $\forall y \in C$. This implies that $u \in \Omega$. Hence $u \in F := \Omega \cap VI(C, B) \cap T^{-1}(0)$.

Finally, we show that $u = \Pi_F x$. Indeed from $x_n = \Pi_{C_n} x$ and Lemma 2.5, we have

$$\langle Jx - Jx_n, x_n - z \rangle \ge 0, \qquad \forall z \in C_n.$$

Since $F \subset C_n$, we also have

$$\langle Jx - Jx_n, x_n - p \rangle \ge 0, \qquad \forall p \in F. \tag{3.35}$$

Taking limit $n \to \infty$, we obtain

$$\langle Jx - Ju, u - p \rangle \ge 0, \qquad \forall p \in F.$$

By again Lemma 2.5, we can conclude that $u = \Pi_F x_0$. This completes the proof. □

Corollary 3.2. *Let E be a 2-uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E. Let $T: E \to E^*$ be a maximal monotone operator satisfying $D(T) \subset C$. Let $J_r = (J + rT)^{-1} J$ for $r > 0$, let A be an α-inverse-strongly monotone operator of C into E^* and S be a relatively nonexpansive mappings from C into itself, with $F := VI(C, A) \cap T^{-1}(0) \cap F(S) \neq \emptyset$. Assume that A an operator of C into E^* that satisfies the conditions (C1)-(C3). Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and,*

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ y_n = J^{-1}(\beta_n J x_n + (1-\beta_n) JSJ_{r_n} z_n), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n J x_1 + (1-\alpha_n) J y_n), \end{cases} \tag{3.36}$$

for all $n \in \mathbb{N}$, where Π_C is the generalized projection from E onto C, J is the duality mapping on E. The coefficient sequence $\{a_n\} \subset [0, 1]$, $\{\beta_n\} \subset (0, 1]$, $\{r_n\} \subset (0, \infty)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\limsup_{n\to\infty} \beta_n < 1$, $\liminf_{n\to\infty} r_n > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < \frac{c^2\alpha}{2}$, $\frac{1}{c}$ is the 2-uniformly convexity constant of E. Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

4. Weak convergence theorem

We next prove a weak convergence theorem under difference condition on data. First we prove the generalized projection sequence $\{\Pi_F x_0\}$ of $\{x_n\}$ is strongly convergent.

Theorem 4.1. *Let E be a 2-uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E. Let Θ be a bifunction from C × C to $\mathbb{R}$ satisfying (A1)-(A4) let $\phi: C \to \mathbb{R}$ be a proper lower semicontinuous and convex function, let $T: E \to E^*$ be a maximal monotone operator satisfying $D(T) \subset C$. Let $J_r = (J + rT)^{-1} J$ for $r > 0$ and let A be an α-inverse-strongly monotone operator of C into E^*, let $B: C \to E^*$ be a continuous and monotone mappings and S be a relatively nonexpansive mapping. from C into itself, with $F := \Omega \cap VI(C, A) \cap T^{-1}(0) \cap F(S) \neq \emptyset$. Assume that A an operator of C into E^* that satisfies the conditions (C1)-(C3). Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and,*

$$\begin{cases} u_n = K_{r_n} x_n, \\ z_n = \Pi_C J^{-1}(Ju_n - \lambda_n A u_n), \\ y_n = J^{-1}(\beta_n J x_n + (1-\beta_n) JSJ_{r_n} z_n), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n J x_1 + (1-\alpha_n) J y_n), \end{cases} \tag{4.1}$$

for all $n \in \mathbb{N}$, *where* Π_C *is the generalized projection from E onto C, J is the duality mapping on E. The coefficient sequence* $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset (0, 1]$, $\{r_n\} \subset (0, \infty)$ *satisfying* $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\limsup_{n\to\infty} \beta_n < 1$, $\liminf_{n\to\infty} r_n > 0$ *and* $\{\lambda_n\} \subset [a, b]$ *for some a, b with* $0 < a < b < \frac{c^2\alpha}{2}$, $\frac{1}{c}$*is the 2-uniformly convexity constant of E. Then the sequence* $\{\Pi_F x_n\}$ *converges strongly to an element v of F , which is a unique element of F satisfying*

$$\lim_{n\to\infty} \phi(v, x_n) = \min_{y\in F} \lim_{n\to\infty} \phi(y, x_n).$$

Proof. Let $H(u_n, y) = \Theta(u_n, y) + \langle Bu_n, y - u_n\rangle + \phi(y) - \phi(u_n)$, $y \in C$ and $K_{r_n} = \{u_n \in C : H(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0.\ \forall y \in C\}$. We first show that $\{x_n\}$ is bounded. Let $p \in F := \Omega \cap VI(C, A) \cap T^{-1}(0) \cap F(S)$ and $u_n = K_{r_n} x_n$. Put $v_n = J^{-1}(Ju_n - \lambda_n A u_n)$ and $w_n = J_{r_n} z_n$ for all $n \geq 0$. Since J_{r_n}, K_{r_n} and S are relatively nonexpansive mappings. By (3.8), we have that, for all $n \in \mathbb{N}$

$$\phi(p, x_{n+1}) \leq \alpha_n \phi(p, x_1) + (1-\alpha_n)\phi(p, x_n). \tag{4.2}$$

From $\sum_{n=1}^{\infty} \alpha_n < \infty$ and Lemma 2.10, we deduce that $\lim_{n\to\infty}\varphi(p, x_n)$ exists. This implies that $\{\varphi(p, x_n)\}$ is bounded. So $\{x_n\}$ is bounded.

Define a function $g: F \to [0, \infty)$ as follows:

$$g(p) = \lim_{n\to\infty} \phi(p, x_n), \qquad \forall p \in F.$$

Then, by the same argument as in proof of [[22], Theorem 3.1], we obtain g is a continuous convex function and if $||z_n|| \to \infty$ then $g(z_n) \to \infty$. Hence, by [[28], Theorem 1.3.11], there exists a point $v \in F$ such that

$$g(v) = \min_{y\in F} g(y) (:= l). \tag{4.3}$$

Put $t_n = \Pi_F x_n$ for all $n \geq 0$. We next prove that $t_n \to v$ as $n \to \infty$. Suppose on the contrary that there exists $\varepsilon_0 > 0$ such that, for each $n \in \mathbb{N}$, there is $n' \geq n$ satisfying $||w_{n'} - v|| \geq \epsilon_0$. Since $v \in F$, we have

$$\phi(t_n, x_n) = \phi(\Pi_F x_n, x_n) \leq \phi(v, \Pi_F x_n) + \phi(\Pi_F x_n, x_n) \leq \phi(v, x_n) \tag{4.4}$$

for all $n \geq 0$. This implies that

$$\limsup_{n\to\infty} \phi(t_n, x_n) \leq \lim_{n\to\infty} \phi(v, x_n) = l. \tag{4.5}$$

Since $(||v|| - ||\Pi_F x_n||)^2 \leq \varphi(v, w_n) \leq \varphi(v, x_n)$ for all $n \geq 0$ and $\{x_n\}$ is bounded, we get $\{w_n\}$ is also bounded. By Lemma 2.3, there exists a stricly increasing, continuous and convex function $K: [0, \infty) \to [0, \infty)$ such that $K(0) = 0$ and

$$\left\| \frac{w_n + v}{2} \right\|^2 \leq \frac{1}{2} \|t_n\|^2 + \frac{1}{2} \|v\|^2 - \frac{1}{4} K(\|t_n - v\|), \tag{4.6}$$

for all $n \geq 0$. Now, choose σ satisfying $0 < \sigma < \frac{1}{4} K(\epsilon_0)$. Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\phi(t_n, x_n) \leq l + \sigma, \phi(v, x_n) \leq l + \sigma, \tag{4.7}$$

for all $n \geq 0$. Thus there exists $k \geq n_0$ satisfying the following:

$$\phi(t_k, x_k) \leq l + \sigma, \phi(v, x_k) \leq l + \sigma, \|t_k - v\| \geq \epsilon_0. \tag{4.8}$$

From (4.2), (4.6) and (4.8), we obtain

$$\begin{aligned} \phi\left(\frac{t_k + v}{2}, x_{n+k}\right) &\leq \phi\left(\frac{t_k + v}{2}, x_k\right) \\ &= \left\| \frac{t_k + v}{2} \right\|^2 - 2\left\langle \frac{t_k + v}{2}, Jx_k \right\rangle + \|x_k\|^2 \\ &\leq \frac{1}{2} \|t_k\|^2 + \frac{1}{2} \|v\|^2 - \frac{1}{4} K(\|t_k - v\|) - \langle t_k + v, Jx_k \rangle + \|x_k\|^2 \\ &= \frac{1}{2} \phi(t_k, x_k) + \frac{1}{2} \phi(v, x_k) - \frac{1}{4} K(\|t_k - v\|) \\ &\leq l + \sigma - \frac{1}{4} K(\epsilon_0), \end{aligned} \tag{4.9}$$

for all $n \geq 0$. Hence

$$l \leq \lim_{n\to\infty} \phi\left(\frac{t_k + v}{2}, x_n\right) = \lim_{n\to\infty} \phi\left(\frac{t_k + v}{2}, x_{n+k}\right) \leq l + \sigma - \frac{1}{4} K(\epsilon_0) < l + \sigma - \sigma = l. \tag{4.10}$$

This is a contradiction. So, $\{w_n\}$ converges strongly to $v \in F := \Omega \cap VI(C, A) \cap T^{-1}(0) \cap F(S)$. Consequently, $v \in F$ is the unique element of F such that

$$\lim_{n\to\infty} \phi(v, x_n) = \min_{y\in F} \lim_{n\to\infty} \phi(y, x_n). \tag{4.11}$$

This completes the proof. □

Theorem 4.2. *Let E be a 2-uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E. Let $T: E \to E^*$ be a maximal monotone operator satisfying $D(T) \subset C$. Let $J_r = (J + rT)^{-1} J$ for $r > 0$, let A be an α-inverse-strongly monotone operator of C into E^* and S be a relatively nonexpansive mappings from C into itself, with $F := VI(C, A) \cap T^{-1}(0) \cap F(S) \neq \emptyset$. Assume that A an operator of C into E^* that satisfies the conditions (C1)-(C3). Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and,*

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n) JSJ_{r_n} z_n), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n) Jy_n), \end{cases} \tag{4.12}$$

for all $n \in \mathbb{N}$, where Π_C is the generalized projection from E onto C, J is the duality mapping on E. The coefficient sequence $\{a_n\} \subset [0, 1]$, $\{\beta_n\} \subset (0, 1]$, $\{r_n\} \subset (0, \infty)$ satisfying $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\limsup_{n\to\infty} \beta_n < 1$, $\liminf_{n\to\infty} r_n > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b

with $0 < a < b < \frac{c^2\alpha}{2}$, $\frac{1}{c}$*is the 2-uniformly convexity constant of E. Then the sequence* $\{\Pi_F x_n\}$ *converges strongly to an element v of F, which is a unique element of F satisfying*

$$\lim_{n\to\infty} \phi(v, x_n) = \min_{y\in F} \lim_{n\to\infty} \phi(y, x_n).$$

Now, we prove a weak convergence theorem for the algorithm (4.13) below under different condition on data.

Theorem 4.3. *Let E be a 2-uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E. Let* Θ *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying (A1)-(A4) let* $\phi: C \to \mathbb{R}$ *be a proper lower semicontinuous and convex function, let* $T: E \to E^*$ *be a maximal monotone operator satisfying* $D(T) \subset C$. *Let* $J_r = (J + rT)^{-1} J$ *for* $r > 0$ *and let A be an* α*-inverse-strongly monotone operator of C into* E^*, *let* $B: C \to E^*$ *be a continuous and monotone mappings and S be a relatively nonexpansive mappings from C into itself, with* $F := \Omega \cap VI(C, A) \cap T^{-1}(0) \cap F(S) \neq \emptyset$. *Assume that A an operator of C into* E^* *that satisfies the conditions (C1)-(C3). Let* $\{x_n\}$ *be a sequence generated by* $x_1 = x \in C$ *and,*

$$\begin{cases} u_n = K_{r_n} x_n, \\ z_n = \Pi_C J^{-1}(Ju_n - \lambda_n A u_n), \\ y_n = J^{-1}(\beta_n J x_n + (1-\beta_n) JSJ_{r_n} z_n), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n J x_1 + (1-\alpha_n) J y_n), \end{cases} \tag{4.13}$$

for all $n \in \mathbb{N}$, *where* Π_C *is the generalized projection from E onto C, J is the duality mapping on E. The coefficient sequence* $\{\alpha_n\} \subset [0, 1]$, $\{\beta n\} \subset (0, 1]$, $\{r_n\} \subset (0, \infty)$ *satisfying* $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\limsup_{n\to\infty} \beta_n < 1$, $\liminf_{n\to\infty} r_n > 0$ *and* $\{\lambda_n\} \subset [a, b]$ *for some a, b with* $0 < a < b < \frac{c^2\alpha}{2}$, $\frac{1}{c}$*is the 2-uniformly convexity constant of E. Then the sequence* $\{x_n\}$ *converges weakly to an element v of F , where* $v = \lim_{n\to\infty} \Pi_F x_n$.

Proof. As in Proof of Theorem 3.1, we have $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u \in C$ and hence $u \in F := \Omega \cap VI(C, A) \cap T^{-1}(0) \cap F(S)$. By Theorem 4.1 the $\{\Pi_F x_n\}$ converges strongly to a point $v \in F$ which is a unique element of F such that

$$\lim_{n\to\infty} \phi(v, x_n) = \min_{y\in F} \lim_{n\to\infty} \phi(y, x_n). \tag{4.14}$$

By the uniform smoothness of E, we also have $\lim_{n\to\infty} \|J\prod_F x_{n_i} - Jv\| = 0$.

Finally, we prove $u = v$. From Lemma 2.5 and $u \in F$, we have

$$\langle \Pi_F x_{n_i} - u, Jx_{n_i} - J\Pi_F x_{n_i} \rangle \geq 0$$

Since J is weakly sequentially continuous, $u_{n_i} \rightharpoonup u$ and $u_n - x_n \to 0$, then

$$\langle v - u, Ju - Jv \rangle \geq 0.$$

On the other hand, since J is monotone, we have

$$\langle v - u, Ju - Jv \rangle \leq 0.$$

Hence,

$$\langle v-u,\ Ju-Jv\rangle = 0.$$

Since E is strict convexity, it follows that $u = v$. Therefore the sequence $\{x_n\}$ converges weakly to $v = \lim_{n\to\infty} \Pi_F x_n$. This completes the proof. □

Theorem 4.4. *Let E be a 2-uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E. Let* $T: E \to E^*$ *be a maximal monotone operator satisfying* $D(T) \subset C$. *Let* $J_r = (J + rT)^{-1} J$ *for* $r > 0$, *let A be an* α*-inverse-strongly monotone operator of C into* E^* *and S be a relatively nonexpansive mappings from C into itself, with* $F := VI(C,A) \cap T^{-1}(0) \cap F(S) \neq \emptyset$. *Assume that A an operator of C into* E^* *that satisfies the conditions (C1)-(C3). Let* $\{x_n\}$ *be a sequence generated by* $x_1 = x \in C$ *and,*

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n) JSJ_{r_n} z_n), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_1 + (1-\alpha_n) Jy_n), \end{cases} \tag{4.15}$$

for all $n \in \mathbb{N}$, *where* Π_C *is the generalized projection from E onto C, J is the duality mapping on E. The coefficient sequence* $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset (0, 1]$, $\{r_n\} \subset (0, \infty)$ *satisfying* $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\limsup_{n\to\infty} \beta_n < 1$, $\liminf_{n\to\infty} r_n > 0$ *and* $\{\lambda_n\} \subset [a, b]$ *for some a, b with* $0 < a < b < \frac{c^2\alpha}{2}$, $\frac{1}{c}$ *is the 2-uniformly convexity constant of E. Then the sequence* $\{x_n\}$ *converges weakly to an element v of F, where* $v = \lim_{n\to\infty} \Pi_F x_n$.

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