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Approximate controllability of impulsive neutral stochastic differential equations with fractional Brownian motion in a Hilbert space

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Abstract

Approximate controllability for impulsive neutral stochastic functional differential equations with finite delay and fractional Brownian motion in a Hilbert space are studied. The results are obtained by using semigroup theory, stochastic analysis, and Banach's fixed point theorem. Finally, an example is given to illustrate the application of our result.

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1 Introduction

The impulsive differential systems are used to describe processes which are subjected to abrupt changes at certain moments. The impulsive effects exist widely in the different areas of the real world such as mechanics, electronics, telecommunications, neural networks, finance and economics, etc. (see [1–7]). On the other hand, it is well known that the stochastic control theory is a stochastic generalization of classical control theory. As one of the fundamental concepts in mathematical control theory, controllability plays an important role both in deterministic and stochastic control theory. Controllability generally means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls (see [8–17]). Moreover, the approximate controllability means that the system can be steered to arbitrary small neighborhood of final state. Approximate controllable systems are more prevalent and very often approximate controllability is completely adequate in applications (see [18–24]).

The purpose of this paper is to investigate the approximate controllability problem for the class of impulsive neutral stochastic functional differential equations with finite delay and fractional Brownian motion in a Hilbert space of the form

$$\begin{cases} d[x(t)+g(t,x(t-r(t)))] = [Ax(t)+f(t,x(t-\nu(t)))+Bu(t)]\,dt+\sigma(t)\,dB^H(t), \\ \quad 0\le t\le T, t\ne t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k=1,2,\dots,m, \\ x(t)=\varphi(t), \quad -\tau\le t\le 0, \end{cases} \tag{1.1}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $S(t)_{t\geq 0}$, in a Hilbert space X, B^H is a fractional Brownian motion on a real and separable Hilbert space Y, the initial data $\varphi \in C([-\tau,0],L^2(\Omega,X))$ and the control function $u(\cdot)$ is given in $L^2([0,T],U)$, the Hilbert space of admissible control functions with U a Hilbert space. The symbol B stands for a bounded linear from U into X. The functions $r,\nu:[0,+\infty)\to[0,\tau]$ $(\tau>0)$ are continuous, $\Delta x|_{t=t_k}=I_k(x(t_k^-))$, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t=t_k$, respectively, and $f,g:[0,+\infty)\times X\to X$, $\sigma:[0,+\infty)\to L_2^0(Y,X)$, are appropriate Lipschitz type functions. Here $R_T:=C([-\tau,T],L^2(\Omega,X))$ be the Banach space of all continuous functions ξ from $[-\tau,T]$ into $L^2(\Omega,X)$, equipped with the supremum norm $\|\xi\|_{R_T}=\sup_{y\in[-\tau,T]}(E\|\xi(y)\|^2)^{1/2}$.

2 Fractional Brownian motion

Fix a time interval $[0,T]$ and let (Ω,F,P) be a complete probability space.

Suppose that $\{\beta^H(t),t\in[0,T]\}$ is the one-dimensional fractional Brownian motion with Hurst parameter $H\in(1/2,1)$. That is, β^H is a centered Gaussian process with covariance function $R_H(s,t)=\frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H})$ (see [25]).

Moreover, β^H has the following Wiener integral representation:

$$\beta^H(t)=\int_0^t K_H(t,s)\,d\beta(s),$$

where $\beta=\{\beta(t),t\in[0,T]\}$ is a Wiener process, and $K_H(t,s)$ is the kernel given by

$$K_H(t,s)=c_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}\,du$$

for $s<t$, where $c_H=\sqrt{\frac{H(2H-1)}{\beta(2-2H,H-\frac{1}{2})}}$ and

$$\beta(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1},\quad p>0,q>0.$$

We put $K_H(t,s)=0$ if $t\leq s$.

We will denote by ζ the reproducing kernel Hilbert space of the fBm. In fact ζ is the closure of set of indicator functions $\{1_{[0,t]},t\in[0,T]\}$ with respect to the scalar product $\langle 1_{[0,t]},1_{[0,s]}\rangle_\zeta=R_H(t,s)$.

The mapping $1_{[0,t]}\to\beta^H(t)$ can be extended to an isometry from ζ onto the first Wiener chaos and we will denote by $\beta^H(\varphi)$ the image of φ under this isometry.

We recall that for $\psi,\varphi\in\zeta$ their scalar product in ζ is given by

$$\langle\psi,\varphi\rangle_\zeta=H(2H-1)\int_0^T\int_0^T\psi(s)\varphi(t)|t-s|^{2H-2}\,ds\,dt.$$

Let us consider the operator K^* from ζ to $L^2([0,T])$ defined by

$$\bigl(K_H^*\varphi\bigr)(s)=\int_S^T\varphi(r)\frac{\partial K}{\partial r}(r,s)\,dr.$$

Moreover, for any $\varphi \in \zeta$, we have

$$\beta^H(\varphi) = \int_0^T (K_H^* \varphi)(t)\, d\beta(t).$$

Let X and Y be two real, separable Hilbert spaces and let $L(Y,X)$ be the space of bounded linear operators from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in X, Y and $L(Y,X)$. Let $Q \in L(Y,Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $\operatorname{tr} Q = \sum_{n=1}^{\infty} \lambda_n < \infty$, where $\lambda_n \geq 0$ $(n = 1, 2, \ldots)$ are non-negative real numbers and $\{e_n\}$ $(n = 1, 2, \ldots)$ is a complete orthonormal basis in Y.

We define the infinite-dimensional fBm on Y with covariance Q as

$$B^H(t) = B_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t),$$

where β_n^H are real, independent fBm's. The Y-valued process is Gaussian, starts from 0, has mean zero and covariance:

$$E\langle B^H(t), x\rangle \langle B^H(s), y\rangle = R(s,t)\langle Q(x), y\rangle \quad \text{for all } x, y \in Y \text{ and } t, s \in [0, T].$$

In order to define Wiener integrals with respect to the Q-fBm, we introduce the space $L_2^0 := L_2^0(Y,X)$ of all Q-Hilbert Schmidt operators $\psi : Y \to X$. We recall that $\psi \in L(Y,X)$ is called a Q-Hilbert-Schmidt operator, if

$$\|\psi\|_{L_2^0}^2 := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \psi e_n\|^2 < \infty$$

and that the space L_2^0 equipped with the inner product $\langle \varphi, \psi \rangle_{L_2^0} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n \rangle$ is a separable Hilbert space.

Let $\phi(s)$; $s \in [0, T]$ be a function with values in $L_2^0(Y,X)$, the Wiener integral of ϕ with respect to B^H is defined by

$$\begin{aligned} \int_0^t \phi(s)\, dB^H(s) &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda} \phi(s) e_n \, d\beta_n^H \\ &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda} K^*(\phi e_n)(s)\, d\beta_n(s), \end{aligned} \tag{2.1}$$

where β_n is the standard Brownian motion.

Lemma 2.1 (see [26]) *If $\psi : [0, T] \to L_2^0(Y,X)$ satisfies $\int_0^T \|\psi(s)\|_{L_2^0}^2 < \infty$ then the above sum in* (2.1) *is well defined as X-valued random variable and we have*

$$E\left\| \int_0^t \psi(s)\, dB^H(s) \right\|^2 \leq 2Ht^{2H-1} \int_0^t \|\psi(s)\|_{L_2^0}^2 \, ds.$$

3 Approximate controllability

Let $A : D(A) \to X$ be the infinitesimal generator of an analytic semigroup, $(S(t))_{t\geq 0}$, of bounded linear operators on X. It is well known that there exist $M \geq 1$ and $\lambda \in R$ such that $\|S(t)\| \leq Me^{\lambda t}$ for every $t \geq 0$.

If $(S(t))_{t\geq 0}$ is uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A, then it is possible to define the fractional power $(-A)^\alpha$ for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X and the expression $\|h\|_\alpha = \|(-A)^\alpha h\|$ defines a norm in $D(-A)^\alpha$. If X_α represents the space $D(-A)^\alpha$ endowed with the norm $\|\cdot\|_\alpha$, then the following properties are well known.

Lemma 3.1 ([27])

(1) *Let* $0 < \alpha \leq 1$, *then* X_α *is a Banach space.*
(2) *If* $0 < \beta \leq \alpha$, *then the injection* $X_\alpha \hookrightarrow X_\beta$ *is continuous.*
(3) *For every* $0 < \alpha \leq 1$ *there exists* $M_\alpha > 0$ *such that*

$$\bigl\|(-A)^\alpha S(t)\bigr\| \leq M_\alpha t^{-\alpha} e^{-\lambda t}, \quad t > 0, \lambda > 0.$$

Now, we present the mild solution of the problem (1.1):

Definition 3.1 An X-valued process $\{x(t), t \in [-\tau, T]\}$ is called a mild solution of equation (1.1) if

(i) $x(\cdot) \in C([-\tau, T], L^2(\Omega, X))$,
(ii) $x(t) = \varphi(t)$, $-\tau \leq t \leq 0$,
(iii) for arbitrary $t \in [0, T]$, we have

$$\begin{aligned} x(t) = {} & S(t)\bigl[\varphi(0) + g\bigl(0, \varphi\bigl(-r(0)\bigr)\bigr)\bigr] - g\bigl(t, x\bigl(t - r(t)\bigr)\bigr) \\ & - \int_0^t AS(t-s) g\bigl(s, x\bigl(s - r(s)\bigr)\bigr)\, ds \\ & + \int_0^t S(t-s) f\bigl(s, x\bigl(s - \nu(s)\bigr)\bigr)\, ds + \int_0^t S(t-s)Bu(s)\, ds \\ & + \int_0^t S(t-s)\sigma(s)\, dB^H(s) + \sum_{0<t_k<t} S(t-t_k) I_k\bigl(x\bigl(t_k^-\bigr)\bigr). \end{aligned} \tag{3.1}$$

In this paper, we will make the following assumptions.

(H1) The operator A is the infinitesimal generator of an analytic semigroup, $(S(t))_{t\geq 0}$, consisting of bounded linear operators on X. Furthermore, there exist constants M and $M_{1-\beta}$ such that for every $t \in [0, T]$ the inequalities $\|S(t)\| \leq M$ and $t^{1-\beta}\|(-A)^{1-\beta} S(t)\| \leq M_{1-\beta}$ hold.

(H2) There exist finite positive constants $C_i = C_i(T)$, $i = 1, 2$, such that the function $f : [0, +\infty) \times X \to X$ satisfies the following Lipschitz conditions: for all $t \in [0, T]$ and $x, y \in X$ the inequalities $\|f(t, x) - f(t, y)\| \leq C_1\|x - y\|$ and $\|f(t, x)\|^2 \leq C_2^2(1 + \|x\|^2)$ are valid.

(H3) The function g is X_β-valued, and there exist constants $\frac{1}{2} < \beta < 1$, $C_i = C_i(T)$, $i = 3, 4$, such that for all $t \in [0, T]$ and $x, y \in X$ the following inequalities are satisfied:

(i) $\|(-A)^\beta g(t, x) - (-A)^\beta g(t, y)\| \leq C_3\|x - y\|$;
(ii) $\|(-A)^\beta g(t, x)\|^2 \leq C_4^2(1 + \|x\|^2)$;
(iii) $C_3\|(-A)^{-\beta}\| < 1$.

(H4) The function $(-A)^\beta g$ is continuous in the quadratic mean sense: for all $x \in C([0,T], L^2(\Omega, X))$, the equality

$$\lim_{t\to s} E\left\|(-A)^\beta g\big(t,x(t)\big) - (-A)^\beta g\big(s,x(s)\big)\right\|^2 = 0$$

is true.

(H5) The function $\sigma : [0,\infty) \to L_2^0(Y,X)$ satisfies $\int_0^T \|\sigma(s)\|^2_{L_2^0}\, ds < \infty$.

(H6) The functions $I_k : X \to X$ are continuous and there exist finite positive constants $C_i = C_i(T)$, $i = 5,6$, such that for all $t \in [0,T]$ and $x, y \in X$ the inequalities $\|I_k(x(t)) - I_k(y(t))\| \le C_5\|x-y\|$ and $\|I_k(x(t))\|^2 \le C_6^2(1 + \|x\|^2)$ are valid.

In order to study the approximate controllability for the system (1.1), we introduce the following linear differential system:

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + Bu(t), & t \in [0,T], \\ x(0) = x_0. \end{cases} \tag{3.2}$$

The controllability operator associated with (3.2) is defined by

$$\Gamma_0^T = \int_0^T S(T-s)BB^*S^*(T-s)\, ds,$$

where B^* and S^* denote the adjoint of B and S, respectively.

Let $x(T;\varphi,u)$ be the state value of (1.1) at terminal state T, corresponding to the control u and the initial value φ. Denote by $R(T,\varphi) = \{x(T;\varphi,u) : u \in L^2([0,T],U)\}$ the reachable set of system (1.1) at terminal time T, its closure in X is denoted by $\overline{R(T,\varphi)}$.

Definition 3.2 The system (1.1) is said to be approximately controllable on the interval $[0,T]$ if $\overline{R(T,\varphi)} = L^2(\Omega,X)$.

Lemma 3.2 (see [19]) *The linear control system* (3.2) *is approximately controllable on* $[0,T]$ *if and only if* $z(zI + \Gamma_0^T)^{-1} \to 0$ *strongly as* $z \to 0^+$.

Lemma 3.3 *For any* $\bar{x}_T \in L^2(\Omega,X)$ *there exists* $\bar{\varphi} \in L^2(\Omega; L^2([0,T]; L_2^0))$ *such that*

$$\bar{x}_T = E\bar{x}_T + \int_0^T \bar{\varphi}(s)\, dB^H(s).$$

Now for any $\delta > 0$ *and* $\bar{x}_T \in L^2(\Omega,X)$, *we define the control function in the following form*:

$$\begin{aligned} u^\delta(t,x) = {} & B^*S^*(T-t)\big(zI + \Gamma_0^T\big)^{-1} \\ & \times \big\{E\bar{x}_T - S(T)\big[\varphi(0) - g\big(0,\varphi(-r(0))\big)\big] + g\big(T,x(T)\big)\big\} \\ & + B^*S^*(T-t)\int_0^t \big(zI + \Gamma_0^T\big)^{-1}\bar{\varphi}(s)\, dB^H(s) \\ & - B^*S^*(T-t)\int_0^t \big(zI + \Gamma_0^T\big)^{-1} AS(T-s)g\big(s,x\big(s-r(s)\big)\big)\, ds \\ & - B^*S^*(T-t)\int_0^t \big(zI + \Gamma_0^T\big)^{-1} S(T-s)f\big(s,x\big(s-\nu(s)\big)\big)\, ds \end{aligned}$$

$$
\begin{aligned}
&- B^*S^*(T-t)\int_0^t \left(zI + \Gamma_0^T\right)^{-1} S(T-s)\sigma(s)\,dB^H(s) \\
&- B^*S^*(T-t)\sum_{0<t_k<t} \left(zI + \Gamma_0^T\right)^{-1} S(T-t_k)I_k\left(x\left(t_k^-\right)\right).
\end{aligned}
$$

Lemma 3.4 *There exists a positive real constant M_C such that, for all $x, y \in R_T$, we have*

$$
E\left\|u^\delta(t,x) - u^\delta(t,y)\right\|^2 \le \frac{M_C}{z^2}\int_0^t E\left\|x(s) - y(s)\right\|^2 ds, \tag{3.3}
$$

$$
E\left\|u^\delta(t,x)\right\|^2 \le \frac{M_C}{z^2}\left(1 + \int_0^t E\left\|x(s)\right\|^2 ds\right). \tag{3.4}
$$

Proof The proof of this lemma similar to the proof of the Lemma 2.5 (see [28]). □

Theorem 3.1 *Assume assumptions* (H1)-(H6) *are satisfied. Then, for all $T > 0$, the system* (1.1) *has a mild solution on* $[-\tau, T]$.

Proof Fix $T > 0$ and let us consider $\Upsilon_T = \{x \in R_T : x(s) = \varphi(s), \text{ for } s \in [-\tau, 0]\}$.

Υ_T is a closed subset of R_T provided with the norm $\|\cdot\|_{R_T}$. For any $\delta > 0$, consider the operator Π_δ on R_T defined as follows:

$$
(\Pi_\delta x)(t) = \begin{cases} \varphi(t), \quad t \in [-\tau, 0], \\ S(t)[\varphi(0) + g(0, \varphi(-r(0)))] - g(t, x(t - r(t))) \\ \quad - \int_0^t AS(t-s)g(s, x(s - r(s)))\,ds \\ \quad + \int_0^t S(t-s)f(s, x(s - \nu(s)))\,ds + \int_0^t S(t-s)Bu^\delta(s,x)\,ds \\ \quad + \int_0^t S(t-s)\sigma(s)\,dB^H(s) + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)), \quad t \in [0, T]. \end{cases} \tag{3.5}
$$

It will be shown that, for all $\delta > 0$, the operator Π_δ has a fixed point. This fixed point is then a solution of equation (1.1). To prove this result, we divide the subsequent proof into two steps.

Step 1: For arbitrary $x \in \Upsilon_T$, let us prove that $t \to \Pi_\delta(x)(t)$ is continuous on the interval $[0, T]$ in the $L^2(\Omega, X)$-sense.

Let $0 < t < t + h < T$, where $t, t + h \in [0, T]\backslash\{t_1, t_2, \ldots, t_m\}$, and let $|h|$ be sufficiently small. Then for any fixed $x \in \Upsilon_T$, it follows from Holder's inequality and the assumptions on the theorem that

$$
\begin{aligned}
&E\left\|(\Pi_\delta x)(t+h) - (\Pi_\delta x)(t)\right\|^2 \\
&\quad \le 8\Big\{E\left\|\left(S(t+h) - S(t)\right)\left(\varphi(0) + g\left(0, \varphi\left(-r(0)\right)\right)\right)\right\|^2 \\
&\qquad + E\left\|g\left(t+h, x\left(t+h-r(t)\right)\right) - g\left(t, x\left(t - r(t)\right)\right)\right\|^2 \\
&\qquad + E\left\|\int_0^t A\left(S(t+h-s) - S(t-s)\right)g\left(s, x\left(s - r(s)\right)\right)ds\right\|^2 \\
&\qquad + E\left\|\int_t^{t+h} (-A)^{1-\beta}S(t+h-s)(-A)^\beta g\left(s, x\left(s - r(s)\right)\right)ds\right\|^2 \\
&\qquad + E\left\|\int_0^t \left(S(t+h-s) - S(t-s)\right)Bu^\delta(s,x)\,ds\right\|^2 + E\left\|\int_t^{t+h} S(t+h-s)Bu^\delta(s,x)\,ds\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ E\left\| \int_0^t \big(S(t+h-s) - S(t-s)\big)\sigma(s)\,dB^H(s)\right\|^2 + E\left\| \int_t^{t+h} S(t+h-s)\sigma(s)\,dB^H(s)\right\|^2 \\
&+ \sum_{0<t_k<t} E\big\| \big(S(t+h-t_k) - S(t-t_k)\big) I_k\big(x\big(t_k^-\big)\big)\big\|^2 \\
&+ \sum_{t<t_k<t+h} E\big\| S(t+h-t_k) I_k\big(x\big(t_k^-\big)\big)\big\|^2 \Big\} \\
\le{}& 8\Big\{ E\big\| \big(S(t+h) - S(t)\big)\big(\varphi(0) + g\big(0,\varphi\big(-r(0)\big)\big)\big)\big\|^2 \\
&+ E\big\| g\big(t+h, x\big(t+h-r(t)\big)\big) - g\big(t, x\big(t-r(t)\big)\big)\big\|^2 \\
&+ t\int_0^t E\big\| A\big(S(t+h-s) - S(t-s)\big) g\big(s, x\big(s-r(s)\big)\big)\big\|^2\,ds \\
&+ \frac{C_4^2 M_{1-\beta}^2}{2\beta-1}(t+h-t)^{2\beta-1}\int_t^{t+h}\big(1 + E\|x\|^2\big)\,ds \\
&+ t\int_0^t E\big\| \big(S(t+h-s) - S(t-s)\big) B u^\delta(s,x)\big\|^2\,ds \\
&+ (t+h-t) M^2 \|B\|^2 \int_t^{t+h} E\big\| u^\delta(s,x)\big\|^2\,ds \\
&+ 2Ht^{2H-1}\int_0^t E\big\| \big(S(t+h-s) - S(t-s)\big)\sigma(s)\big\|_{L_2^0}^2\,ds \\
&+ 2H(t+h-t)^{2H-1} M^2 \int_t^{t+h} E\big\|\sigma(s)\big\|_{L_2^0}^2\,ds \\
&+ \sum_{0<t_k<t} E\big(\big\| \big(S(t+h-t_k) - S(t-t_k)\big)\big\|^2 \big\| I_k\big(x\big(t_k^-\big)\big)\big\|^2\big) + M^2 \sum_{t<t_k<t+h} E\big\| I_k\big(x\big(t_k^-\big)\big)\big\|^2 \Big\}.
\end{aligned}
$$

Hence using the strong continuity of $S(t)$ and Lebesgue's dominated convergence theorem, we conclude that the right-hand side of the above inequalities tends to zero as $h \to 0$. Thus we conclude $\Pi_\delta(x)(t)$ is continuous from the right in $[0,T]$. A similar argument shows that it is also continuous from the left in $(0,T]$. Thus $\Pi_\delta(x)(t)$ is continuous on $[0,T]$ in the L^2-sense.

Step 2: Now, we are going to show that Π_δ is a contraction mapping in Υ_{T_1} with some $T_1 \le T$ to be specified latter. Let $x, y \in \Upsilon_T$, we obtain for any fixed $t \in [0,T]$

$$
\begin{aligned}
&\big\| \Pi_\delta(x)(t) - \Pi_\delta(y)(t)\big\|^2 \\
&\le 5\big\| g\big(t, x\big(t-r(t)\big) - g\big(t, y\big(t-r(t)\big)\big\|^2 + 5\left\| \int_0^t S(t-s) B\big[u^\delta(s,x) - u^\delta(s,y)\big]\,ds\right\|^2 \\
&\quad + 5\left\| \int_0^t (-A)^{1-\beta} S(t-s)\big((-A)^\beta g\big(s, x\big(s-r(s)\big) - (-A)^\beta g\big(s, y\big(s-r(s)\big)\big)\big)\,ds\right\|^2 \\
&\quad + 5\left\| \int_0^t S(t-s)\big(f\big(s, x\big(s-\nu(s)\big)\big) - f\big(s, y\big(s-\nu(s)\big)\big)\big)\,ds\right\|^2 \\
&\quad + 5\left\| \sum_{0<t_k<t} S(t-t_k)\big(I_k\big(x\big(t_k^-\big)\big) - I_k\big(y\big(t_k^-\big)\big)\big)\right\|^2.
\end{aligned}
$$

By the Lipschitz property of $(-A)^{\beta}g$ and f combined with Holder's inequality, we obtain

$$\begin{aligned}
E\|\Pi_\delta(x)(t)-\Pi_\delta(y)(t)\|^2 \le{}& 5E\|x(t-r(t))-y(t-r(t))\|^2 \\
&+\frac{5M^2\|B\|^2M_C}{z^2}\int_0^t E\|x(s)-y(s)\|^2\,ds \\
&+5C_3^2M_{1-\beta}^2\frac{T^{2\beta-1}}{2\beta-1}\int_0^t E\|x(s-r(s))-y(s-r(s))\|^2\,ds \\
&+5TC_1^2M^2\int_0^t E\|x(s-\nu(s))-y(s-\nu(s))\|^2\,ds \\
&+5m^2M^2C_5^2E\|x(t)-y(t)\|^2.
\end{aligned}$$

Hence

$$\sup_{t\in[-\tau,T]} E\|\Pi_\delta(x)(t)-\Pi_\delta(y)(t)\|^2 \le \gamma(T)\sup_{t\in[-\tau,T]} E\|x(t)-y(t)\|^2,$$

where

$$\gamma(T)=5\left[1+\frac{M^2\|B\|^2M_C}{z^2}T+\frac{C_3^2M_{1-\beta}^2}{2\beta-1}T^{2\beta}+M^2C_1^2T^2+m^2M^2C_5^2\right].$$

Then there exists $0<T_1\le T$ such that $0<\gamma(T_1)<1$ and Π_δ is a contraction mapping on S_{T_1} and therefore has a unique fixed point, which is a mild solution of equation (1.1) on $[-\tau,T_1]$. This procedure can be repeated in order to extend the solution to the entire interval $[-\tau,T]$ in finitely many steps. This completes the proof. □

Theorem 3.2 *Assume that* (H1)-(H6) *are satisfied. Further, if the functions f and g are uniformly bounded, and $S(t)$ is compact, then the system* (1.1) *is approximately controllable on* $[0,T]$.

Proof Let x_δ be a fixed point of Π_δ. By using the stochastic Fubini theorem, it can easily be seen that

$$\begin{aligned}
x_\delta(T)={}&\bar{x}_T-z(zI+\Gamma_0^T)^{-1}\Big\{E\bar{x}_T-S(T)[\varphi(0)-g(0,\varphi(-r(0)))] \\
&+g(T,x_\delta(T))+\int_0^T\bar{\varphi}(s)\,dB^H(s)\Big\} \\
&+z\int_0^T(zI+\Gamma_0^T)^{-1}AS(T-s)g(s,x_\delta(s-r(s)))\,ds \\
&+z\int_0^T(zI+\Gamma_0^T)^{-1}S(T-s)f(s,x_\delta(s-\nu(s)))\,ds \\
&+z\int_0^T(zI+\Gamma_0^T)^{-1}S(T-s)\sigma(s)\,dB^H(s) \\
&+\sum_{0<t_k<T}z(zI+\Gamma_0^T)^{-1}S(T-t_k)I_k(x_\delta(t_k^-)).
\end{aligned}$$

It follows from the assumption on f and g that there exists $\bar{D} > 0$ such that

$$\left\| f\big(s, x_\delta\big(s - \nu(s)\big)\big)\right\|^2 + \left\| g\big(s, x_\delta\big(s - r(s)\big)\big)\right\|^2 \leq \bar{D} \tag{3.6}$$

for all $(s, \omega) \in [0, T] \times \Omega$. Then there is a subsequence still denoted by $\{f(s, x_\delta(s - \nu(s))), g(s, x_\delta(s - r(s)))\}$ which converges weakly to, say, $\{f(s), g(s)\}$ in $X \times L_2^0$.

From the above equation, we have

$$\begin{aligned}
&E\left\|x_\delta(T) - \bar{x}_T\right\|^2 \\
&\quad\leq 7E\left\| z\big(zI + \Gamma_0^T\big)^{-1}\Big\{\bar{x}_T - S(T)\big[\varphi(0) - g\big(0, \varphi\big(-r(0)\big)\big)\big] \right. \\
&\qquad\left. + g\big(T, x_\delta(T)\big) + \int_0^T \bar{\varphi}(s)\, dB^H(s)\Big\}\right\|^2 \\
&\qquad + 7E\left(\int_0^T \left\| z\big(zI + \Gamma_0^T\big)^{-1}\right\| \left\| AS(T-s)\big[g\big(s, x_\delta\big(s - r(s)\big)\big) - g(s)\big]\right\| ds\right)^2 \\
&\qquad + 7E\left(\int_0^T \left\| z\big(zI + \Gamma_0^T\big)^{-1} AS(T-s) g(s)\right\| ds\right)^2 \\
&\qquad + 7E\left(\int_0^T \left\| z\big(zI + \Gamma_0^T\big)^{-1}\right\| \left\| S(T-s) f\big(s, x_\delta\big(s - \nu(s)\big)\big) - f(s)\right\| ds\right)^2 \\
&\qquad + 7E\left(\int_0^T \left\| z\big(zI + \Gamma_0^T\big)^{-1} S(T-s) f(s)\right\| ds\right)^2 \\
&\qquad + 14HT^{2H-1}\int_0^T \left\| z\big(zI + \Gamma_0^T\big)^{-1} S(T-s)\sigma(s)\right\|_{L_2^0}^2 ds \\
&\qquad + 7E\left\| \sum_{0<t_k<T} z\big(zI + \Gamma_0^T\big)^{-1} S(T - t_k) I_k\big(x_\delta\big(t_k^-\big)\big)\right\|^2.
\end{aligned}$$

On the other hand, by Lemma 3.2, the operator $z(zI + \Gamma_0^T)^{-1} \to 0$ strongly as $z \to 0^+$ for all $0 \leq s \leq T$, and, moreover, $\|z(zI + \Gamma_0^T)^{-1}\| \leq 1$. Thus, by the Lebesgue dominated convergence theorem the compactness of $S(t)$ implies that $E\|x_\delta(T) - \bar{x}_T\|^2 \to 0$ as $z \to 0^+$. This gives the approximate controllability of (1.1). □

4 Example

In this section, we present an example to illustrate our main result.

Let us consider the following stochastic control partial neutral functional differential equation with finite variable delays driven by a fractional Brownian motion:

$$\begin{cases}
d[x(t,\xi) - G(t, x(t - r(t)), \xi)] = [\frac{\partial^2 x(t,\xi)}{\partial \xi^2} + F(t, x(t - \nu(t)), \xi) + \mu(t,\xi)]\, dt \\
\quad + \sigma(t)\, dB^H(t), \quad 0 \leq \xi \leq \pi, 0 \leq t \leq T, t \neq t_k, \\
x(t, 0) = x(t, \pi) = 0, \quad t \geq 0, \\
x(t_k^+, \xi) - x(t_k^-, \xi) = I_k(x(t_k^-, \xi)), \quad k = 1, 2, \ldots, m, \\
x(t, \xi) = \varphi(t, \xi), \quad t \in [-\tau, 0], 0 \leq \xi \leq \pi,
\end{cases} \tag{4.1}$$

where B^H is a fractional Brownian motion and $F, G : R^+ \times R \to R$ are continuous functions.

To study this system, let $X = Y = U = L^2([0,\pi], R)$ and let the operator $A : D(A) \subset X \to X$ be given by $Ay = y''$ with

$$D(A) = \{y \in X : y'' \in X, y(0) = y(\pi) = 0\}.$$

It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ on X. Furthermore, A has discrete spectrum with eigenvalues $-n^2$, $n \in N$ and the corresponding normalized eigenfunctions are given by

$$e_n = \sqrt{\frac{2}{\pi}} \sin nx, \quad n = 1, 2, \ldots.$$

In addition $(e_n)_{n\in N}$ is a complete orthonormal basis in X and

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, e_n \rangle_{e_n}$$

for $x \in X$ and $t \geq 0$. It follows from this representation that $T(t)$ is compact for every $t > 0$ and that $\|T(t)\| \leq e^{-t}$ for every $t \geq 0$.

In order to define the operator $Q : Y \to R$, we choose a sequence $\{\lambda_n\}_{n\in N} \subset R^+$, set $Qe_n = \lambda_n e_n$, and assume that

$$\mathrm{tr}(Q) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} < \infty.$$

Define the fractional Brownian motion in Y by

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta^H(t) e_n,$$

where $H \in (\frac{1}{2}, 1)$ and $\{\beta_n^H\}_{n\in N}$ is a sequence of one-dimensional fractional Brownian motions mutually independent.

Define $x(t)(\cdot) = x(t,\cdot)$, $f(t,x)(\cdot) = F(t,x(\cdot))$, and $g(t,x)(\cdot) = G(t,x(\cdot))$. Define the bounded operator $B : U \to X$ by $Bu(t)(\xi) = \mu(t,\xi)$, $0 \leq \xi \leq \pi$, $u \in U$. Therefore, with the above choice, the system (4.1) can be written into the abstract form (1.1) and all conditions of Theorem 3.2 are satisfied. Thus by Theorem 3.2, the stochastic partial neutral functional differential equation with finite variable delays driven by a fractional Brownian motion is approximately controllable on $[0,\pi]$.

Competing interests
The author declares that he has no competing interests.

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