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Remarks on monotone multivalued mappings on a metric space with a graph

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Abstract

Let (X,d) be a metric space and $J:X\to 2^X$ be a multivalued mapping. In this work, we discuss the definition of G-contraction mappings introduced by Beg *et al.* (Comp. Math. Appl. 60:1214-1219, 2010) and show that it is restrictive and fails to give the main result of (Beg *et al.* in Comp. Math. Appl. 60:1214-1219, 2010). In this work, we give a new definition of the G-contraction and obtain sufficient conditions for the existence of fixed points for such mappings.

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Keywords: directed graph; connected/weakly connected graph; fixed point; metric space; monotone multivalued contraction mapping; Pompeiu-Hausdorff distance

1 Introduction

Fixed point theorems for monotone single-valued mappings in a metric space endowed with a partial ordering have been widely investigated. These theorems are hybrids of the two most fundamental and useful theorems in fixed point theory: the Banach contraction principle [1], Theorem 2.1, and Tarski's fixed point theorem [2, 3]. Generalizing the Banach contraction principle for multivalued mapping to metric spaces, Nadler [4] obtained the following result.

Theorem 1.1 ([4]) *Let (X,d) be a complete metric space. Denote by $\mathcal{CB}(X)$ the set of all nonempty closed bounded subsets of X. Let $F:X\to\mathcal{CB}(X)$ be a multivalued mapping. If there exists $k\in[0,1)$ such that*

$$H\big(F(x),F(y)\big)\leq kd(x,y)$$

for all $x,y\in X$, where H is the Pompeiu-Hausdorff metric on $CB(X)$, then F has a fixed point in X.

A number of extensions and generalizations of Nadler's theorem were obtained by different authors; see for instance [5, 6] and references cited therein. The Tarski theorem was extended to multivalued mappings by different authors; see [7–9]. The existence of fixed points for single-valued mappings in partially ordered metric spaces was initially considered by Ran and Reurings in [10], who proved the following result.

Theorem 1.2 ([10]) *Let $(X,\preceq)$ be a partially ordered set such that every pair $x,y \in X$ has an upper and lower bound. Let d be a metric on X such that (X,d) is a complete metric space. Let $f : X \to X$ be a continuous monotone (either order preserving or order reversing) mapping. Suppose that the following conditions hold:*

1. *There exists $k \in [0,1)$ with*

$$d\big(f(x),f(y)\big) \leq kd(x,y), \quad \text{for all } x \succeq y.$$

2. *There exists an $x_0 \in X$ with $x_0 \preceq f(x_0)$ or $x_0 \succeq f(x_0)$.*

Then f is a Picard Operator (PO), that is, f has a unique fixed point $x^* \in X$ and for each $x \in X$, $\lim_{n\to\infty} f^n(x) = x^*$.

After this, various authors considered the problem of existence of a fixed point for contraction mappings in partially ordered metric spaces; see [11–14] and references cited therein. Nieto *et al.* in [14] extended the ideas of [10] to prove the existence of solutions to some differential equations. Recently, two results have appeared, giving sufficient conditions for f to be a PO, if (X,d) is endowed with a graph. The first result in this direction was given by Jachymski and Lukawska [15, 16], who generalized the results of [12, 14, 17, 18] to single-valued mapping in metric spaces with a graph instead of partial ordering.

The aim of this paper is twofold: first to give a correct definition of monotone multivalued mappings, second to extend the conclusion of Theorem 1.2 to the case of monotone multivalued mappings in metric spaces endowed with a graph.

2 Preliminaries

It seems that the terminology of graph theory instead of partial ordering gives a clearer picture and yields an interesting generalization of the Banach contraction principle. Let us begin this section with such a terminology for metric spaces as will be used throughout.

Let G be a directed graph (digraph) with the set of vertices $V(G)$ and the set of edges $E(G)$ contains all the loops, *i.e.* $(x,x) \in E(G)$ for any $x \in V(G)$. We also assume that G has no parallel edges (arcs) and so we can identify G with the pair $(V(G),E(G))$. Our graph theory notations and terminology are standard and can be found in all graph theory books, like [19] and [20]. Moreover, we may treat G as a weighted graph (see [20], p.309]) by assigning to each edge the distance between its vertices. By G^{-1} we denote the conversion of a graph G, *i.e.*, the graph obtained from G by reversing the direction of edges. Thus we have

$$E\big(G^{-1}\big) = \big\{(y,x) | (x,y) \in E(G)\big\}.$$

A digraph G is called an oriented graph if whenever $(u,v) \in E(G)$, then $(v,u) \notin E(G)$. The letter $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\widetilde{G}) = E(G) \cup E\big(G^{-1}\big).$$

We call (V',E') a subgraph of G if $V' \subseteq V(G)$, $E' \subseteq E(G)$, and for any edge $(x,y) \in E'$, $x,y \in V'$.

If x and y are vertices in a graph G, then a (directed) path in G from x to y of length N is a sequence $(x_i)_{i=1}^{i=N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$, and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, \ldots, N$. A graph G is connected if there is a directed path between any two vertices. G is weakly connected if $\widetilde{G}$ is connected. If G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the relation $\mathcal{R}$ defined on $V(G)$ by the rule

$y \mathcal{R} z$ if there is a (directed) path in G from y to z.

Clearly G_x is connected.

Definition 2.1 ([21]) Let (X, d) be a metric space and $\mathcal{CB}(X)$ be the class of all nonempty closed and bounded subsets of X. The Pompeiu-Hausdorff distance [21] on $\mathcal{CB}(X)$ is defined by

$$H(U, W) := \max\left\{\sup_{w\in W} d(w, A), \sup_{u\in U} d(u, W)\right\},$$

for $U, W \in \mathcal{CB}(X)$, where $d(u, W) := \inf_{w\in W} d(u, w)$. The mapping H is said to be a Pompeiu-Hausdorff metric induced by d.

Definition 2.2 ([4]) Let (X, d) be a metric space and $\mathcal{CB}(X)$ be the class of all nonempty closed and bounded subsets of X. A multivalued map $J : X \to \mathcal{CB}(X)$ is called contractive if there exists $k \in [0, 1)$ such that

$$H\bigl(J(x), J(y)\bigr) \le kd(x, y),$$

for all $x, y \in X$.

Example 2.1 Let $I = [0, 1]$ denote the unit interval of real numbers (with the usual metric) and let $f : I \to I$ be given by

$$f(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2}, & 0 \le x \le \frac{1}{2}, \\ -\frac{1}{2}x + \frac{1}{2}, & \frac{1}{2} \le x \le 1. \end{cases}$$

Define $F : I \to 2^I$ by $F(x) = \{0\} \cup \{f(x)\}$ for each $x \in I$. It is easy to verify that F is a multivalued contraction mapping with set of fixed points $\{0, \frac{2}{3}\}$.

Example 2.2 Let $I^2 = \{(x, y) : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$, and let $F : I^2 \to \mathcal{CB}(I^2)$ be defined by $F(x, y)$ is the line segment in I^2 from the point $(\frac{1}{2}x, 0)$ to the point $(\frac{1}{2}x, 1)$ for each $(x, y) \in I^2$. It is easy to see that F is a multivalued contraction mapping with the set of fixed points $\{(0, y) : 0 \le y \le 1\}$.

Next we introduce the concept of monotone multivalued mappings. In [22], the authors offered the following definition.

Definition 2.3 ([22], Def. 2.6) Let $F : X \rightsquigarrow X$ be a set valued mapping with nonempty closed and bounded values. The mapping F is said to be a G-contraction if there exists $k \in [0,1)$ such that

$$H\bigl(F(x), F(y)\bigr) \le kd(x,y), \quad \text{for all } (x,y) \in E(G)$$

and such that if $u \in F(x)$ and $v \in F(y)$ are such that

$$d(u,v) \le kd(x,y) + \alpha, \quad \text{for each } \alpha > 0,$$

then $(u,v) \in E(G)$.

In particular, this definition implies that if $u \in F(x)$ and $v \in F(y)$ are such that

$$d(u,v) \le kd(x,y),$$

then $(u,v) \in E(G)$, which is very restrictive. In fact, in the proof of Theorem 3.1 in [22], there is absolutely no reason for $(x_1, x_2) \in E(G)$. Definition 2.4 of G-contraction multivalued mappings, inspired by the definition of contraction multivalued mappings in [23, 24], is more appropriate. In the sequel, we assume that (X,d) is a metric space, and G is a directed graph (digraph) with the set of vertices $V(G) = X$ and the set of edges $E(G)$ contains all the loops, *i.e.* $(x,x) \in E(G)$, for any $x \in X$.

Definition 2.4 ([23, 24]) A multivalued mapping $T : X \to 2^X$ is said to be monotone increasing G-contraction if there exists $\alpha \in [0,1)$ such that for any $u, w \in X$ with $(u,w) \in E(G)$ and any $U \in T(u)$ there exists $W \in T(w)$ such that

$$(U, W) \in E(G) \quad \text{and} \quad d(U, W) \le \alpha d(u,v).$$

Property 1 For any sequence $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$.

3 Main results

We begin with the following theorem, which gives the existence of a fixed point for monotone multivalued mappings in metric spaces endowed with a graph.

Theorem 3.1 *Let (X,d) be a complete metric space and suppose that the triple (X,d,G) has property* 1. *Let $T : X \to \mathcal{C}B(X)$ be a monotone increasing G-contraction mapping and $X_T := \{x \in X; (x,u) \in E(G) \text{ for some } u \in T(x)\}$. If $X_T \neq \emptyset$, then the following statements hold*:

(1) *For any $x \in X_T$, $T|_{[x]_{\widetilde{G}}}$ has a fixed point.*
(2) *If G is weakly connected, then T has a fixed point in G.*
(3) *If $X' := \bigcup\{[x]_{\widetilde{G}} : x \in X_T\}$, then $T|_{X'}$ has a fixed point in X.*
(4) *If $T(X) \subseteq E(G)$ then T has a fixed point.*
(5) Fix $T \neq \emptyset$ *if and only if* $X_T \neq \emptyset$.

Proof 1. Let $x_0 \in X_T$, then there exists $x_1 \in T(x_0)$ such that $(x_0, x_1) \in E(G)$. Since T is monotone increasing G-contraction, there exists $x_2 \in T(x_1)$, $(x_1, x_2) \in E(G)$, such that

$$d(x_1, x_2) \le \alpha d(x_0, x_1),$$

where $\alpha < 1$ is associated to the definition of T being monotone increasing G-contraction. Without loss of generality, we may assume $\alpha > 0$. By induction, we construct a sequence $\{x_n\}$ such that $x_{n+1} \in T(x_n)$, $(x_n, x_{n+1}) \in E(G)$, and

$$d(x_n, x_{n+1}) \le \alpha d(x_n, x_{n-1}) \le \alpha^n d(x_0, x_1),$$

for any $n \ge 1$. Since $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \le d(x_0, x_1) \sum_{n=0}^{\infty} \alpha^n < \infty$, we conclude that $\{x_n\}$ is a Cauchy sequence, and hence converges to some $x \in X$ since X is a complete metric space. We claim that $x \in T(x)$, *i.e.* x is a fixed point of T. Indeed using the definition of G-contraction of T, there exists $y_n \in T(x)$ such that $(x_{n+1}, y_n) \in E(G)$ and

$$d(x_{n+1}, y_n) \le \alpha d(x_n, x),$$

for any $n \ge 1$. Hence

$$d(y_n, x) \le d(y_n, x_{n+1}) + d(x_{n+1}, x) \le \alpha d(x_n, x) + d(x_{n+1}, x),$$

for any $n \ge 1$. This implies that $\{y_n\}$ converges to x. Since $T(x)$ is closed, we get $x \in T(x)$ as claimed. As $(x_n, x) \in E(G)$, for every $n \ge 0$, we conclude that $(x_0, x_1, \dots, x_n, x)$ is a path in G and so $x \in [x_0]_{\widetilde{G}}$.

2. Since $X_T \neq \emptyset$, there exists an $x_0 \in X_T$, and since G is weakly connected, then $[x_0]_{\widetilde{G}} = X$ and by 1, mapping T has a fixed point.

3. It follows easily from 1 and 2.

4. $T(X) \subseteq E(G)$ implies that all $x \in X$ are such that there exists some $y \in T(x)$ with $(x, y) \in E(G)$; so $X_T = X$ and by 2 and 3, T has a fixed point.

5. Assume Fix $T \neq \emptyset$. This implies that there exists an $x \in \operatorname{Fix} T$ such that $x \in T(x)$. $\triangle \subseteq E(G)$ therefore $(x, x) \in E(G)$, which implies that $x \in X_T$. So $X_T \neq \emptyset$. Conversely if $X_T \neq \emptyset$, then Fix $T \neq \emptyset$, follows from 2 and 3. □

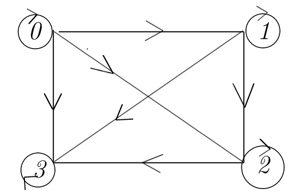
Remark 3.1 The missing information in Theorem 3.1 is the uniqueness of the fixed point. In fact, we do have a partial positive answer to this question. Indeed if $\bar{u}$ and $\bar{w}$ are two fixed points of T such that $(\bar{u}, \bar{w}) \in E(G)$, then we must have $\bar{u} = \bar{w}$. In general T may have more than one fixed point.

Remark 3.2 If we assume G is such that $E(G) := X \times X$ then clearly G is connected and our Theorem 3.1 gives Nadler's theorem [4].

The following is a direct consequence of Theorem 3.1.

Corollary 3.1 *Let (X, d) be a complete metric space and the triple (X, d, G) have the Property* 1. *If G is weakly connected then every G-contraction $T : X \to \mathcal{CB}(X)$ such that $(x_0, x_1) \in E(G)$, for some $x_1 \in T(x_0)$, has a fixed point.*

Figure 1 ***G*: Pompeiu-Hausdorff weighted graph.**



Example 3.1 Let $X = \{0,1,2,3,4\} = V(G)$ and

$$E(G) = \{(0,0),(1,1),(2,2),(3,3),(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)\}.$$

Let $V(G)$ be endowed with metric $d : X \times X \to \mathbb{R}^+$ defined by

$$d(0,0) = d(1,1) = d(2,2) = d(3,3) = 0,$$
$$d(0,1) = d(1,0) = \frac{1}{4},$$
$$d(0,2) = d(2,0) = d(1,2) = d(2,1) = d(1,3) = \cdots = d(3,2) = \frac{4}{5}.$$

The graph of G is shown in Figure 1.

The Pompeiu-Hausdorff weights assigned to $U, W \in CB(X)$ are

$$H(U,W) = \begin{cases} \frac{1}{4} & \text{if } U, W \subseteq \{0,1\} \text{ with } U \neq W, \\ \frac{4}{5} & \text{if } U \text{ or } W \text{ (or both)} \nsubseteq \{0,1\} \text{ with } U \neq W, \\ 0 & \text{if } U = W. \end{cases}$$

Define $T : X \to \mathcal{CB}(X)$ as follows:

$$T(x) = \begin{cases} \{0\} & \text{if } x \in \{0,1\}, \\ \{1\} & \text{if } x \in \{2,3\}. \end{cases}$$

Note that, for all $x, y \in X$ with edge between x and y, there is an edge between $T(x)$ and $T(y)$. Also there is a path between x and y implies that there is a path between $T(x)$ and $T(y)$. Moreover, T is a G-contraction with all other assumptions of Theorem 3.1 satisfied and T has 0 as a fixed point.

Competing interests
The author declares that he has no competing interests.

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