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Weak convergence of a Mann-like algorithm for nonexpansive and accretive operators

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Abstract

Zero point problems of two accretive operators and fixed point problems of a nonexpansive mappings are investigated based on a Mann-like iterative algorithm. Weak convergence theorems are established in a Banach space.

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Keywords: accretive operator; fixed point; nonexpansive mapping; resolvent; zero point

1 Introduction and preliminaries

Let E be a real Banach space and let E^* be the dual space of E. Let R^+ be the set of nonnegative real numbers. Given a continuous strictly increasing function: $h: R^+ \to R^+$, where R^+ denotes the set of nonnegative real numbers, such that $\lim_{r\to\infty} h(r) = \infty$ and $h(0) = 0$, we associate with it a (possibly multivalued) generalized duality map $\mathfrak{J}_\varphi : E \to 2^{E^*}$, defined as

$$\mathfrak{J}_\varphi(x) := \{x^* \in E^* : x^*(x) = h(\|x\|)\|x\|, h(\|x\|) = \|x^*\|\}, \quad \forall x \in E.$$

In this paper, we use the generalized duality map associated with the gauge function $h(t) = t^{q-1}$ for $q > 1$,

$$\mathfrak{J}_q(x) := \{x^* \in E^* : \langle x^*, x\rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}, \quad \forall x \in E.$$

The modulus of convexity of E is the function $\delta_E(\epsilon) : (0,2] \to [0,1]$ defined by $\delta_E(\epsilon) = \inf\{1 - \frac{\|x+t\|}{2} : \|x\| = \|y\| = 1, \|x-y\| \ge \epsilon\}$. Recall that E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0,2]$. Let $p > 1$. We say that E is p-uniformly convex if there exists a constant $c_q > 0$ such that $\delta_E(\epsilon) \ge c_p\epsilon^p$ for any $\epsilon \in (0,2]$.

Let $B_E = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if the limit $\lim_{t\to 0}(\|x+ty\| - \|x\|)/t$ exists for each $x, y \in B_E$. In this case, E is said to be smooth. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in B_E$, the limit is attained uniformly for all $x \in B_E$. The norm of E is said to be Fréchet differentiable if for each $x \in B_E$, the limit is attained uniformly for all $y \in B_E$. The norm of E is said to be uniformly Fréchet differentiable if the limit is attained uniformly for all $x, y \in B_E$.

Let $\rho_E : [0,\infty) \to [0,\infty)$ be the modulus of smoothness of E by

$$\rho_E(t) = \sup\left\{ \frac{\|y+x\| - \|y-x\|}{2} - 1 : \|y\| \leq t, x \in B_E \right\}.$$

A Banach space E is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. Let $q > 1$. E is said to be q-uniformly smooth if there exists a fixed constant $c > 0$ such that $\rho_E(t) \leq ct^q$. It is well known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable. If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth [1], and hence the norm of E is uniformly Fréchet differentiable, in particular, the norm of E is Fréchet differentiable.

Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. To be more precise, L^p is mini$\{p,2\}$-uniformly smooth for every $p > 1$. It is well known that E is p-uniformly convex if and only if E^* is q-uniformly smooth, where p and q satisfy the relation $\frac{1}{p} + \frac{1}{q} = 1$.

Let T be a mapping on E. The fixed point set of T is denoted by $\mathrm{Fix}(T)$. Recall that T is said to be nonexpansive iff

$$\|Tx - Ty\| \leq \|x, y\|, \quad \forall x, y \in E.$$

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$ is said to be accretive if, for $t > 0$ and $x, y \in D(A)$,

$$\|x - y\| \leq \left\| t(u - v) + (x - y) \right\|, \quad \forall u \in Ax, v \in Ay.$$

It follows from Kato [2] that A is accretive if and only if, for $x, y \in D(A)$, there exists $j_q(x - y) \in \mathfrak{J}_q(x - y)$ such that

$$\left\langle u - v, j_q(x - y) \right\rangle \geq 0.$$

An accretive operator A is said to be m-accretive if $R(I + rA) = E$ for all $r > 0$. In Hilbert spaces, an operator A is m-accretive if and only if A is maximal monotone. In this paper, we use $A^{-1}(0)$ to denote the set of zero points of A.

Recall that a single valued operator A on E is said to be α-strongly accretive if there exists a constant $\alpha > 0$ and some $j_q(x - y) \in \mathfrak{J}_q(x - y)$ such that

$$\left\langle Ax - Ay, j_q(x - y) \right\rangle \geq \alpha \|x - y\|^q, \quad \forall x, y \in E.$$

A is said to be α-inverse strongly accretive if there exists a constant $\alpha > 0$ and some $j_q(x - y) \in \mathfrak{J}_q(x - y)$ such that

$$\left\langle Ax - Ay, j_q(x - y) \right\rangle \geq \alpha \|Ax - Ay\|^q, \quad \forall x, y \in E.$$

For a multi-valued accretive operator A, we can define a nonexpansive single valued mapping $J_r^A : R(I + rA) \to D(A)$ by $J_r^A = (I + rA)^{-1}$ for each $r > 0$, which is called the resolvent operator of A.

The convex feasibility problem asks to find a point in the intersection of convex sets. This is an important problem in mathematics and engineering; see, *e.g.*, [3–6] and the references therein. Oftentimes, the convex sets are given as fixed point sets of projections or (more generally) averaged nonexpansive operators. In this paper, we will focus our attention on the problem of finding a common element in $\mathrm{Fix}(T) \cap (A+B)^{-1}(0)$, where T is a nonexpansive mapping, A is an α-inverse strongly accretive operator and B is an m-accretive operator, in the framework of uniformly convex and q-uniformly smooth Banach spaces. The problem is quite general in the sense that it includes: split feasibility problems, convexly constrained linear inverse problems, fixed point problems, variational inequalities, convexly constrained minimization problems, and Nash equilibrium problems in noncooperative games, as special cases; see, for instance, [7–12] and the references therein. Recently, mean valued iterative algorithms have been introduced by many authors to investigate this problem; see, for instance, [13–18] and the references therein. Related work can also be found, *e.g.*, in [19–22]. However, there is little work in the existing literature in the setting of Banach spaces. The aim of this paper is to establish a weak convergence theorem in the framework of Banach spaces based on a Mann-like iterative algorithm. Applications are also provided to support the main results of this article.

In order to obtain our main results, we also need the following lemmas.

Lemma 1.1 *Let E be a real Banach space. Let $A : E \to E$ be a single valued operator and let $B : E \to 2^E$ be an m-accretive operator. Then*

$$(A+B)^{-1}(0) = \mathrm{Fix}\big(J_r^B(I - rA)\big),$$

where $J_r^B(I-rA)$ is the resolvent of B for $a > 0$.

Proof

$$\begin{aligned} p \in \mathrm{Fix}\big(J_r^B(I - rA)\big) \quad &\Longleftrightarrow \quad p = J_r^B(I - rA)p \\ &\Longleftrightarrow \quad p + rBp = p - rAp \\ &\Longleftrightarrow \quad p \in (A+B)^{-1}(0). \end{aligned}$$

□

Lemma 1.2 [1] *Let E be a real q-uniformly smooth Banach space. Then the following inequality holds*:

$$\|x+y\|^q \le \|x\|^q + q\big\langle y, \mathfrak{J}_q(x+y)\big\rangle, \quad \forall x, y \in E,$$

and

$$\|x+y\|^q \le \|x\|^q + q\big\langle y, \mathfrak{J}_q(x)\big\rangle + K_q\|y\|^q, \quad \forall x, y \in E,$$

where K_q is some fixed positive constant.

Lemma 1.3 [1] *Let $r > 0$ and $q > 1$ be two fixed real numbers. Then a Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex function*

$\varphi : [0,\infty) \to [0,\infty)$ *with* $\varphi(0) = 0$ *such that*

$$\big\|ax + (1-a)y\big\|^q \le (1-a)\|y\|^q + a\|x\|^q - w(a)\varphi\big(\|y-x\|\big),$$

where $w(a) = a^q(1-a) + (1-a)^q a$, *for all* $x, y \in B_r(0) := \{x \in E : \|x\| \le r\}$ *and* $a \in [0,1]$.

Lemma 1.4 [23] *Let E be a real uniformly convex Banach space and let C be a nonempty closed convex and bounded subset of E. Then there is a strictly increasing and continuous convex function* $\psi : [0,\infty) \to [0,\infty)$ *with* $\varphi(0) = 0$ *such that, for every nonexpansive mapping* $T : C \to C$ *and, for all* $x, y \in C$ *and* $t \in [0,1]$, *the following inequality holds*:

$$\big\|\big(tTx + (1-t)Ty\big) - T\big(tx + (1-t)y\big)\big\| \le \psi^{-1}\big(\|y-x\| - \|Ty - Tx\|\big).$$

Lemma 1.5 [24] *Let E be a real uniformly convex Banach space, and let T be a nonexpansive mapping on E. Then* $I - T$ *is demiclosed at zero.*

Lemma 1.6 [25] *Let E be a real uniformly convex Banach space. Let* E^* *the dual space of E such that it has the Kadec-Klee property. Suppose that* $\{x_n\}$ *is a bounded sequence such that* $\lim_{n\to\infty} \|(1-a)p_1 - p_2 + ax_n\|$ *exists for all* $a \in [0,1]$ *and* $p_1, p_2 \in \omega_w(x_n)$, *where* $\omega_w(x_n) : \{x : \exists x_{n_i} \rightharpoonup x\}$ *denotes the weak* ω*-limit set of* $\{x_n\}$ *Then* $\omega_w(x_n)$ *is a singleton.*

2 Main results

Theorem 2.1 *Let E be a real uniformly convex and q-uniformly smooth Banach space with constant* K_q. *Let* $B : D(B) \subset E \to 2^E$ *be an m-accretive operator,* $A : E \to E$ *an* α*-inverse strongly accretive operator and* $T : E \to E$ *a nonexpansive mapping such that* $\operatorname{Fix}(T) \cap (A+B)^{-1}(0) \neq \emptyset$. *Let* $\{r_n\}$ *be a positive number sequence and let* $\{\alpha_n\}$ *be a real number sequence in* $(0,1)$ *such that* $\{\alpha_n\} \subset [\alpha, \alpha']$, *where* $0 < \alpha < \alpha' < 1$ *and* $\{r_n\} \subset [r, r']$, *where* $0 < r < r' < (\frac{q\alpha}{K_q})^{\frac{1}{q-1}}$. *Let* $\{x_n\}$ *be a sequence generated in the following manner*: $x_0 \in E$ *and* $x_{n+1} = \alpha_n Tx_n + (1-\alpha_n)(I + r_nB)^{-1}(x_n - r_nAx_n)$, $\forall n \ge 0$, *Then* $\{x_n\}$ *converges weakly to some point in* $\operatorname{Fix}(T) \cap (A+B)^{-1}(0)$.

Proof First, we show that $\{x_n\}$ is bounded. From Lemma 1.2, we have

$$\begin{aligned}
&\big\|(I - r_nA)x - (I - r_nA)y\big\|^q \\
&\quad\le \|x-y\|^q - qr_n\big\langle Ax - Ay, \mathfrak{J}_q(x-y)\big\rangle + K_q r_n^q\|Ax - Ay\|^q \\
&\quad\le \|x-y\|^q - qr_n\alpha\|Ax - Ay\|^q + K_q r_n^q\|Ax - Ay\|^q \\
&\quad= \|x-y\|^q - \big(\alpha q - K_q r_n^{q-1}\big)r_n\|Ax - Ay\|^q.
\end{aligned} \tag{2.1}$$

In view of the restriction imposed on $\{r_n\}$, one sees that $I - r_nA$ is nonexpansive. Put $J_{r_n}^B = (I + r_nB)^{-1}$ and fix $p \in (A+B)^{-1}(0) \cap \operatorname{Fix}(T)$. By using Lemma 1.1, we find from (2.1) that

$$\begin{aligned}
\|x_{n+1} - p\| &\le \alpha_n\|Tx_n - p\| + (1-\alpha_n)\big\|J_{r_n}^B(x_n - r_nAx_n) - p\big\| \\
&\le \alpha_n\|x_n - p\| + (1-\alpha_n)\big\|(x_n - r_nAx_n) - (p - r_nAp)\big\| \\
&\le \|x_n - p\|.
\end{aligned}$$

It follows that $\lim_{n\to\infty}\|x_n - p\|$ exists and, in particular, that $\{x_n\}$ is bounded. Put $y_n = J^B_{r_n}(x_n - r_nAx_n)$. Since B is m-accretive, we find from Lemma 1.3 and (2.1) that

$$\begin{aligned}
\|y_n - p\|^q &\leq \left\| \frac{r_n}{2}\left(\frac{x_n - r_nAx_n - y_n}{r_n} - \frac{(I - r_nA)p - p}{r_n}\right) + (y_n - p)\right\|^q \\
&= \left\| \frac{1}{2}\big((I - r_nA)x_n - (I - r_nA)p\big) + \frac{1}{2}(y_n - p)\right\|^q \\
&\leq \frac{1}{2}\|y_n - p\|^q + \frac{1}{2}\big\|(I - r_nA)x_n - (I - r_nA)p\big\|^q \\
&\quad - \frac{1}{2^q}\varphi\big(\big\|(y_n - p) - \big((I - r_nA)x_n - (I - r_nA)p\big)\big\|\big) \\
&\leq \big\|(I - r_nA)x_n - (I - r_nA)p\big\|^q \\
&\quad - \frac{1}{2^q}\varphi\big(\big\|(y_n - p) - \big((I - r_nA)x_nn - (I - r_nA)p\big)\big\|\big) \\
&\leq \|x_n - p\|^q - \big(\alpha q - K_q r_n^{q-1}\big)r_n\|Ax_n - Ap\|^q \\
&\quad - \frac{1}{2^q}\varphi\big(\big\|(y_n - p) - \big((I - r_nA)x_nn - (I - r_nA)p\big)\big\|\big).
\end{aligned} \tag{2.2}$$

Since $\|\cdot\|^q$ is convex, we find from (2.1) and (2.2) that

$$\begin{aligned}
\|x_{n+1} - p\|^q &\leq \alpha_n\|Tx_n - p\|^q + (1 - \alpha_n)\|y_n - p\|^q \\
&\leq \|x_n - p\|^q - \big(\alpha q - K_q r_n^{q-1}\big)r_n(1 - \alpha_n)\|Ax_n - Ap\|^q \\
&\quad - (1 - \alpha_n)\frac{1}{2^q}\varphi\big(\big\|(y_n - p) - \big((I - r_nA)x_n - (I - r_nA)p\big)\big\|\big).
\end{aligned}$$

It follows from the restrictions imposed on $\{\alpha_n\}$ and $\{r_n\}$ that

$$\lim_{n\to\infty}\big\|(y_n - x_n) - (r_nAp - r_nAx_n)\big\| = 0 \tag{2.3}$$

and

$$\lim_{n\to\infty}\|Ax_n - Ap\| = 0. \tag{2.4}$$

Since $\|y_n - x_n\| \leq \|(y_n - x_n) - (r_nAp - r_nAx_n)\| + r_n\|Ap - Ax_n\|$, we find from (2.3) and (2.4) that

$$\lim_{n\to\infty}\big\|x_n - J^B_{r_n}(x_n - r_nAx_n)\big\| = 0. \tag{2.5}$$

Since B is an m-accretive operator, we have

$$\left\langle \frac{x_n - J^B_r(I - rA)x_n}{r} - \frac{x_n - J^B_{r_n}(I - r_nA)x_n}{r_n}, \mathfrak{J}_q\big(J^B_r(I - rA)x_n - J^B_{r_n}(I - r_nA)x_n\big)\right\rangle \geq 0.$$

It follows that

$$\begin{aligned}&\left\|x_n - J_{r_n}^B(I - r_nA)x_n\right\| \left\|J_r^B(I - rA)x_n - J_{r_n}^B(I - r_nA)x_n\right\|^{q-1}\\&\geq \frac{r_n - r}{r_n}\left\langle x_n - J_{r_n}^B(I - r_nA)x_n, \mathfrak{J}_q\left(J_r^B(I - rA)x_n - J_{r_n}^B(I - r_nA)x_n\right)\right\rangle\\&\geq \left\|J_r^B(I - rA)x_n - J_{r_n}^B(I - r_nA)x_n\right\|^q,\end{aligned}$$

which implies

$$\left\|x_n - J_{r_n}^B(I - r_nA)x_n\right\| \geq \left\|J_r^B(I - rA)x_n - J_{r_n}^B(I - r_nA)y_n\right\|. \tag{2.6}$$

From (2.5), one sees that

$$\lim_{n\to\infty}\left\|J_r^B(x_n - rAx_n) - x_n\right\| = 0. \tag{2.7}$$

In view of Lemma 1.3, one has

$$\begin{aligned}\|x_{n+1} - p\|^q &\leq \alpha_n\|Tx_n - p\|^q + (1 - \alpha_n)\|y_n - p\|^q\\&\quad - \left(\alpha_n^q(1 - \alpha_n) + (1 - \alpha_n)^q\alpha_n\right)\varphi\left(\|Tx_n - y_n\|\right)\\&\leq \|x_n - p\|^q - \left(\alpha_n^q(1 - \alpha_n) + (1 - \alpha_n)^q\alpha_n\right)\varphi\left(\|Tx_n - y_n\|\right),\end{aligned}$$

that is,

$$\left(\alpha_n^q(1 - \alpha_n) + (1 - \alpha_n)^q\alpha_n\right)\varphi\left(\|Tx_n - y_n\|\right) \leq \|x_n - p\|^q - \|x_{n+1} - p\|^q.$$

This shows that $\lim_{n\to\infty}\|Tx_n - y_n\| = 0$. From (2.5), we find $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. In view of Lemma 1.5, we see that $\omega_w(x_n) \subset \operatorname{Fix}(J_r^B(I + rA)) \cap \operatorname{Fix}(T) = (A + B)^{-1}(0) \cap \operatorname{Fix}(T)$.

Next, we show that $\omega_w(x_n)$ is a singleton set. Define mappings $S_n : E \to E$ by

$$S_nx := \alpha_nTx + (1 - \alpha_n)J_{r_n}^B(I - r_nA)x, \quad \forall x \in C.$$

Set

$$S_{n,m} = S_{n+m-1}S_{n+m-2}\cdots S_n, \quad \forall n, m \geq 1.$$

Since S_n is nonexpansive, we find that $S_{n,m}$ is also nonexpansive and $S_{n,m}x_n = x_{n+m}$. For all $t \in [0,1]$ and $n, m \geq 1$, put

$$b_n(t) = \left\|tx_n + (1 - t)p_1 - p_2\right\|$$

and

$$c_{n,m} = \left\|S_{n,m}\left(tx_n + (1 - t)p_1\right) - \left(tx_{n+m} + (1 - t)p_1\right)\right\|,$$

where p_1 and p_2 are in $(A + B)^{-1}(0) \cap \operatorname{Fix}(T)$. Using Lemma 1.4, we find that

$$
\begin{aligned}
c_{n,m} &\le \psi^{-1}\big(\|x_n - p_1\| - \|S_{n,m}x_n - S_{n,m}p_1\|\big) \\
&= \psi^{-1}\big(\|x_n - p_1\| - \|x_{n+m} - p_1 + p_1 - S_{n,m}p_1\|\big) \\
&\le \psi^{-1}\big(\|x_n - p_1\| - \big(\|x_{n+m} - p_1\| - \|p_1 - S_{n,m}p_1\|\big)\big).
\end{aligned}
$$

This implies that $\{c_{n,m}\}$ converges uniformly to zero as $n \to \infty$ for all $m \ge 1$. On the other hand, we have

$$
\begin{aligned}
b_{n+m}(t) &\le c_{n,m} + \big\|S_{n,m}\big(tx_n + (1-t)p_1\big) - p_2\big\| \\
&\le c_{n,m} + \big\|S_{n,m}\big(tx_n + (1-t)p_1\big) - S_{n,m}p_2\big\| + \|S_{n,m}p_2 - p_2\| \\
&\le c_{n,m} + b_n(t) + \|S_{n,m}p_2 - p_2\|.
\end{aligned}
$$

Taking $\limsup$ as $m \to \infty$ and then the $\liminf$ as $n \to \infty$, we find that $\limsup_{n\to\infty} b_n(t) \le \liminf_{n\to\infty} b_n(t)$. This proves that $\lim_{n\to\infty} b_n(t)$ exists for any $t \in [0,1]$. This implies from Lemma 1.6 that $\omega_w(x_n)$ is a singleton set. This proves the proof. □

Note that, in the framework of Hilbert spaces, the concept of monotonicity coincides with the concept of accretivity. Next, we apply our main results to solve variational inequality problems and minimizer problems of convex functions in the framework of Hilbert spaces.

Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and its induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and let Proj_C^H be the metric projection from H onto C. Recall the following classical variational inequality: find $x \in C$ such that $\langle y - x, Ax\rangle \ge 0$, $\forall y \in C$. The solution set of the variational inequality is denoted by $\mathrm{VI}(C,A)$. Projection-gradient methods have been recently investigated for solving the variational inequality. It is well known that x is a solution to the variational inequality iff x is a fixed point of $\mathrm{Proj}_C^H(I - rA)$, where I denotes the identity on H and r is a positive real number. If A is inverse strongly monotone, then $\mathrm{Proj}_C^H(I - rA)$ is a nonexpansive mapping. Moreover. If C is also bounded, then the existence of solutions of the variational inequality is guaranteed by the nonexpansivity of mapping $\mathrm{Proj}_C^H(I - rA)$. Let i_C be a function defined by

$$
i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}
$$

It is easy to see that i_C is a proper lower and semicontinuous convex function on H, and the subdifferential ∂i_C of i_C is maximal monotone. Define the resolvent $J_r := (I + r\partial i_C)^{-1}$ of subdifferential operator ∂i_C. Letting $x = J_r y$, we find that

$$
\begin{aligned}
y \in x + r\partial i_C x \quad &\Longleftrightarrow \quad y \in x + rN_C^H x \\
&\Longleftrightarrow \quad \langle y - x, v - x\rangle \le 0, \quad \forall v \in C \\
&\Longleftrightarrow \quad x = \mathrm{Proj}_C^H y,
\end{aligned}
$$

where $N_C^H x := \{y \in H : \langle y, v - x\rangle, \forall v \in C\}$. Putting $B = \partial i_C$ in Theorem 2.1, we find the following results immediately.

Corollary 2.2 *Let H be a real Hilbert space. Let C be a nonempty closed and convex subset of E and let Proj_C^H be the metric projection from H onto C. Let A an α-inverse strongly monotone operator on H and T a nonexpansive mapping on C such that $\mathrm{Fix}(T) \cap \mathrm{VI}(C,A) \neq \emptyset$. Let $\{r_n\}$ be a positive number sequence and let $\{\alpha_n\}$ be a real number sequence in $(0,1)$ such that $\{\alpha_n\} \subset [\alpha,\alpha']$, where $0 < \alpha < \alpha' < 1$ and $\{r_n\} \subset [r,r']$, where $0 < r < r' < 2\alpha$. Let $\{x_n\}$ be a sequence generated in the following manner: $x_0 \in C$ and $x_{n+1} = \alpha_n Tx_n + (1-\alpha_n)\mathrm{Proj}_C^H(x_n - r_nAx_n)$, $\forall n \geq 0$. Then $\{x_n\}$ converges weakly to some point in $\mathrm{Fix}(T) \cap \mathrm{VI}(C,A)$.*

Now, we are in a position to consider the problem of finding minimizers of proper lower semicontinuous convex functions. For a proper lower semicontinuous convex function $g : H \to (-\infty,\infty]$, the subdifferential mapping ∂g of g is defined by $\partial g(x) = \{x^* \in H : g(x) + \langle y-x,x^*\rangle \leq g(y), \forall y \in H\}$, $\forall x \in H$. Rockafellar [26] proved that ∂g is a maximal monotone operator. It is easy to verify that $0 \in \partial g(v)$ if and only if $g(v) = \min_{x\in H} g(x)$.

Corollary 2.3 *Let H be a real Hilbert space. Let $g : H \to (-\infty,\infty]$ be a proper convex and lower semicontinuous function and let $T : H \to H$ be a nonexpansive mapping such that $\mathrm{Fix}(T) \cap (\partial g)^{-1}(0) \neq \emptyset$. Let $\{r_n\}$ be a positive number sequence and let $\{\alpha_n\}$ be a real number sequence in $(0,1)$ such that $\{\alpha_n\} \subset [\alpha,\alpha']$, where $0 < \alpha < \alpha' < 1$ and $\{r_n\} \subset [r,r']$, where $0 < r < r' < (\frac{q\alpha}{K_q})^{\frac{1}{q-1}}$. Let $\{x_n\}$ be a sequence generated in the following manner: $x_0 \in H$ and $x_{n+1} = \alpha_n Tx_n + (1-\alpha_n)y_n$, $\forall n \geq 0$, where $y_n = \min_{z\in H}\{g(z) + \frac{\|z-x_n+e_n\|^2}{2r_n}\}$. Then $\{x_n\}$ converges weakly to some point in $\mathrm{Fix}(T) \cap (A+B)^{-1}(0)$.*

Proof Since $g : H \to (-\infty,\infty]$ is a proper convex and lower semicontinuous function, we see that subdifferential ∂g of g is maximal monotone. Putting $A = 0$, we have $y_n = \arg\min_{z\in H}\{g(z) + \frac{\|z-x_n\|^2}{2r_n}\}$ is equivalent to $0 \in \partial g(y_n) + \frac{1}{r_n}(y_n - x_n)$. Hence, we have $x_n \in y_n + r_n\partial g(y_n)$. By use of Theorem 2.1, we find the desired conclusion immediately. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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