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Fractional Navier boundary value problems

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Abstract

We study the following fractional Navier boundary value problem:

$$\begin{cases} D^{\alpha}(D^{\beta}u)(x) + u(x)f(x,u(x)) = 0, \quad 0 < x < 1, \\ \lim_{x\to 0^+} D^{\beta-1}u(x) = 0, \qquad \lim_{x\to 0^+} D^{\alpha-1}(D^{\beta}u)(x) = \xi, \\ u(1) = 0, \qquad D^{\beta}u(1) = -\zeta, \end{cases}$$

where $\alpha, \beta \in (1,2]$, D^{α} and D^{β} stand for the standard Riemann-Liouville fractional derivatives, and $\xi, \zeta \ge 0$ are such that $\xi + \zeta > 0$.

Our purpose is to prove the existence, uniqueness, and global asymptotic behavior of a positive continuous solution, where $f : (0,1) \times [0,\infty) \to [0,\infty)$ is continuous and dominated by a function p satisfying appropriate integrability condition.

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1 Introduction

The existence, uniqueness, and global asymptotic behavior of positive continuous solutions related to fractional differential equations have been studied by many researchers. Many fractional differential equations subject to various boundary conditions have been addressed; see, for instance, [1–12] and the references therein. It is known that fractional differential equations serve as a good tool to model many phenomena in various fields of science and engineering (see [13–24] and references therein for discussions of various applications).

In [2], the authors proved the existence and uniqueness of a positive solution to the following fractional boundary value problem:

$$\begin{cases} D^{\alpha}u(x) = u(x)\varphi(x,u(x)), \quad 0 < x < 1, \\ \lim_{x\to 0^+} D^{\alpha-1}u(x) = -\xi, \qquad u(1) = \zeta, \end{cases} \tag{1.1}$$

where $1 < \alpha \le 2$, $\xi, \zeta \ge 0$ are such that $\xi + \zeta > 0$, and $\varphi(x,s) \in C^{+}((0,1) \times [0,\infty))$ satisfies appropriate conditions. Inspired by the above-mentioned paper, we aim at studying similar problem in the case of fractional Navier boundary value problem. More precisely, we are concerned with the problem

$$\begin{cases} D^\alpha(D^\beta u)(x) + u(x)f(x,u(x)) = 0, & 0 < x < 1, \\ \lim_{x\to 0^+} D^{\beta-1}u(x) = 0, & \lim_{x\to 0^+} D^{\alpha-1}(D^\beta u)(x) = \xi, \\ u(1) = 0, & D^\beta u(1) = -\zeta, \end{cases} \tag{1.2}$$

where $\alpha,\beta \in (1,2]$, and $\xi,\zeta \geq 0$ are such that $\xi + \zeta > 0$. The nonlinear term $f(x,s)$ is required to be a nonnegative continuous function in $(0,1) \times [0,\infty)$ dominated by a function p belonging to the class $\mathcal{J}_{\alpha,\beta}$ defined as follows.

Definition 1.1 Let $\alpha,\beta \in (1,2]$. A nonnegative measurable function p on $(0,1)$ belongs to the class $\mathcal{J}_{\alpha,\beta}$ iff

$$\int_0^1 t^{\beta-2}(1-t)^\alpha p(t)\,dt < \infty. \tag{1.3}$$

Next, we introduce the following notation.

(i) $B^+((0,1))$ is the set of nonnegative measurable functions in $(0,1)$.

(ii) Let X be a metric space, we denote by $C(X)$ (resp. $C^+(X)$) the set of continuous (resp. nonnegative continuous) functions in X.

(iii) For $\gamma \in (1,2]$, $C_{2-\gamma}([0,1]) = \{w \in C((0,1]) : x \to x^{2-\gamma}w(x) \in C([0,1])\}$.

(iv) For $\gamma \in (1,2]$, $G_\gamma(x,s)$ is the Green function of the operator $u \to -D^\gamma u$, with boundary data $\lim_{x\to 0^+} D^{\gamma-1}u(x) = u(1) = 0$. From [8], Lemma 7, we have

$$G_\gamma(x,s) = \frac{1}{\Gamma(\gamma)}\bigl(x^{\gamma-2}(1-s)^{\gamma-1} - \bigl((x-s)^+\bigr)^{\gamma-1}\bigr), \tag{1.4}$$

where $x^+ = \max(x,0)$.

Proposition 1.2 (see [8]) *Let $1 < \gamma \leq 2$ and $\varphi \in B^+((0,1))$. Then we have*

(i) *For $(x,s) \in (0,1] \times [0,1]$,*

$$\frac{(\gamma-1)}{\Gamma(\gamma)}H(x,s) \leq G_\gamma(x,s) \leq \frac{1}{\Gamma(\gamma)}H(x,s), \tag{1.5}$$

where $H(x,s) := x^{\gamma-2}(1-s)^{\gamma-2}(1-\max(x,s))$.

(ii) *The function $x \to G_\gamma\varphi(x) := \int_0^1 G_\gamma(x,s)\varphi(s)\,ds$ belongs to $C_{2-\gamma}([0,1])$ if and only if $\int_0^1 (1-s)^{\gamma-1}\varphi(s)\,ds < \infty$.*

(iii) *If the map $s \to (1-s)^{\gamma-1}\varphi(s) \in C((0,1)) \cap L^1((0,1))$, then $G_\gamma\varphi$ belongs to $C_{2-\gamma}([0,1])$, and it is the unique solution of the problem*

$$\begin{cases} D^\gamma u(x) = -\varphi(x), & 0 < x < 1, \\ \lim_{x\to 0^+} D^{\gamma-1}u(x) = u(1) = 0. \end{cases}$$

Throughout this paper, for $\alpha,\beta \in (1,2]$, let $G(x,s)$ be the Green function of the operator $u \to D^\alpha(D^\beta u)$ with Navier boundary conditions $\lim_{x\to 0^+} D^{\beta-1}u(x) = \lim_{x\to 0^+} D^{\alpha-1}(D^\beta u)(x) =$

$u(1) = D^\beta u(1) = 0$. Then we have

$$G(x,s) = \int_0^1 G_\beta(x,t)G_\alpha(t,s)\,dt. \tag{1.6}$$

For a given function p in $B^+((0,1))$, we put

$$\kappa_p := \sup_{x,s\in(0,1)} \int_0^1 \frac{G(x,t)G(t,s)}{G(x,s)} p(t)\,dt, \tag{1.7}$$

and we will prove that $\kappa_p < \infty$ if and only if $p \in \mathcal{J}_{\alpha,\beta}$.

From here on, let ξ, ζ be two nonnegative constants such that $\xi + \zeta > 0$, and $\theta(x)$ be the unique solution of the problem

$$\begin{cases} D^\alpha(D^\beta u)(x) = 0, \quad 0 < x < 1, \\ \lim_{x\to 0^+} D^{\beta-1}u(x) = 0, \qquad \lim_{x\to 0^+} D^{\alpha-1}(D^\beta u)(x) = \xi, \\ u(1) = 0, \qquad D^\beta u(1) = -\zeta. \end{cases} \tag{1.8}$$

We can easily verify that, for $x \in (0,1]$, $\theta(x) = \xi h_1(x) + \zeta h_2(x)$, where

$$\begin{aligned} h_1(x) &= \int_0^1 G_\beta(x,t)G_\alpha(t,0)\,dt \\ &= \frac{1}{(\alpha-1)\Gamma(\alpha+\beta-1)} x^{\beta-2}\bigl(1-x^\alpha\bigr) + \frac{1}{\Gamma(\alpha+\beta)} x^{\beta-2}\bigl(x^{\alpha+1}-1\bigr) \end{aligned} \tag{1.9}$$

and

$$h_2(x) = \int_0^1 G_\beta(x,t)t^{\alpha-2}\,dt = \frac{\Gamma(\alpha-1)}{\Gamma(\alpha+\beta-1)} x^{\beta-2}\bigl(1-x^\alpha\bigr). \tag{1.10}$$

Note that from (1.5), (1.9), and (1.10) it follows that there exists a constant $c > 0$ such that, for each $x \in (0,1]$,

$$\frac{1}{c} x^{\beta-2}(1-x) \le \theta(x) \le c x^{\beta-2}(1-x). \tag{1.11}$$

To state our existence results, a combination of the following hypotheses are required.

(A_1) f is in $C^+((0,1)\times[0,\infty))$.
(A_2) There exists $p \in \mathcal{J}_{\alpha,\beta} \cap C^+((0,1))$ with $\kappa_p \le \frac{1}{2}$ such that, for each $x \in (0,1)$, the map $s \to s(p(x) - f(x, s\theta(x)))$ is nondecreasing on $[0,1]$.
(A_3) For each $x \in (0,1)$, the function $s \to sf(x,s)$ is nondecreasing on $[0,\infty)$.

Our main results are the following.

Theorem 1.3 *Under conditions* (A_1)-(A_2), *problem* (1.2) *admits a solution* $u \in C_{2-\beta}([0,1])$ *such that*

$$c_0\theta(x) \le u(x) \le \theta(x), \quad 0 < x \le 1, \tag{1.12}$$

where $c_0 \in (0,1)$.

Moreover, this solution is unique if hypothesis (A_3) *is also satisfied.*

Corollary 1.4 *Let* $\alpha, \beta \in (1,2]$, *and* h *be a nonnegative function in* $C^1([0,\infty))$ *such that the map* $s \to \varrho(s) = sh(s)$ *is nondecreasing on* $[0,\infty)$. *Let* $q \in C^+((0,1))$ *and assume that the function* $\tilde{q}(x) := q(x) \max_{0\leq t\leq \theta(x)} \varrho'(t)$ *belongs to* $\mathcal{J}_{\alpha,\beta}$. *Then for* $\lambda \in [0, \frac{1}{2\kappa_{\tilde{q}}})$, *the problem*

$$\begin{cases} D^\alpha(D^\beta u)(x) + \lambda q(x)u(x)h(u(x)) = 0, \quad 0 < x < 1, \\ \lim_{x\to 0^+} D^{\beta-1}u(x) = 0, \qquad \lim_{x\to 0^+} D^{\alpha-1}(D^\beta u)(x) = \xi, \\ u(1) = 0, \qquad D^\beta u(1) = -\zeta, \end{cases} \tag{1.13}$$

admits a unique solution $u \in C_{2-\beta}([0,1])$ *such that*

$$(1 - \lambda\kappa_{\tilde{q}})\theta(x) \leq u(x) \leq \theta(x), \quad 0 < x \leq 1.$$

Our paper is organized as follows. In Section 2, we establish some properties of $G(x,s)$. In particular, we prove the existence of a constant $c > 0$ such that, for all $x, t, s \in (0,1)$,

$$\frac{1}{c}t^{\beta-2}(1-t)^\alpha \leq \frac{G(x,t)G(t,s)}{G(x,s)} \leq ct^{\beta-2}(1-t)^\alpha.$$

This implies that $\kappa_p < \infty$ if and only if $p \in \mathcal{J}_{\alpha,\beta}$. In Section 3, for a given function $p \in \mathcal{J}_{\alpha,\beta}$ with $\kappa_p \leq \frac{1}{2}$, we construct the Green function $\mathcal{H}(x,s)$ of the operator $u \to D^\alpha(D^\beta u) + p(x)u$ with boundary conditions $\lim_{x\to 0^+} D^{\beta-1}u(x) = \lim_{x\to 0^+} D^{\alpha-1}(D^\beta u)(x) = u(1) = D^\beta u(1) = 0$, and we derive some of its properties including the following:

$$(1 - \kappa_p)G(x,s) \leq \mathcal{H}(x,s) \leq G(x,s) \quad \text{for all } (x,s) \in (0,1] \times [0,1]$$

and

$$W\varphi = W_p\varphi + W_p(pW\varphi) = W_p\varphi + W(pW_p\varphi) \quad \text{for } \varphi \in \mathcal{B}^+\big((0,1)\big),$$

where W and W_p are defined by

$$W\varphi(x) := \int_0^1 G(x,s)\varphi(s)\,ds \quad \text{and} \quad W_p\varphi(x) := \int_0^1 \mathcal{H}(x,s)\varphi(s)\,ds, \quad x \in (0,1].$$

Exploiting these results, we prove our main results by means of a perturbation argument.

2 Estimates on the Green function

We recall the definition of the Riemann-Liouville derivative.

Definition 2.1 (see [17, 22, 23]) The Riemann-Liouville derivative of fractional order $\gamma > 0$ of a function g is defined as

$$D^\gamma g(x) := \frac{1}{\Gamma(n-\gamma)}\left(\frac{d}{dx}\right)^n \int_0^x (x-s)^{n-\gamma-1}g(s)\,ds,$$

where $n - 1 \leq \gamma < n \in \mathbb{N}$.

Next, we prove some properties of $G(x,s)$.

Proposition 2.2 *Let $\alpha,\beta \in (1,2]$. Then there exist two constants $m > 0$ and $M > 0$ such that, for all $(x,s) \in (0,1] \times [0,1]$, we have*

$$mx^{\beta-2}(1-x)(1-s)^{\alpha-1} \leq G(x,s) \leq Mx^{\beta-2}(1-x)(1-s)^{\alpha-1}. \tag{2.1}$$

Proof Using (1.6) and (1.5), we have

$$\begin{aligned} G(x,s) &= \int_0^1 G_\beta(x,t)G_\alpha(t,s)\,dt \\ &\leq \frac{1}{\Gamma(\beta)}\int_0^1 x^{\beta-2}(1-x)(1-t)^{\beta-2}G_\alpha(t,s)\,dt \\ &\leq \frac{x^{\beta-2}(1-x)}{\Gamma(\beta)\Gamma(\alpha)}\int_0^1 (1-t)^{\beta-2}t^{\alpha-2}(1-s)^{\alpha-1}\,dt \\ &= Mx^{\beta-2}(1-x)(1-s)^{\alpha-1}. \end{aligned}$$

On the other hand, using again (1.6), (1.5), and the inequality $1-\max(x,s) \geq (1-x)(1-s)$, we get

$$\begin{aligned} G(x,s) &= \int_0^1 G_\beta(x,t)G_\alpha(t,s)\,dt \\ &\geq \frac{(\beta-1)}{\Gamma(\beta)}\frac{(\alpha-1)}{\Gamma(\alpha)}\int_0^1 x^{\beta-2}(1-x)(1-t)^{\beta}t^{\beta-2}(1-s)^{\alpha-1}\,dt \\ &= mx^{\beta-2}(1-x)(1-s)^{\alpha-1}. \end{aligned}$$

□

Using Proposition 2.2, we deduce the following.

Corollary 2.3 *Let $\alpha,\beta \in (1,2]$. Then there exists a constant $c > 0$ such that, for all $x,t,s \in (0,1)$, we have*

$$\frac{1}{c}t^{\beta-2}(1-t)^{\alpha} \leq \frac{G(x,t)G(t,s)}{G(x,s)} \leq ct^{\beta-2}(1-t)^{\alpha}. \tag{2.2}$$

Proposition 2.4 *Let $\alpha,\beta \in (1,2]$, and p be a function in $B^+((0,1))$.*

(i) *There exists a constant $c > 0$ such that*

$$\frac{1}{c}\int_0^1 t^{\beta-2}(1-t)^{\alpha}p(t)\,dt \leq \kappa_p \leq c\int_0^1 t^{\beta-2}(1-t)^{\alpha}p(t)\,dt, \tag{2.3}$$

where κ_p is given by (1.7).

In particular,

$$\kappa_p < \infty \quad \text{if and only if} \quad p \in \mathcal{J}_{\alpha,\beta}. \tag{2.4}$$

(ii) *For $x \in (0,1]$, we have*

$$W(\theta p)(x) \leq \kappa_p\theta(x), \tag{2.5}$$

where $\theta(x) := \xi h_1(x) + \zeta h_2(x)$, and h_1 and h_2 are given respectively in (1.9) *and* (1.10).

Proof Let p be a function in $B^+((0,1))$.

(i) Inequalities in (2.3) follow immediately from (2.2) and (1.7).

(ii) Since $\theta(x) := \xi h_1(x) + \zeta h_2(x)$, it suffices to prove (2.5) for h_1 and h_2. To this end, observe that from (1.6) it follows that, for each $x, s \in (0,1)$,

$$\lim_{r\to 0} \frac{G(s,r)}{G(x,r)} = \frac{G(s,0)}{G(x,0)} = \frac{h_1(s)}{h_1(x)}.$$

So by Fatou's lemma and (1.7) we deduce that

$$\int_0^1 G(x,s)\frac{h_1(s)}{h_1(x)}p(s)\,ds \le \liminf_{r\to 0} \int_0^1 G(x,s)\frac{G(s,r)}{G(x,r)}p(s)\,ds \le \kappa_p,$$

that is,

$$W(h_1 p)(x) \le \kappa_p h_1(x) \quad \text{for } x \in (0,1].$$

Similarly, we prove that $W(h_2 p)(x) \le \kappa_p h_2(x)$ by observing that

$$\lim_{r\to 1} \frac{G(s,r)}{G(x,r)} = \frac{h_2(s)}{h_2(x)}.$$

This ends the proof. □

Corollary 2.5 *Let $\alpha, \beta \in (1,2]$ and $\varphi \in \mathcal{B}^+((0,1))$. Then $x \to W\varphi(x) \in C_{2-\beta}([0,1])$ if and only if $\int_0^1 (1-s)^{\alpha-1}\varphi(s)\,ds < \infty$.*

Proof The assertion follows from (2.1) and the dominated convergence theorem. □

Proposition 2.6 *Let $\alpha, \beta \in (1,2]$ and $\varphi \in \mathcal{B}^+((0,1))$ be such that $s \to (1-s)^{\alpha-1}\varphi(s) \in C((0,1)) \cap L^1((0,1))$. Then $W\varphi$ is the unique nonnegative solution in $C_{2-\beta}([0,1])$ of*

$$\begin{cases} D^\alpha(D^\beta u)(x) = \varphi(x), \quad 0 < x < 1, \\ \lim_{x\to 0^+} D^{\beta-1}u(x) = \lim_{x\to 0^+} D^{\alpha-1}(D^\beta u)(x) = u(1) = D^\beta u(1) = 0. \end{cases}$$

Proof Let $\varphi \in \mathcal{B}^+((0,1))$. From (1.6) and the Fubini-Tonelli theorem we obtain

$$W\varphi(x) = \int_0^1 G_\beta(x,t)G_\alpha\varphi(t)\,dt, \tag{2.6}$$

where $G_\alpha\varphi(t) = \int_0^1 G_\alpha(t,s)\varphi(s)\,ds$.

Since the function $s \to (1-s)^{\alpha-1}\varphi(s) \in C((0,1)) \cap L^1((0,1))$, we deduce by Proposition 1.2 that $G_\alpha\varphi$ is the unique solution in $C_{2-\alpha}([0,1])$ of

$$\begin{cases} D^\alpha v(x) = -\varphi(x), \quad 0 < x < 1, \\ \lim_{x\to 0^+} D^{\alpha-1}v(x) = v(1) = 0. \end{cases} \tag{2.7}$$

On the other hand, by using (1.5) we deduce that

$$\int_0^1 (1-t)^{\beta-1} G_\alpha \varphi(t)\,dt \le \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\beta-1} \left(\int_0^1 t^{\alpha-2}(1-s)^{\alpha-1}\varphi(s)\,ds \right) dt$$
$$\le \frac{\Gamma(\beta)}{(\alpha-1)\Gamma(\alpha+\beta-1)} \int_0^1 (1-s)^{\alpha-1}\varphi(s)\,ds < \infty.$$

Hence, the function $t \to (1-t)^{\beta-1} G_\alpha \varphi(t) \in C((0,1)) \cap L^1((0,1))$. Therefore, using (2.6) and Proposition 1.2, we deduce that $W\varphi$ is the unique solution in $C_{2-\beta}([0,1])$ of

$$\begin{cases} D^\beta u(x) = -G_\alpha \varphi(x), & 0 < x < 1, \\ \lim_{x\to 0^+} D^{\beta-1} u(x) = u(1) = 0. \end{cases} \tag{2.8}$$

Combining (2.7) and (2.8), we obtain the required result. □

3 Proofs of main results

Let $\alpha, \beta \in (1,2]$. For $(x,s) \in (0,1] \times [0,1]$, put $H_0(x,s) = G(x,s)$ and

$$H_n(x,s) = \int_0^1 G(x,t) H_{n-1}(t,s) p(t)\,dt, \quad n \ge 1. \tag{3.1}$$

Now, let $\mathcal{H} : (0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$\mathcal{H}(x,s) = \sum_{n=0}^{\infty} (-1)^n H_n(x,s), \tag{3.2}$$

provided that the series converges.

Lemma 3.1 *Let $\alpha, \beta \in (1,2]$ and $m, M > 0$ be as in (2.1). Let $p \in \mathcal{J}_{\alpha,\beta}$ with $\kappa_p < 1$. Then on $(0,1] \times [0,1]$, we have*

(i) *$H_n(x,s) \le \kappa_p^n G(x,s)$ for each $n \in \mathbb{N}$.*
So, $\mathcal{H}(x,s)$ is well defined in $(0,1] \times [0,1]$.

(ii) *For each $n \in \mathbb{N}$,*

$$l_n x^{\beta-2}(1-x)(1-s)^{\alpha-1} \le H_n(x,s) \le r_n x^{\beta-2}(1-x)(1-s)^{\alpha-1}, \tag{3.3}$$

where

$$l_n = m^{n+1} \left(\int_0^1 t^{\beta-2}(1-t)^\alpha p(t)\,dt \right)^n \quad \text{and} \quad r_n = M^{n+1} \left(\int_0^1 t^{\beta-2}(1-t)^\alpha p(t)\,dt \right)^n.$$

(iii) *$H_{n+1}(x,s) = \int_0^1 H_n(x,t) G(t,s) p(t)\,dt$ for each $n \in \mathbb{N}$.*

(iv) *$\int_0^1 \mathcal{H}(x,t) G(t,s) p(t)\,dt = \int_0^1 G(x,t) \mathcal{H}(t,s) p(t)\,dt$.*

Proof By simple induction we prove (i), (ii), and (iii).

(iv) By Lemma 3.1(i) we have

$$0 \le H_n(x,t) G(t,s) p(t) \le \kappa_p^n G(x,t) G(t,s) p(t) \quad \text{for } n \ge 0 \text{ and all } x,t,s \in (0,1].$$

Therefore, the series $\sum_{n\geq 0}\int_0^1 H_n(x,t)G(t,s)p(t)\,dt$ converges.

So by applying the dominated convergence theorem we deduce that

$$\begin{aligned}\int_0^1 \mathcal{H}(x,t)G(t,s)p(t)\,dt &= \sum_{n=0}^{\infty}\int_0^1 (-1)^n H_n(x,t)G(t,s)p(t)\,dt \\ &= \sum_{n=0}^{\infty}\int_0^1 (-1)^n G(x,t)H_n(t,s)p(t)\,dt \\ &= \int_0^1 G(x,t)\mathcal{H}(t,s)p(t)\,dt.\end{aligned}$$

□

Proposition 3.2 *Let $\alpha,\beta \in (1,2]$ and $p \in \mathcal{J}_{\alpha,\beta}$ with $\kappa_p < 1$. Then the function $(x,s) \to x^{2-\beta}\mathcal{H}(x,s) \in C([0,1]\times[0,1])$.*

Proof Clearly, the function $(x,s) \to x^{2-\beta}H_0(x,s) \in C([0,1]\times[0,1])$.

Assume that the function $(x,s) \to x^{2-\beta}H_{n-1}(x,s) \in C([0,1]\times[0,1])$.

Using Lemma 3.1(i) and (2.1), we have, for all $(x,s,t) \in [0,1]\times[0,1]\times(0,1]$,

$$\begin{aligned}x^{2-\beta}G(x,t)H_{n-1}(t,s)p(t) &\leq \kappa_p^{n-1}x^{2-\beta}G(x,t)G(t,s)p(t) \\ &\leq M^2(1-x)(1-t)^{\alpha-1}t^{\beta-2}(1-t)(1-s)^{\alpha-1}p(t) \\ &\leq M^2 t^{\beta-2}(1-t)^{\alpha}p(t).\end{aligned}$$

So by (3.1) and the dominated convergence theorem we conclude that the function $(x,s) \to x^{2-\beta}H_n(x,s) \in C([0,1]\times[0,1])$.

From Lemma 3.1(i) and (2.1) we deduce that

$$x^{2-\beta}H_n(x,s) \leq \kappa_p^n x^{2-\beta}G(x,s) \leq M\kappa_p^n. \tag{3.4}$$

Therefore, the series $\sum_{n\geq 0}(-1)^n x^{2-\beta}H_n(x,s)$ is uniformly convergent on $[0,1]\times[0,1]$, and so the function $(x,s) \to x^{2-\beta}\mathcal{H}(x,s) \in C([0,1]\times[0,1])$. □

Lemma 3.3 *Let $\alpha,\beta \in (1,2]$ and $p \in \mathcal{J}_{\alpha,\beta}$ with $\kappa_p \leq \frac{1}{2}$. Then for $(x,s) \in (0,1]\times[0,1]$, we have*

$$(1-\kappa_p)G(x,s) \leq \mathcal{H}(x,s) \leq G(x,s). \tag{3.5}$$

Proof Let $p \in \mathcal{J}_{\alpha,\beta}$ with $\kappa_p \leq \frac{1}{2}$. By Lemma 3.1(i) we deduce that

$$\left|\mathcal{H}(x,s)\right| \leq \sum_{n=0}^{\infty}(\kappa_p)^n G(x,s) = \frac{1}{1-\kappa_p}G(x,s). \tag{3.6}$$

Now, from the expression of $\mathcal{H}$ we have

$$\mathcal{H}(x,s) = G(x,s) - \sum_{n=0}^{\infty}(-1)^n H_{n+1}(x,s). \tag{3.7}$$

Since the series $\sum_{n\geq 0}\int_0^1 G(x,t)H_n(t,s)p(t)\,dt$ converges, we conclude by (3.7) and (3.1) that

$$\begin{aligned}\mathcal{H}(x,s) &= G(x,s) - \sum_{n=0}^{\infty}(-1)^n\int_0^1 G(x,t)H_n(t,s)p(t)\,dt\\ &= G(x,s) - \int_0^1 G(x,t)\Bigl(\sum_{n=0}^{\infty}(-1)^n H_n(t,s)\Bigr)p(t)\,dt,\end{aligned}$$

namely,

$$\mathcal{H}(x,s) = G(x,s) - W\bigl(p\mathcal{H}(\cdot,s)\bigr)(x). \tag{3.8}$$

On the other hand, since

$$\begin{aligned}W\bigl(p\mathcal{H}(\cdot,s)\bigr)(x) &\leq \frac{1}{1-\kappa_p}W\bigl(pG(\cdot,s)\bigr)(x)\\ &= \frac{1}{1-\kappa_p}H_1(x,s) \leq \frac{\kappa_p}{1-\kappa_p}G(x,s),\end{aligned} \tag{3.9}$$

we deduce that

$$\mathcal{H}(x,s) \geq G(x,s) - \frac{\kappa_p}{1-\kappa_p}G(x,s) = \frac{1-2\kappa_p}{1-\kappa_p}G(x,s) \geq 0.$$

Hence, $\mathcal{H}(x,s) \leq G(x,s)$, and by (3.8) we have

$$\mathcal{H}(x,s) \geq G(x,s) - W\bigl(pG(\cdot,s)\bigr)(x) \geq (1-\kappa_p)G(x,s).$$ □

Corollary 3.4 *Let $\alpha,\beta \in (1,2]$ and $p \in \mathcal{J}_{\alpha,\beta}$ with $\kappa_p \leq \frac{1}{2}$.*
Let $\varphi \in \mathcal{B}^+((0,1))$. Then

$$W_p\varphi \in C_{2-\beta}\bigl([0,1]\bigr) \quad \textit{if and only if} \quad \int_0^1 (1-s)^{\alpha-1}\varphi(s)\,ds < \infty.$$

Proof The assertion follows from Proposition 3.2, (3.5), and (2.1). □

Lemma 3.5 *Let $\alpha,\beta \in (1,2]$ and $p \in \mathcal{J}_{\alpha,\beta}$ with $\kappa_p \leq \frac{1}{2}$.*
Let $h \in \mathcal{B}^+((0,1))$. Then we have, for $x \in (0,1]$,

$$Wh(x) = W_ph(x) + W_p(pWh)(x) = W_ph(x) + W(pW_ph)(x). \tag{3.10}$$

In particular, if $W(ph) < \infty$, then

$$\bigl(I - W_p(p\cdot)\bigr)\bigl(I + W(p\cdot)\bigr)h = \bigl(I + W(p\cdot)\bigr)\bigl(I - W_p(p\cdot)\bigr)h = h. \tag{3.11}$$

Proof Let $(x,s) \in (0,1] \times [0,1]$. Then by (3.8) we have

$$G(x,s) = \mathcal{H}(x,s) + W\bigl(p\mathcal{H}(\cdot,s)\bigr)(x).$$

Let $h \in \mathcal{B}^+((0,1))$. Using the Fubini theorem, we obtain

$$\begin{aligned} Wh(x) &= \int_0^1 \bigl(\mathcal{H}(x,s) + W\bigl(p\mathcal{H}(\cdot,s)\bigr)(x)\bigr)h(s)\,ds \\ &= W_p h(x) + W(pW_p h)(x). \end{aligned}$$

Using Lemma 3.1(iv) and again the Fubini theorem, we have

$$\int_0^1 \int_0^1 \mathcal{H}(x,t)G(t,s)p(t)h(s)\,dt\,ds = \int_0^1 \int_0^1 G(x,t)\mathcal{H}(t,s)p(t)h(s)\,dt\,ds,$$

that is,

$$W_p(pWh)(x) = W(pW_p h)(x).$$

So

$$Wh(x) = W_p h(x) + W(pW_p h)(x) = W_p h(x) + W_p(pWh)(x). \qquad \square$$

Proposition 3.6 *Let $\alpha, \beta \in (1,2]$ and $p \in \mathcal{J}_{\alpha,\beta} \cap C((0,1))$ with $\kappa_p \leq \frac{1}{2}$. Let $\varphi \in \mathcal{B}^+((0,1))$ be such that $s \to (1-s)^{\alpha-1}\varphi(s) \in C((0,1)) \cap L^1((0,1))$. Then $W_p\varphi \in C_{2-\beta}([0,1])$, and it is the unique nonnegative solution of the problem*

$$\begin{cases} D^\alpha(D^\beta u)(x) + p(x)u(x) = \varphi(x), \quad 0 < x < 1, \\ \lim_{x\to 0^+} D^{\beta-1}u(x) = \lim_{x\to 0^+} D^{\alpha-1}(D^\beta u)(x) = u(1) = D^\beta u(1) = 0, \end{cases} \tag{3.12}$$

satisfying

$$(1-\kappa_p)W\varphi \leq u \leq W\varphi. \tag{3.13}$$

Proof By Corollary 3.4 the function $x \to p(x)W_p\varphi(x) \in C((0,1))$.

Using (3.10) and (2.1), we have that there exists $c \geq 0$ such that

$$W_p\varphi(x) \leq W\varphi(x) \leq M \int_0^1 x^{\beta-2}(1-x)(1-s)^{\alpha-1}\varphi(s)\,ds = cx^{\beta-2}(1-x). \tag{3.14}$$

Therefore,

$$\int_0^1 (1-s)^{\alpha-1}p(s)W_p\varphi(s)\,ds \leq c\int_0^1 s^{\beta-2}(1-s)^\alpha p(s)\,ds < \infty.$$

Hence, by Proposition 2.6 the function $u = W_p\varphi = W\varphi - W(pW_p\varphi)$ satisfies the equation

$$\begin{cases} D^\alpha(D^\beta u)(x) = \varphi(x) - p(x)u(x), \quad 0 < x < 1, \\ \lim_{x\to 0^+} D^{\beta-1}u(x) = \lim_{x\to 0^+} D^{\alpha-1}(D^\beta u)(x) = u(1) = D^\beta u(1) = 0. \end{cases}$$

By integration of inequalities (3.5) we obtain (3.13).

Let us prove the uniqueness. Let $v \in C_{2-\beta}([0,1])$ be another solution of problem (3.12) satisfying $v \leq W\varphi$.

Put $\tilde{v} := v + W(pv)$. Since the function $s \to (1-s)^{\alpha-1}p(s)v(s) \in C((0,1)) \cap L^1((0,1))$, then by Proposition 2.6 it follows that

$$\begin{cases} D^\alpha(D^\beta\tilde{v})(x) = \varphi(x), & 0 < x < 1, \\ \lim_{x\to 0^+} D^{\beta-1}\tilde{v}(x) = \lim_{x\to 0^+} D^{\alpha-1}(D^\beta\tilde{v})(x) = \tilde{v}(1) = D^\beta\tilde{v}(1) = 0. \end{cases}$$

From the uniqueness in Proposition 2.6 we conclude that

$$\tilde{v} := v + W(pv) = W\varphi.$$

So

$$\big(I + W(p\cdot)\big)\big((v-u)^+\big) = \big(I + W(p\cdot)\big)\big((v-u)^-\big),$$

where $(v-u)^+ = \max(v-u,0)$ and $(v-u)^- = \max(u-v,0)$.

From (3.13), (3.14), (1.11), and (2.5), there exists a constant $\tilde{c} > 0$, such that

$$W\big(p|v-u|\big) \leq 2\tilde{c}W(p\theta) \leq 2\tilde{c}\kappa_p\theta < \infty.$$

Therefore, $u = v$ by Lemma 3.5. □

Proof of Theorem 1.3 Consider $\xi \geq 0$ and $\zeta \geq 0$ with $\xi + \zeta > 0$. Let $\alpha, \beta \in (1,2]$ and $p \in \mathcal{J}_{\alpha,\beta} \cap C((0,1))$ be such that (A_2) is satisfied.

Let

$$\mathcal{S} := \big\{u \in \mathcal{B}^+\big((0,1)\big) : (1-\kappa_p)\theta \leq u \leq \theta\big\},$$

where $\theta(x) := \xi h_1(x) + \zeta h_2(x)$, and h_1 and h_2 are defined respectively by (1.9) and (1.10).

Define the operator $\mathcal{F}$ on $\mathcal{S}$ by

$$\mathcal{F}u = \theta - W_p(p\theta) + W_p\big((p - f(\cdot,u))u\big).$$

By (3.10) and (2.5) we have

$$W_p(p\theta) \leq W(p\theta) \leq \kappa_p\theta \leq \theta. \tag{3.15}$$

Using (A_2), we get

$$0 \leq f(\cdot,u) \leq p \quad \text{for all } u \in \mathcal{S}. \tag{3.16}$$

Next, we prove that $\mathcal{F}\mathcal{S} \subseteq \mathcal{S}$. Indeed, using (3.16) and (3.15), we have, for $u \in \mathcal{S}$,

$$\mathcal{F}u \leq \theta - W_p(p\theta) + W_p(pu) \leq \theta$$

and

$$\begin{aligned}\mathcal{F}u &\geq \theta - W_p(p\theta)\\ &\geq (1-\kappa_p)\theta.\end{aligned}$$

Observe that, by (A_2), $\mathcal{F}$ becomes nondecreasing on $\mathcal{S}$.

Define the sequence $\{v_n\}$ by $v_0 = (1-\kappa_p)\theta$ and $v_{n+1} = \mathcal{F}v_n$ for $n \in \mathbb{N}$. Since $\mathcal{F}\mathcal{S} \subseteq \mathcal{S}$, we have $v_1 = \mathcal{F}v_0 \geq v_0$, and by the monotonicity of $\mathcal{F}$ we deduce that

$$(1-\kappa_p)\theta = v_0 \leq v_1 \leq \cdots \leq v_n \leq v_{n+1} \leq \theta.$$

Using (A_1)-(A_2) and the dominated convergence theorem, we deduce that the sequence $\{v_n\}$ converges to a function $u \in \mathcal{S}$ satisfying

$$u = \big(I - W_p(p\cdot)\big)\theta + W_p\big(\big(p - f(\cdot,u)\big)u\big),$$

that is,

$$\big(I - W_p(p\cdot)\big)u = \big(I - W_p(p\cdot)\big)\theta - W_p\big(uf(\cdot,u)\big),$$

and by (3.15) we have $W(pu) \leq W(p\theta) \leq \theta < \infty$. Therefore, by Lemma 3.5 we deduce that

$$u = \theta - W\big(uf(\cdot,u)\big). \tag{3.17}$$

We claim that u is a solution.

Indeed, from (3.16) and (1.11), there exists a constant $c > 0$ such that

$$(1-s)^{\alpha-1}u(s)f\big(s,u(s)\big) \leq (1-s)^{\alpha-1}\theta(s)p(s) \leq cs^{\beta-2}(1-s)^{\alpha}p(s). \tag{3.18}$$

So, by Proposition 2.6 the function $W(uf(\cdot,u)) \in C_{2-\beta}([0,1])$. This implies by (3.17) that $u \in C_{2-\beta}([0,1])$.

Now, since the function $s \to (1-s)^{\alpha-1}u(s)f(s,u(s)) \in C((0,1)) \cap L^1((0,1))$, we deduce by Proposition 2.6 that u is a solution.

It remains to prove the uniqueness. Let v be another solution in $C_{2-\beta}([0,1])$ to problem (1.2) satisfying (1.12). Since $v \leq \theta$, we deduce by (3.18) that

$$0 \leq v(s)f\big(s,v(s)\big) \leq \theta(s)p(s) \leq cs^{\beta-2}(1-s)p(s).$$

This implies that $s \to (1-s)^{\alpha-1}v(s)f(s,v(s)) \in C((0,1)) \cap L^1((0,1))$. Let $\tilde{v} := v + W(vf(\cdot,v))$. By Proposition 2.6, we have

$$\begin{cases} D^{\alpha}(D^{\beta}\tilde{v})(x) = 0, \quad 0 < x < 1,\\ \lim_{x\to 0^+} D^{\beta-1}\tilde{v}(x) = 0, \qquad \lim_{x\to 0^+} D^{\alpha-1}(D^{\beta}\tilde{v})(x) = \xi,\\ \tilde{v}(1) = 0, \qquad D^{\beta}\tilde{v}(1) = -\zeta.\end{cases}$$

Hence,

$$v = \theta - W\big(vf(\cdot, v)\big). \tag{3.19}$$

Let $\omega : (0,1) \to \mathbb{R}$ be defined by

$$\omega(z) = \begin{cases} \frac{v(z)f(z,v(z))-u(z)f(z,u(z))}{v(z)-u(z)} & \text{if } v(z) \neq u(z), \\ 0 & \text{if } v(z) = u(z). \end{cases}$$

By (A_3), $\omega \in \mathcal{B}^+((0,1))$ and from (3.17) and (3.19) we deduce that

$$\big(I + W(\omega\cdot)\big)\big((v-u)^+\big) = \big(I + W(\omega\cdot)\big)\big((v-u)^-\big),$$

where $(v-u)^+ = \max(v-u, 0)$ and $(v-u)^- = \max(u-v, 0)$.

From (A_2) we have $\omega \le p$. So by using (1.12) and (2.5) we obtain

$$W\big(\omega|v-u|\big) \le 2W(p\theta) \le 2\kappa_p \theta < \infty.$$

Hence, $u = v$ by (3.11). □

Proof of Corollary 1.4 The statement follows from Theorem 1.3 with $f(x,t) = \lambda q(x)h(t)$, $\varrho(t) = th(t)$ and $p(x) := \lambda q(x)\max_{0\le t\le\theta(x)} \varrho'(t)$. □

Example 3.7 Let $\sigma \ge 0$, $\nu \ge 0$, and $q \in C^+((0,1))$ be such that

$$\int_0^1 t^{(\beta-2)(1+\sigma+\nu)}(1-t)^{\alpha+\sigma+\nu} q(t)\,dt < \infty.$$

Let $\varrho(t) = t^{\sigma+1}\ln(1+t^\nu)$ and $\tilde{q}(t) := q(t)\max_{0\le s\le\theta(t)} \varrho'(s)$. Since $\tilde{q} \in \mathcal{J}_{\alpha,\beta}$, then for $\xi \ge 0$, $\zeta \ge 0$ with $\xi + \zeta > 0$ and $\lambda \in [0, \frac{1}{2\kappa_{\tilde{q}}})$, the problem

$$\begin{cases} D^\alpha(D^\beta u)(x) + \lambda q(x)u^{\sigma+1}(x)\ln(1+u^\nu(x)) = 0, \quad 0 < x < 1, \\ \lim_{x\to 0^+} D^{\beta-1}u(x) = 0, \qquad \lim_{x\to 0^+} D^{\alpha-1}(D^\beta u)(x) = \xi, \\ u(1) = 0, \qquad D^\beta u(1) = -\zeta, \end{cases}$$

has a unique solution $u \in C_{2-\beta}([0,1])$ such that

$$(1 - \lambda\alpha_{\tilde{q}})\theta(x) \le u(x) \le \theta(x) \quad \text{for } x \in (0,1].$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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