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A posteriori error estimates of fully discrete finite-element schemes for nonlinear parabolic integro-differential optimal control problems

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Abstract

The aim of this work is to study the optimality conditions and the adaptive multi-mesh fully discrete finite-element schemes for quadratic nonlinear parabolic integro-differential optimal control problems. We derive a posteriori error estimates in $L^2(J;H^1(\Omega))$-norm and $L^2(J;L^2(\Omega))$-norm for both the coupled state and control approximation. Such estimates can be used to construct reliable adaptive finite-element approximation for nonlinear parabolic integro-differential optimal control problems.

MSC: 49J20; 65N30

Keywords: nonlinear parabolic integro-differential optimal control problems; adaptive multi-mesh finite-element methods; a posteriori error estimates; fully discrete

1 Introduction

Parabolic integro-differential optimal control problems are very important for modeling in science. They have various physical backgrounds in many practical applications such as population dynamics, visco-elasticity, and heat conduction in materials with memory. The finite-element approximation of parabolic integro-differential optimal control problems plays a very important role in the numerical methods for these problems. The finite-element approximation of an optimal control problem by piecewise constant functions has been investigated by Falk [1] and Geveci [2]. The discretization for semilinear elliptic optimal control problems is discussed by Arada *et al.* in [3]. In [4], Brunner and Yan analyzed the finite-element Galerkin discretization for a class of optimal control problems governed by integral equations and integro-differential equations. Systematic introductions of the finite-element method for optimal control problems can be found in [5–10].

The adaptive finite-element approximation is the most important method to boost the accuracy of the finite-element discretization. It ensures a higher density of nodes in a certain area of the given domain, where the solution is discontinuous or more difficult to approximate, using an a posteriori error indicator. A posteriori error estimates are computable quantities in terms of the discrete solution and measure the actual discrete errors without the knowledge of exact solutions. They are essential in designing algorithms for a mesh which equidistribute the computational effort and optimize the computation. The literature for this is huge. Some techniques directly relevant to our work can be found in [11, 12]. Recently, in [13–16], we derived a priori error estimates and superconvergence for linear quadratic optimal control problems using mixed finite-element methods. A posteriori error estimates of mixed finite-element methods for general semilinear optimal control problems were addressed in [17].

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p = \sum_{|\alpha|\leq m}\|D^\alpha v\|_{L^p(\Omega)}^p$, a semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p = \sum_{|\alpha|=m}\|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. We denote by $L^s(0,T;W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,p}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,p}(\Omega))} = (\int_0^T \|v\|_{W^{m,p}(\Omega)}^s\,dt)^{\frac{1}{s}}$ for $s \in [1,\infty)$, and the standard modification for $s = \infty$. The details can be found in [18].

The problems that we are interested in are the following nonlinear parabolic integro-differential optimal control problems:

$$\min_{u(t)\in K}\left\{\int_0^T\left(\frac{1}{2}\|y-y_0\|^2+\frac{\alpha}{2}\|u\|^2\right)dt\right\} \tag{1.1}$$

subject to the state equations

$$y_t - \operatorname{div}\bigl(A\nabla y(x,t)\bigr) - \int_0^t \operatorname{div}\bigl(\psi(t,\tau)\nabla y(x,\tau)\bigr)\,d\tau + \phi(y) = f + Bu, \quad x\in\Omega, t\in J, \tag{1.2}$$

$$y(x,t) = 0, \quad x \in \partial\Omega, t \in J, \tag{1.3}$$

$$y(x,0) = y_0(x), \quad x \in \Omega, \tag{1.4}$$

where the bounded open set $\Omega \subset \mathbb{R}^2$ is a 2 regular convex polygon with boundary $\partial\Omega$, $J = (0,T]$, $f \in L^2(\Omega)$, $\psi = \psi(x,t,\tau) = \psi_{i,j}(x,t,\tau)_{2\times2} \in C^\infty(0,T;L^2(\bar{\Omega}))^{2\times2}$, $y_0 \in H^1(\Omega)$, α is a positive constant, and B is a continuous linear operator from K to $L^2(\Omega)$. For any $R > 0$ the function $\phi(\cdot) \in W^{2,\infty}(-R,R)$, $\phi'(y) \in L^2(\Omega)$ for any $y \in L^2(J;H_0^1(\Omega))$, and $\phi'(y) \geq 0$. We assume the coefficient matrix $A(x) = (a_{i,j}(x))_{2\times2} \in (W^{1,\infty}(\Omega))^{2\times2}$ is a symmetric positive definite matrix and there is a constant $c > 0$ satisfying for any vector $\mathbf{X} \in \mathbb{R}^2$, $\mathbf{X}^t A\mathbf{X} \geq c\|\mathbf{X}\|_{\mathbb{R}^2}^2$. Here, K denotes the admissible set of the control variable, defined by

$$K = \left\{u(x,t) \in L^2\bigl(J;L^2(\Omega)\bigr) : \int_\Omega u(x,t)\,dx \geq 0, t \in J\right\}. \tag{1.5}$$

The plan of this paper is as follows. In the next section, we construct the optimality conditions and present the finite-element discretization for nonlinear parabolic integro-differential optimal control problems. A posteriori error estimates of finite-element solutions for those problems are established in Section 3. Finally, we analyze the conclusion and future work in Section 4.

2 Finite elements for integro-differential optimal control

We shall now construct the optimality conditions and the finite element discretization of the nonlinear parabolic integro-differential optimal control problem (1.1)-(1.4). Let $V = H_0^1(\Omega)$, $W = L^2(\Omega)$. Let

$$a(y,w) = \int_\Omega (A\nabla y)\cdot\nabla w, \quad \forall y,w \in V, \tag{2.1}$$

$$\psi(t,\tau;z,w) = \big(\psi(t,\tau)\nabla z, \nabla w\big), \quad \forall z, w \in V, \tag{2.2}$$

$$(u,v) = \int_\Omega uv, \quad \forall u, v \in W, \tag{2.3}$$

$$(f_1, f_2) = \int_\Omega f_1 f_2, \quad \forall f_1, f_2 \in W. \tag{2.4}$$

Then the nonlinear parabolic integro-differential optimal control problem (1.1)-(1.4) can be restated as

$$\min_{u(t)\in K} \left\{ \int_0^T \left(\frac{1}{2}\|y - y_0\|^2 + \frac{\alpha}{2}\|u\|^2 \right) dt \right\} \tag{2.5}$$

subject to

$$(y_t, w) + a(y,w) + \int_0^t \psi\big(t,\tau; y(\tau), w\big)\, d\tau + \big(\phi(y), w\big)$$

$$= (f + Bu, w), \quad \forall w \in V, t \in J, \tag{2.6}$$

$$y(x,0) = y_0(x), \quad x \in \Omega, \tag{2.7}$$

where the inner product in $L^2(\Omega)$ or $L^2(\Omega)^2$ is indicated by $(\cdot,\cdot)$.

It is well known (see, *e.g.*, [4]) that the optimal control problem has a solution (y,u), and that if a pair (y,u) is the solution of equations (2.5)-(2.7), then there is a co-state $p \in V$ such that the triplet (y,p,u) satisfies the following optimality conditions:

$$(y_t, w) + a(y,w) + \int_0^t \psi\big(t,\tau; y(\tau), w\big)\, d\tau + \big(\phi(y), w\big) = (f + Bu, w), \quad \forall w \in V, \tag{2.8}$$

$$y(x,0) = y_0(x), \quad x \in \Omega, \tag{2.9}$$

$$-(p_t, w) + a(q,p) + \int_t^T \psi\big(\tau, t; q, p(\tau)\big)\, d\tau + \big(\phi'(y)p, q\big) = (y - y_0, q), \quad \forall q \in V, \tag{2.10}$$

$$p(x,T) = 0, \quad x \in \Omega, \tag{2.11}$$

$$\int_0^T \big(\alpha u + B^* p, v - u\big)\, dt \geq 0, \quad \forall v \in K, \tag{2.12}$$

where B^* is the adjoint operator of B.

Let us consider the finite-element approximation of the optimal control problem (2.5)-(2.7). Again here we consider only n-simplex elements and conforming finite elements.

For ease of exposition we will assume that Ω is a polygon. Let $\mathcal{T}^h$ be regular partition of Ω. Associated with $\mathcal{T}^h$ is a finite-dimensional subspace V^h of $C(\bar{\Omega})$, such that $\chi|_\tau$ are polynomials of order m ($m \geq 1$) $\forall \chi \in V^h$ and $\tau \in \mathcal{T}^h$. It is easy to see that $V^h \subset V$. Let h_τ denote the maximum diameter of the element τ in $\mathcal{T}^h$, $h = \max_{\tau \in \mathcal{T}^h}\{h_\tau\}$. In addition C or c denotes a general positive constant independent of h.

Due to the limited regularity of the optimal control u in general, there will be no advantage in considering higher-order finite element spaces rather than the piecewise constant space for the control. So, we only consider piecewise constant finite elements for the approximation of the control, though higher-order finite elements will be used to approximate the state and the co-state.

Let $P_0(\tau)$ denote the piecewise constant space over τ. Then we take $K^h = \{u \in K : u(x,t)|_\tau \in P_0(\tau)\}$. By the definition of the finite-element subspace, the finite-element discretization of equations (2.5)-(2.7) is as follows: compute $(y_h, u_h) \in V^h \times K^h$ such that

$$\min_{u_h \in K^h} \left\{ \int_0^T \left(\frac{1}{2} \|y_h - y_0\|^2 + \frac{\alpha}{2} \|u_h\|^2 \right) dt \right\}, \tag{2.13}$$

$$(y_{ht}, w_h) + a(y_h, w_h) + \int_0^t \psi\big(t, \tau; y_h(\tau), w_h\big) \, d\tau + \big(\phi(y_h), w_h\big) = (f + Bu_h, w_h), \tag{2.14}$$

$$y_h(x, 0) = y_0^h(x), \quad x \in \Omega, \tag{2.15}$$

where $w_h \in V^h$, $y_0^h \in V^h$ is an approximation of y_0.

Again, it follows that the optimal control problem (2.13)-(2.15) has a solution (y_h, u_h), and that if a pair (y_h, u_h) is the solution of equations (2.13)-(2.15), then there is a co-state $p_h \in V^h$ such that triplet (y_h, p_h, u_h) satisfies the following optimality conditions:

$$(y_{ht}, w_h) + a(y_h, w_h) + \int_0^t \psi\big(t, \tau; y_h(\tau), w_h\big) \, d\tau + \big(\phi(y_h), w_h\big) = (f + Bu_h, w_h), \tag{2.16}$$

$$y_h(x, 0) = y_0^h(x), \quad x \in \Omega, \tag{2.17}$$

$$-(p_{ht}, w_h) + a(q_h, p_h) + \int_t^T \psi\big(\tau, t; q_h, p_h(\tau)\big) \, d\tau + \big(\phi'(y_h) p_h, q_h\big) = (y_h - y_0, q_h), \tag{2.18}$$

$$p_h(x, T) = 0, \quad x \in \Omega, \tag{2.19}$$

$$\big(\alpha u_h + B^* p_h, v_h - u_h\big)_U \geq 0, \tag{2.20}$$

where $w_h, q_h \in V^h$, $v_h \in K^h$.

We now consider the fully discrete finite-element approximation for the semidiscrete problem. Let $\triangle t > 0$, $N = T / \triangle t \in \mathbb{Z}$, and $t^i = i \triangle t$, $i \in \mathbb{R}$. Also, let

$$\xi^i = \xi^i(x) = \xi\big(x, t^i\big), \qquad d_t \xi^i = \frac{\xi^i - \xi^{i-1}}{\triangle t}.$$

For $i = 1, 2, \ldots, N$, construct the finite-element spaces $V_i^h \in V$ with the mesh $\mathcal{T}_h^i$ (similar to V_h). Similarly, construct the finite-element spaces $K_i^h \in L^2(\Omega)$ with the mesh $\mathcal{T}_h^i$ (similar as I_h). Let h_τ^i denote the maximum diameter of the element τ^i in $\mathcal{T}_h^i$. Define mesh functions $\tau(\cdot)$ and mesh size functions $h_\tau(\cdot)$ such that $\tau(t)|_{t \in (t_{i-1}, t_i]} = \tau^i$, $h_\tau(t)|_{t \in (t_{i-1}, t_i]} = h_{\tau_i}$. For ease of exposition, we shall denote $\tau(t)$ and $h_\tau(t)$ by τ and h_τ, respectively. Then the fully discrete finite-element approximation of equations (2.13)-(2.15) is as follows: compute $(y_h^i, u_h^i) \in V_i^h \times K_i^h$, $i = 1, 2, \ldots, N$, such that

$$\min_{u_h^i \in K_i^h} \left\{ \sum_{i=1}^N \Delta t \left(\frac{1}{2} \big\|y_h^i - y_0\big\|^2 + \frac{\alpha}{2} \big\|u_h^i\big\|^2 \right) \right\}, \tag{2.21}$$

$$\begin{aligned} &\big(d_t y_h^i, w_h\big) + a\big(y_h^i, w_h\big) + \int_0^t \psi\big(t, \tau; y_h^i(\tau), w_h\big) \, d\tau + \big(\phi\big(y_h^i\big), w_h\big) \\ &\quad = \big(f(x, t_i) + Bu_h^i, w_h\big), \end{aligned} \tag{2.22}$$

$$\forall w_h \in V_i^h, i = 1, 2, \ldots, N, \qquad y_h^0(x) = y_0^h(x), \quad x \in \Omega, \tag{2.23}$$

where $y_0^h \in V^h$ is an approximation of y_0.

Now, it follows that the optimal control problem (2.21)-(2.23) has a solution (Y_h^i, U_h^i), $i = 1, 2, \dots, N$, and that if a pair (Y_h^i, U_h^i), $i = 1, 2, \dots, N$, is the solution of (2.21)-(2.23), then there is a co-state $P_h^{i-1} \in V_i^h$, $i = N, \dots, 2, 1$, such that triplet $(Y_h^i, P_h^{i-1}, U_h^i)$ satisfies the following optimality conditions:

$$\begin{aligned} &\big(d_t Y_h^i, w_h\big) + a\big(Y_h^i, w_h\big) + \int_0^t \psi\big(t, \tau; Y_h^i(\tau), w_h\big)\, d\tau + \big(\phi\big(Y_h^i\big), w_h\big) \\ &\quad = \big(f + BU_h^i, w_h\big), \qquad (2.24) \\ &\quad \forall w_h \in V_i^h, i = 1, 2, \dots, N, \qquad Y_h^0(x) = y_0^h(x), \quad x \in \Omega, \qquad (2.25) \\ &-\big(d_t P_h^i, q_h\big) + a\big(q_h, P_h^{i-1}\big) + \int_t^T \psi\big(\tau, t; q_h, P_h^{i-1}(\tau)\big)\, d\tau + \big(\phi'\big(Y_h^{i-1}\big)P_h^{i-1}, q_h\big) \qquad (2.26) \\ &\quad = \big(Y_h^i - y_0, q_h\big), \quad \forall q_h \in V_i^h, i = N, \dots, 2, 1, \qquad P_h^N(x) = 0, \quad x \in \Omega, \qquad (2.27) \\ &\big(\alpha U_h^i + B^* P_h^i, v_h - U_h^i\big) \ge 0, \quad \forall v_h \in K_i^h, i = 1, 2, \dots, N. \qquad (2.28) \end{aligned}$$

For $i = 1, 2, \dots, N$, let

$$Y_h|_{(t_{i-1}, t_i]} = \big((t_i - t)Y_h^{i-1} + (t - t_{i-1})Y_h^i\big)/\Delta t, \qquad (2.29)$$

$$P_h|_{(t_{i-1}, t_i]} = \big((t_i - t)P_h^{i-1} + (t - t_{i-1})P_h^i\big)/\Delta t, \qquad (2.30)$$

$$U_h|_{(t_{i-1}, t_i]} = U_h^i. \qquad (2.31)$$

For any function $w \in C(0, T; L^2(\Omega))$, let $\hat{w}(x, t)|_{t \in (t_{i-1}, t_i]} = w(x, t_i)$, $\tilde{w}(x, t)|_{t \in (t_{i-1}, t_i]} = w(x, t_{i-1})$. Then the optimality conditions (2.24)-(2.28) can be restated as

$$\begin{aligned} &(Y_{ht}, w_h) + a(\hat{Y}_h, w_h) + \int_0^t \psi\big(t, \tau; \hat{Y}_h(\tau), w_h\big)\, d\tau + \big(\phi(\hat{Y}_h), w_h\big) = (\hat{f} + BU_h, w_h), \qquad (2.32) \\ &\quad \forall w_h \in V_i^h, i = 1, 2, \dots, N, \qquad Y_h^0(x) = y_0^h(x), \quad x \in \Omega, \qquad (2.33) \\ &-(P_{ht}, q_h) + a(q_h, \tilde{P}_h) + \int_t^T \psi\big(\tau, t; q_h, \tilde{P}_h(\tau)\big)\, d\tau + \big(\phi'(\tilde{Y}_h)\tilde{P}_h, q_h\big) \\ &\quad = (\hat{Y}_h - y_0, q_h), \qquad (2.34) \\ &\quad \forall q_h \in V_i^h, i = N, \dots, 2, 1, \qquad P_h(x, T) = 0, \quad x \in \Omega, \qquad (2.35) \\ &\big(\alpha U_h + B^* \tilde{P}_h, v_h - U_h\big) \ge 0, \quad \forall v_h \in K_i^h, i = 1, 2, \dots, N. \qquad (2.36) \end{aligned}$$

In the rest of the paper, we shall use some intermediate variables. For any control function $U_h \in K$, we first define the state solution $(y(U_h), p(U_h))$ which satisfies

$$\begin{aligned} &\big(y_t(U_h), w\big) + a\big(y(U_h), w\big) + \int_0^t \psi\big(t, \tau; y(U_h)(\tau), w\big)\, d\tau + \big(\phi\big(y(U_h)\big), w\big) \\ &\quad = (f + BU_h, w), \qquad (2.37) \\ &\quad \forall w \in V, \qquad y(U_h)(x, 0) = y_0(x), \quad x \in \Omega, \qquad (2.38) \\ &-\big(p_t(U_h), q\big) + a\big(q, p(U_h)\big) + \int_t^T \psi\big(\tau, t; q, p(U_h)(\tau)\big)\, d\tau + \big(\phi'\big(y(U_h)\big)p(U_h), q\big) \qquad (2.39) \end{aligned}$$

$$= \big(y(U_h) - y_0, q\big), \quad \forall q \in V, \qquad p(U_h)(x,T) = 0, \quad x \in \Omega. \tag{2.40}$$

Now we restate the following well-known estimates in [18].

Lemma 2.1 *Let $\hat{\pi}_h$ be the Clément type interpolation operator defined in* [18]. *Then for any $v \in H^1(\Omega)$ and all elements τ,*

$$\|v - \hat{\pi}_h v\|_{L^2(\tau)} + h_\tau \big\|\nabla(v - \hat{\pi}_h v)\big\|_{L^2(\tau)} \le Ch_\tau \sum_{\bar{\tau}' \cap \bar{\tau} \neq \emptyset} |v|_{L^2(\tau')}, \tag{2.41}$$

$$\|v - \hat{\pi}_h v\|_{L^2(l)} \le Ch_l^{1/2} \sum_{l \subset \bar{\tau}'} |\nabla v|_{L^2(\tau')}, \tag{2.42}$$

where l is the edge of the element.

3 A posteriori error estimates

In this section we will obtain a posteriori error estimates in $L^2(J;H^1(\Omega))$ and $L^2(J;L^2(\Omega))$ for the coupled state and control approximation. Firstly, we estimate the error $\|y(U_h) - \hat{Y}_h\|_{L^2(J;H^1(\Omega))}$.

Theorem 3.1 *Let $(y(U_h), p(U_h))$ and (Y_h, P_h) be the solutions of equations* (2.37)-(2.39) *and equations* (2.32)-(2.34), *respectively. Then*

$$\big\|y(U_h) - \hat{Y}_h\big\|^2_{L^2(J;H^1(\Omega))} \le C \sum_{i=1}^{6} \eta_i^2, \tag{3.1}$$

where

$$\eta_1^2 = \int_0^T \sum_{\tau \in \mathcal{T}^h} h_\tau^2 \int_\tau \bigg(\hat{f} + BU_h - Y_{ht} + \operatorname{div}(A\nabla\hat{Y}_h) + \int_0^t \operatorname{div}\big(\psi(t,\tau)\nabla\hat{Y}_h(\tau)\big)\,d\tau - \phi(\hat{Y}_h)\bigg)^2\,dt,$$

$$\eta_2^2 = \int_0^T \sum_{l \cap \partial\Omega = \varnothing} h_l \int_l \bigg[A\nabla\hat{Y}_h \cdot n + \int_0^t \Big(\big(\psi(t,\tau)\nabla\hat{Y}_h(\tau)\big) \cdot n\Big)\,d\tau\bigg]^2\,dt,$$

$$\eta_3^2 = \int_0^T \sum_{l \subset \partial\Omega} h_l \int_l \bigg[A\nabla\hat{Y}_h \cdot n + \int_0^t \Big(\big(\psi(t,\tau)\nabla\hat{Y}_h(\tau)\big) \cdot n\Big)\,d\tau\bigg]^2\,dt,$$

$$\eta_4^2 = \|f - \hat{f}\|^2_{L^2(J;L^2(\Omega))},$$

$$\eta_5^2 = \|Y_h - \hat{Y}_h\|^2_{L^2(J;H^1(\Omega))},$$

$$\eta_6^2 = \big\|Y_h(x,0) - y_0(x)\big\|^2_{L^2(\Omega)},$$

where l is a face of an element τ, h_l is the size of face l, $[A\nabla y_h \cdot n]$ is the A-normal derivative jump over the interior face l, defined by

$$[A\nabla Y_h \cdot n]_l = (A\nabla Y_h|_{\tau_l^1} - A\nabla Y_h|_{\tau_l^2}) \cdot n,$$

where n is the unit normal vector on $l = \bar{\tau}_l^1 \cap \bar{\tau}_l^2$ outwards of τ_l^1.

Proof Let $e^y = y(U_h) - Y_h$, and let e^y_I be the Clément type interpolator of e^y defined in Lemma 2.1. Note that

$$\begin{aligned}
&\int_0^T \big(y_t(U_h) - Y_{ht}, e^y\big)\,dt \\
&\quad= \int_0^T \int_\Omega \big(y_t(U_h) - Y_{ht}\big)e^y\,dx\,dt \\
&\quad= \frac{1}{2}\int_\Omega \big(\big(y(U_h) - Y_h\big)(x,T)\big)^2\,dx - \frac{1}{2}\big\|Y_h(x,0) - y_0(x)\big\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.2}$$

From equation (3.2), we have

$$\int_0^T \big(y_t(U_h) - Y_{ht}, e^y\big)\,dt + \frac{1}{2}\big\|Y_h(x,0) - y_0(x)\big\|^2_{L^2(\Omega)} \geq 0. \tag{3.3}$$

By using the assumptions of A and ϕ, thus we can obtain the following result:

$$\begin{aligned}
&c\big\|e^y\big\|^2_{L^2(J;H^1(\Omega))} \\
&\quad\leq \int_0^T \big(A\nabla\big(y(U_h) - Y_h\big), \nabla e^y\big)\,dt + \int_0^T \big(\phi\big(y(U_h)\big) - \phi(Y_h), e^y\big)\,dt \\
&\quad= \int_0^T \big(A\nabla\big(y(U_h) - Y_h\big), \nabla\big(e^y - e^y_I\big)\big)\,dt + \int_0^T \big(\phi\big(y(U_h)\big) - \phi(Y_h), e^y - e^y_I\big)\,dt \\
&\qquad+ \int_0^T \big(A\nabla\big(y(U_h) - Y_h\big), \nabla e^y_I\big)\,dt + \int_0^T \big(\phi\big(y(U_h)\big) - \phi(Y_h), e^y_I\big)\,dt \\
&\quad\leq \int_0^T \big(A\nabla\big(y(U_h) - Y_h\big), \nabla\big(e^y - e^y_I\big)\big)\,dt + \int_0^T \big(\phi\big(y(U_h)\big) - \phi(Y_h), e^y - e^y_I\big)\,dt \\
&\qquad+ \int_0^T \big(y_t(U_h) - Y_{ht}, e^y - e^y_I\big)\,dt + \frac{1}{2}\big\|Y_h(x,0) - y_0(x)\big\|^2_{L^2(\Omega)} \\
&\qquad+ \int_0^T \big(A\nabla\big(y(U_h) - Y_h\big), \nabla e^y_I\big)\,dt + \int_0^T \big(\phi\big(y(U_h)\big) - \phi(Y_h), e^y_I\big)\,dt \\
&\qquad+ \int_0^T \big(y_t(U_h) - Y_{ht}, e^y_I\big)\,dt.
\end{aligned} \tag{3.4}$$

By using equations (2.32), (2.37), and (3.4), we infer that

$$\begin{aligned}
&c\big\|e^y\big\|^2_{L^2(J;H^1(\Omega))} \\
&\quad\leq \int_0^T \sum_{\tau\in\mathcal{T}^h} \int_\tau \bigg(\hat{f} - Y_{ht} + \operatorname{div}(A\nabla\hat{Y}_h) + \int_0^t \operatorname{div}\big(\psi(t,\tau)\nabla\hat{Y}_h(\tau)\big)\,d\tau - \phi(\hat{Y}_h)\bigg) \\
&\qquad\times \big(e^y - e^y_I\big)\,dt \\
&\qquad+ \int_0^T \sum_{l\cap\partial\Omega=\varnothing} \int_l \bigg(A\nabla\hat{Y}_h\cdot n + \int_0^t \big(\big(\psi(t,\tau)\nabla\hat{Y}_h(\tau)\big)\cdot n\big)\,d\tau\bigg)\big(e^y - e^y_I\big)\,dt \\
&\qquad+ \int_0^T \sum_{l\subset\partial\Omega} \int_l \bigg(A\nabla\hat{Y}_h\cdot n + \int_0^t \big(\big(\psi(t,\tau)\nabla\hat{Y}_h(\tau)\big)\cdot n\big)\,d\tau\bigg)\big(e^y - e^y_I\big)\,dt
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^T \big(A\nabla(Y_h - \hat{Y}_h), \nabla e^y\big)\,dt + \int_0^T \big(\phi(Y_h) - \phi(\hat{Y}_h), e^y\big)\,dt \\
&+ \int_0^T \big(f - \hat{f}, e^y\big)\,dt + \frac{1}{2}\big\|Y_h(x,0) - y_0(x)\big\|^2_{L^2(\Omega)} \\
&\equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + \frac{1}{2}\big\|Y_h(x,0) - y_0(x)\big\|^2_{L^2(\Omega)}. \qquad (3.5)
\end{aligned}
$$

Let us bound each of the terms on the right-hand side of equation (3.5). By Lemma 2.1 we have

$$
\begin{aligned}
I_1 &= \int_0^T \sum_{\tau\in\mathcal{T}^h} \int_\tau \Big(\hat{f} - Y_{ht} + \operatorname{div}(A\nabla\hat{Y}_h) + \int_0^t \operatorname{div}\big(\psi(t,\tau)\nabla\hat{Y}_h(\tau)\big)\,d\tau - \phi(\hat{Y}_h)\Big) \\
&\quad \times \big(e^y - e^y_I\big)\,dt \\
&\le C\int_0^T \sum_{\tau\in\mathcal{T}^h} h_\tau^2 \int_\tau \Big(\hat{f} - Y_{ht} + \operatorname{div}(A\nabla\hat{Y}_h) + \int_0^t \operatorname{div}\big(\psi(t,\tau)\nabla\hat{Y}_h(\tau)\big)\,d\tau - \phi(\hat{Y}_h)\Big)^2\,dt \\
&\quad + C\delta\int_0^T \sum_{\tau\in\mathcal{T}^h} h_\tau^{-2}\int_\tau \big|e^y - e^y_I\big|^2\,dt \\
&\le C\int_0^T \sum_{\tau\in\mathcal{T}^h} h_\tau^2 \int_\tau \Big(\hat{f} - Y_{ht} + \operatorname{div}(A\nabla\hat{Y}_h) + \int_0^t \operatorname{div}\big(\psi(t,\tau)\nabla\hat{Y}_h(\tau)\big)\,d\tau - \phi(\hat{Y}_h)\Big)^2\,dt \\
&\quad + C\delta\big\|e^y\big\|^2_{L^2(J;H^1(\Omega))}. \qquad (3.6)
\end{aligned}
$$

Next, using Lemma 2.1, we get

$$
\begin{aligned}
I_2 &= \int_0^T \sum_{l\cap\partial\Omega=\varnothing} \int_l \Big(A\nabla\hat{Y}_h\cdot n + \int_0^t \big(\big(\psi(t,\tau)\nabla\hat{Y}_h(\tau)\big)\cdot n\big)\,d\tau\Big)\big(e^y - e^y_I\big)\,dt \\
&\le C\int_0^T \sum_{l\cap\partial\Omega=\varnothing} h_l \int_l \Big[A\nabla\hat{Y}_h\cdot n + \int_0^t \big(\big(\psi(t,\tau)\nabla\hat{Y}_h(\tau)\big)\cdot n\big)\,d\tau\Big]^2\,dt \\
&\quad + C\delta\int_0^T \sum_{\tau\in\mathcal{T}^h} h_\tau^{-2}\int_\tau \big|e^y - e^y_I\big|^2\,dt + C\delta\int_0^T \sum_{\tau\in\mathcal{T}^h}\int_\tau \big|\nabla\big(e^y - e^y_I\big)\big|^2\,dt \\
&\le C\int_0^T \sum_{l\cap\partial\Omega=\varnothing} h_l \int_l \Big[A\nabla\hat{Y}_h\cdot n + \int_0^t \big(\big(\psi(t,\tau)\nabla\hat{Y}_h(\tau)\big)\cdot n\big)\,d\tau\Big]^2\,dt \\
&\quad + C\delta\big\|e^y\big\|^2_{L^2(J;H^1(\Omega))}, \qquad (3.7)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \int_0^T \sum_{l\subset\partial\Omega} \int_l \Big(A\nabla\hat{Y}_h\cdot n + \int_0^t \big(\big(\psi(t,\tau)\nabla\hat{Y}_h(\tau)\big)\cdot n\big)\,d\tau\Big)\big(e^y - e^y_I\big)\,dt \\
&\le C\int_0^T \sum_{l\subset\partial\Omega} h_l \int_l \Big(A\nabla\hat{Y}_h\cdot n + \int_0^t \big(\big(\psi(t,\tau)\nabla\hat{Y}_h(\tau)\big)\cdot n\big)\,d\tau\Big)^2\,dt \\
&\quad + C\delta\int_0^T \sum_{\tau\in\mathcal{T}^h} h_\tau^{-2}\int_\tau \big|e^y - e^y_I\big|^2\,dt + C\delta\int_0^T \sum_{\tau\in\mathcal{T}^h}\int_\tau \big|\nabla\big(e^y - e^y_I\big)\big|^2\,dt
\end{aligned}
$$

$$\leq C\int_0^T \sum_{l\subset\partial\Omega} h_l \int_l \left(A\nabla\hat{Y}_h\cdot n + \int_0^t \big(\big(\psi(t,\tau)\nabla\hat{Y}_h(\tau)\big)\cdot n\big)\,d\tau\right)^2 dt$$
$$+ C\delta\left\|e^y\right\|^2_{L^2(J;H^1(\Omega))}. \tag{3.8}$$

For the right-hand terms I_4-I_6 of equation (3.5), the Schwarz inequality implies

$$I_4 = \int_0^T \big(A\nabla(Y_h - \hat{Y}_h), \nabla e^y\big)\,dt$$
$$\leq C\|Y_h - \hat{Y}_h\|^2_{L^2(J;H^1(\Omega))} + C\delta\left\|e^y\right\|^2_{L^2(J;H^1(\Omega))}, \tag{3.9}$$
$$I_5 = \int_0^T \big(\phi(Y_h) - \phi(\hat{Y}_h), e^y\big)\,dt$$
$$= \int_0^T \big(\tilde{\phi}'(Y_h)(Y_h - \hat{Y}_h), e^y\big)\,dt$$
$$\leq C\|Y_h - \hat{Y}_h\|^2_{L^2(J;H^1(\Omega))} + C\delta\left\|e^y\right\|^2_{L^2(J;H^1(\Omega))}, \tag{3.10}$$

and

$$I_6 = \int_0^T \big(f - \hat{f}, e^y\big)\,dt$$
$$\leq C\|f - \hat{f}\|^2_{L^2(J;L^2(\Omega))} + C\delta\left\|e^y\right\|^2_{L^2(J;H^1(\Omega))}. \tag{3.11}$$

Let δ be small enough, and add inequalities (3.5)-(3.11) to obtain

$$\left\|y(U_h) - \hat{Y}_h\right\|^2_{L^2(J;H^1(\Omega))} \leq C\sum_{i=1}^{6}\eta_i^2. \tag{3.12}$$

This completes the proof. □

Analogously to the proof of Theorem 3.1, we can obtain the following estimates.

Theorem 3.2 *Let* $(y(U_h), p(U_h))$ *and* (Y_h, P_h) *be the solutions of equations* (2.37)-(2.39) *and equations* (2.32)-(2.34), *respectively. Then*

$$\left\|p(U_h) - \tilde{P}_h\right\|^2_{L^2(J;H^1(\Omega))} \leq C\sum_{i=1}^{11}\eta_i^2, \tag{3.13}$$

where

$$\eta_7^2 = \int_0^T \sum_{\tau\in\mathcal{T}^h} h_\tau^2$$
$$\times \int_\tau \left(P_{ht} + \operatorname{div}\big(A^*\nabla\tilde{P}_h\big) + \int_t^T \operatorname{div}\big(\psi^*(\tau,t)\nabla\tilde{P}_h(\tau)\big)\,d\tau + \hat{Y}_h - y_0 - \phi'(\tilde{Y}_h)\tilde{P}_h\right)^2 dt,$$
$$\eta_8^2 = \int_0^T \sum_{l\cap\partial\Omega=\varnothing} h_l \int_l \left[A^*\nabla\tilde{P}_h\cdot n + \int_t^T \big(\psi^*(\tau,t)\nabla\tilde{P}_h(\tau)\big)\cdot n\,d\tau\right]^2 dt,$$

$$\eta_9^2 = \int_0^T \sum_{l\subset\partial\Omega} h_l \int_l \left[A^*\nabla\tilde{P}_h \cdot n + \int_t^T \big(\psi^*(\tau,t)\nabla\tilde{P}_h(\tau)\big)\cdot n\,d\tau\right]^2 dt,$$

$$\eta_{11}^2 = \|P_h - \tilde{P}_h\|^2_{L^2(J;H^1(\Omega))},$$

$$\eta_{10}^2 = \|Y_h - \tilde{Y}_h\|^2_{L^2(J;H^1(\Omega))},$$

where η_1-η_6 are defined in Theorem 3.1, l is a face of an element τ, $[A^*\nabla\tilde{P}_h \cdot n]$ is the A-normal derivative jump over the interior face l, defined by

$$\big[A^*\nabla\tilde{P}_h \cdot n\big]_l = \big(A^*\nabla\tilde{P}_h|_{\tau_l^1} - A^*\nabla\tilde{P}_h|_{\tau_l^2}\big)\cdot n,$$

where n is the unit normal vector on $l = \bar{\tau}_l^1 \cap \bar{\tau}_l^2$ outwards of τ_l^1.

For given $u \in K$, let M be the inverse operator of the state equation (2.8), such that $y(u) = MBu$ is the solution of the state equation (2.8). Similarly, for given $U_h \in K^h$, $Y_h(U_h) = M_hBU_h$ is the solution of the discrete state equation (2.32). Let

$$S(u) = \frac{1}{2}\|MBu - y_0\|^2 + \frac{\alpha}{2}\|u\|^2,$$

$$S_h(U_h) = \frac{1}{2}\|M_hBU_h - y_0\|^2 + \frac{\alpha}{2}\|U_h\|^2.$$

It is clear that S and S_h are well defined and continuous on K and K^h. Also the functional S_h can be naturally extended on K. Then equations (2.5) and (2.21) can be represented as

$$\min_{u\in K}\big\{S(u)\big\}, \tag{3.14}$$

$$\min_{U_h\in K^h}\big\{S_h(U_h)\big\}. \tag{3.15}$$

It can be shown that

$$\big(S'(u), v\big) = \big(\alpha u + B^*p, v\big),$$

$$\big(S'(U_h), v\big) = \big(\alpha U_h + B^*p(U_h), v\big),$$

$$\big(S_h'(U_h), v\big) = \big(\alpha U_h + B^*\tilde{P}_h, v\big),$$

where $p(U_h)$ is the solution of equations (2.37)-(2.39).

In many applications, $S(\cdot)$ is uniform convex near the solution u (see, *e.g.*, [19]). The convexity of $S(\cdot)$ is closely related to the second-order sufficient conditions of the control problems, which are assumed in many studies on numerical methods of the problems. If $S(\cdot)$ is uniformly convex, then there is a $c > 0$, such that

$$\int_0^T \big(S'(u) - S'(U_h), u - U_h\big) \geq c\|u - U_h\|^2_{L^2(J;L^2(\Omega))}, \tag{3.16}$$

where u and U_h are the solutions of equations (3.14) and (3.15), respectively. We will assume the above inequality throughout this paper.

In order to have sharp a posteriori error estimates, we divide Ω into some subsets:

$$\Omega_i^- = \{x \in \Omega : (B^*\tilde{P}_h)(x,t_i) \le -\alpha U_h^i\},$$
$$\Omega_i = \{x \in \Omega : (B^*\tilde{P}_h)(x,t_i) > -\alpha U_h^i, U_h^i = 0\},$$
$$\Omega_i^+ = \{x \in \Omega : (B^*\tilde{P}_h)(x,t_i) > -\alpha U_h^i, U_h^i > 0\}.$$

Then, it is clear that the three subsets do not intersect, and $\Omega = \Omega_i^- \cup \Omega_i \cup \Omega_i^+$, $i = 1, 2, \ldots, N$.

Let $p(U_h)$ be the solution of equations (2.37)-(2.39); we establish the following error estimate, which can be proved similarly to the proofs given in [10].

Theorem 3.3 *Let u and U_h be the solutions of equations* (2.5) *and* (2.36), *respectively. Then*

$$\|u - U_h\|^2_{L^2(J;L^2(\Omega))} \le C\big(\eta_{12}^2 + \|\tilde{P}_h - p(U_h)\|^2_{L^2(J;H^1(\Omega))}\big), \tag{3.17}$$

where

$$\eta_{12}^2 = \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \int_{\Omega_i^-} |B^*\tilde{P}_h + \alpha U_h|^2 \, dx \, dt.$$

Proof It follows from the inequality (3.16) that

$$\begin{aligned} c\|u - U_h\|^2_{L^2(J;L^2(\Omega))} &\le \int_0^T \big(S'(u), u - U_h\big) - \big(S'(U_h), u - U_h\big) \, dt \\ &\le -\int_0^T \big(S'(U_h), u - U_h\big) \, dt \\ &= \int_0^T \big(S_h'(U_h), U_h - u\big) \, dt + \int_0^T \big(S_h'(U_h) - S'(U_h), u - U_h\big) \, dt. \end{aligned} \tag{3.18}$$

Note that

$$\begin{aligned} \int_0^T \big(S_h'(U_h), U_h - u\big) \, dt = {}& \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \int_{\Omega_i^-} \big(B^*\tilde{P}_h + \alpha U_h\big)(U_h - u) \, dx \, dt \\ &+ \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \int_{\Omega_i} \big(B^*\tilde{P}_h + \alpha U_h\big)(U_h - u) \, dx \, dt \\ &+ \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \int_{\Omega_i^+} \big(B^*\tilde{P}_h + \alpha U_h\big)(-u) \, dx \, dt. \end{aligned} \tag{3.19}$$

It is easy to see that

$$\begin{aligned} \int_{\Omega_i^-} \big(B^*\tilde{P}_h + \alpha U_h\big)(U_h - u) \, dx &\le \int_{\Omega_i^-} |B^*\tilde{P}_h + \alpha U_h|^2 \, dx + \delta\|u - U_h\|^2_{L^2(J;L^2(\Omega))} \\ &= C\eta_{12}^2 + \delta\|u - U_h\|^2_{L^2(J;L^2(\Omega))}. \end{aligned} \tag{3.20}$$

Since U_h is piecewise constant, $U_h|_s > 0$ if $s \cap \Omega_i^+$ is not empty. If $u_h|_s > 0$, there exist $\varepsilon > 0$ and $\beta \in U_h$, such that $\beta \geq 0$, $\|\beta\|_{L^\infty(s)} = 1$ and $(u_h - \varepsilon\beta)|_s \geq 0$. For example, one can always find such a required β from one of the shape functions on s. Hence, $\hat{u}_h \in K^h$, where $\hat{u}_h = U_h - \varepsilon\beta$ as $x \in s$ and otherwise $\hat{u} = U_h$. Then, it follows from equation (2.36) that

$$\begin{aligned}\int_s \big(B^*\tilde{P}_h + \alpha U_h\big)\beta\,dx &= \varepsilon^{-1}\int_s \big(B^*\tilde{P}_h + \alpha U_h\big)\big(U_h - (U_h - \varepsilon\beta)\big)\,dx \\ &\leq \varepsilon^{-1}\int_\Omega \big(B^*\tilde{P}_h + \alpha U_h\big)\big(U_h - (U_h - \varepsilon\beta)\big)\,dx \leq 0. \end{aligned} \tag{3.21}$$

Note that on Ω_i^+, $B^*\tilde{P}_h + \alpha U_h \geq B^*\tilde{P}_h > 0$ and from equation (3.20) we have

$$\begin{aligned}\int_{s\cap\Omega_i^+} \big|B^*\tilde{P}_h + \alpha U_h\big|\beta\,dx &= \int_{s\cap\Omega_i^+} \big(B^*\tilde{P}_h + \alpha U_h\big)\beta\,dx \leq -\int_{s\cap\Omega_i^-} \big(B^*\tilde{P}_h + \alpha U_h\big)\beta\,dx \\ &\leq \int_{s\cap\Omega_i^-} \big|B^*\tilde{P}_h + \alpha U_h\big|\,dx. \end{aligned} \tag{3.22}$$

Let $\hat{s}$ be the reference element of s, $s^0 = s \cap \Omega_i^+$, and $\hat{s}^0 \subset \hat{s}$ be a part mapped from $\hat{s}^0$. Note that $(\int_s |\cdot|^2)^{1/2}$, $\int_s |\cdot|\beta$ are both norms on $L^2(s)$. In such a case for the function β fixed above, it follows from the equivalence of the norm in the finite-dimensional space that

$$\begin{aligned}&\int_{s\cap\Omega_i^+} \big|B^*\tilde{P}_h + \alpha U_h\big|^2\,dx \\ &\quad = \int_{s^0} \big|B^*\tilde{P}_h + \alpha U_h\big|^2\,dx \leq Ch_s^2\int_{\hat{s}^0} \big|B^*\tilde{P}_h + \alpha U_h\big|^2\,dx \\ &\quad \leq Ch_s^2\left(\int_{\hat{s}^0} \big|B^*\tilde{P}_h + \alpha U_h\big|\beta\,dx\right)^2 \leq Ch_s^{-2}\left(\int_{s\cap\Omega_i^+} \big|B^*\tilde{P}_h + \alpha U_h\big|\beta\,dx\right)^2 \\ &\quad \leq Ch_s^{-2}\left(\int_{s\cap\Omega_i^-} \big|B^*\tilde{P}_h + \alpha U_h\big|\,dx\right)^2 \leq C\int_{s\cap\Omega_i^-} \big|B^*\tilde{P}_h + \alpha U_h\big|^2\,dx, \end{aligned} \tag{3.23}$$

where the constant C can be made independent of β since it is always possible to find the required β from the shape functions on s. Thus

$$\begin{aligned}\int_{\Omega_i^+} \big(B^*\tilde{P}_h + \alpha U_h\big)(U_h - u)\,dx &\leq C\int_{\Omega_i^+} \big|B^*\tilde{P}_h + \alpha U_h\big|^2\,dx + \delta\|u - U_h\|^2_{L^2(J;L^2(\Omega))} \\ &\leq C\int_{\Omega_i^-} \big|B^*\tilde{P}_h + \alpha U_h\big|^2\,dx + \delta\|u - U_h\|^2_{L^2(J;L^2(\Omega))} \\ &\leq C\eta_{12}^2 + \delta\|u - U_h\|^2_{L^2(J;L^2(\Omega))}. \end{aligned} \tag{3.24}$$

It follows from the definition of Ω_i that $B^*\tilde{P}_h + \alpha U_h > 0$ on Ω_i. Note that $-u \leq 0$, we have

$$\int_{\Omega_i} \big(B^*\tilde{P}_h + \alpha U_h\big)(-u)\,dx \leq 0. \tag{3.25}$$

It is easy to show that

$$\big(S_h'(U_h) - S'(U_h), u - U_h\big)$$

$$
\begin{aligned}
&= \big(B^*\tilde{P}_h + \alpha U_h, u - U_h\big) - \big(B^* p(U_h) + \alpha U_h, u - U_h\big) \\
&= \big(B^*\big(\tilde{P}_h - p(U_h)\big), u - U_h\big) \\
&\leq C\big\|\tilde{P}_h - p(U_h)\big\|^2_{L^2(J;L^2(\Omega))} + \delta\|u - U_h\|^2_{L^2(J;L^2(\Omega))} \\
&\leq C\big\|\tilde{P}_h - p(U_h)\big\|^2_{L^2(J;H^1(\Omega))} + \delta\|u - U_h\|^2_{L^2(J;L^2(\Omega))}. \qquad (3.26)
\end{aligned}
$$

Therefore, equation (3.17) follows from equations (3.18)-(3.20) and (3.24)-(3.26). □

Hence, we combine Theorems 3.1-3.3 to conclude that

Theorem 3.4 *Let* (y, p, u) *and* (Y_h, P_h, U_h) *be the solutions of equations* (2.8)-(2.12) *and equations* (2.32)-(2.36), *respectively. Then*

$$
\|u - U_h\|^2_{L^2(J;L^2(\Omega))} + \|y - Y_h\|^2_{L^2(J;H^1(\Omega))} + \|p - P_h\|^2_{L^2(J;H^1(\Omega))} \leq C\sum_{i=1}^{12}\eta_i^2, \qquad (3.27)
$$

where η_1-η_{12} *are defined in Theorems* 3.1-3.3, *respectively.*

Proof From equations (2.8)-(2.11) and (2.37)-(2.40), we obtain the error equations

$$
\begin{aligned}
&\big(y_t - y_t(U_h), w\big) + a\big(y - y(U_h), w\big) + \int_0^t \psi\big(t, \tau; \big(y - y(U_h)\big)(\tau), w\big)\, d\tau \\
&\quad + \big(\phi(y) - \phi\big(y(U_h)\big), w\big) = \big(B(u - U_h), w\big), \qquad (3.28) \\
&-\big(p_t - p_t(U_h), q\big) + a\big(q, p - p(U_h)\big) + \int_t^T \psi\big(\tau, t; q, \big(p - p(U_h)\big)(\tau)\big)\, d\tau \\
&\quad + \big(\phi'(y)p - \phi'\big(y(U_h)\big)p(U_h), q\big) = \big(y - y(U_h), q\big), \qquad (3.29)
\end{aligned}
$$

for all $w \in V$ and $q \in V$. Thus it follows from equations (3.28)-(3.29) that

$$
\begin{aligned}
&\big(y_t - y_t(U_h), w\big) + a\big(y - y(U_h), w\big) + \int_0^t \psi\big(t, \tau; \big(y - y(U_h)\big)(\tau), w\big)\, d\tau \\
&\quad + \big(\tilde{\phi}'(y)\big(y - y(U_h)\big), w\big) = \big(B(u - U_h), w\big), \qquad (3.30) \\
&-\big(p_t - p_t(U_h), q\big) + a\big(q, p - p(U_h)\big) + \int_t^T \psi\big(\tau, t; q, \big(p - p(U_h)\big)(\tau)\big)\, d\tau \\
&\quad + \big(\phi'\big(y(U_h)\big)\big(p - p(U_h)\big), q\big) = \big(\tilde{\phi}''\big(y(U_h)\big)\big(y(U_h) - y\big)p, q\big). \qquad (3.31)
\end{aligned}
$$

By using the stability results in [20, 21], then we obtain

$$
\big\|y - y(U_h)\big\|^2_{L^2(J;H^1(\Omega))} \leq C\|u - U_h\|^2_{L^2(J;L^2(\Omega))}, \qquad (3.32)
$$

and

$$
\big\|p - p(U_h)\big\|^2_{L^2(J;H^1(\Omega))} \leq \big\|y - y(U_h)\big\|^2_{L^2(J;H^1(\Omega))} \leq C\|u - U_h\|^2_{L^2(J;L^2(\Omega))}. \qquad (3.33)
$$

Finally, combining Theorems 3.1-3.3 and equations (3.32)-(3.33) leads to equation (3.27). □

4 Conclusion and future work

In this paper we discuss the finite-element methods of the nonlinear parabolic integro-differential optimal control problems (1.1)-(1.4). We have established a posteriori error estimates for both the state, the co-state, and the control variables. The posteriori error estimates for those problems by finite-element methods seem to be new.

In our future work, we shall use the mixed finite-element method to deal with nonlinear parabolic integro-differential optimal control problems. Furthermore, we shall consider a posteriori error estimates and superconvergence of mixed finite-element solution for nonlinear parabolic integro-differential optimal control problems.

Competing interests
The author declares that they have no competing interests.

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