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Some common fixed-point and invariant approximation results with generalized almost contractions

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Abstract

In this paper, the concept of a generalized almost (f,g)-contraction is introduced and we establish some common fixed-point results for the noncommuting generalized almost (f,g)-contraction in the setup of metric spaces and normed linear spaces, where the set of fixed points of f and g need not be starshaped. As applications, invariant approximation results are proved. Supporting examples are also given.

Keywords: best approximation; Banach operator pair; generalized almost contraction; property (N); jointly continuous contractive family

1 Introduction

The classical Banach contraction principle is a very popular tool for solving problems in nonlinear analysis. It has various applications to operator theory, variational analysis, and approximation theory, so it has been extended in many ways (see, *e.g.*, [1–30]).

In 2004, Berinde [1] defined the notion of a weak contraction mapping, which is more general than a contraction mapping. However, in [2] Berinde renamed it as an almost contraction, which is more appropriate.

Definition 1.1 Let (X,d) be a complete metric space. A map $T : X \to X$ is called an almost contraction if there exist a constant $\delta \in (0,1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x,y) + L d(y, Tx) \quad \text{for all } x, y \in X. \tag{1.1}$$

Berinde [1] proved some fixed-point theorems for almost contractions in a complete metric space which generalized the results of Kannan [3], Chatterjea [4], and Zamfirescu [5].

In 2008, Babu *et al.* [6] defined the class of mappings satisfying 'condition (B)' as follows.

Definition 1.2 Let (X,d) be a metric space. A map $T : X \to X$ is said to satisfy 'condition (B)' if there exist a constant $\delta \in (0,1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x,y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \tag{1.2}$$

for all $x, y \in X$.

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They prove that any map T satisfying 'condition (B)' has a unique fixed point in complete metric spaces. They also discuss quasi-contraction, almost contraction, and the class of mappings that satisfy 'condition (B)' in detail.

Afterwards Berinde [7] generalized the above definition and proved the following fixed-point result.

Theorem 1.3 *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping for which there exist $\delta \in (0, 1)$ and some $L \geq 0$ such that for all $x, y \in X$*

$$d(Tx, Ty) \leq \delta M_1(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \tag{1.3}$$

where

$$M_1(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}.$$

Then T has a unique fixed point.

The contractive condition (1.3) is termed as generalized almost contraction.

Recently, Abbas and Ilić in [15] introduced the following definition.

Definition 1.4 Let T and f be two self-maps of a metric space (X, d). A map T is called a generalized almost f-contraction if there exist $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta M_1(x, y) + L \min\{d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}, \tag{1.4}$$

where

$$M_1(x, y) = \max\left\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{2}[d(fx, Ty) + d(fy, Tx)]\right\}.$$

If f = identity map, then condition (1.3) can be obtained as particular case of condition (1.4). However, in [15] Abbas and Ilić obtained various common fixed-point and invariant approximation results for such mappings under the assumption of weak compatibility of maps.

Recently, Chen and Li [10] introduced the class of Banach operator pairs, as a new class of noncommuting mappings and obtained some common fixed-point and invariant approximation results for this class of maps. This class of noncommuting maps is different from the class of noncommuting maps (*viz. R*-subcommuting, R-sub-weakly commuting, C_q-commuting, compatible, weakly compatible *etc.*) studied in [11–13, 15, 17–19, 27–29]. So, it has been further studied by various authors (see, *e.g.*, [16, 21, 22, 24]).

In this article, we introduce the class of generalized almost (f, g)-contraction and consequently establish some common fixed-point results for the noncommuting generalized almost (f, g)-contraction in the framework of metric spaces and normed linear spaces, where the set of fixed points of f and g need not be starshaped. As an application, invariant approximation results are proved. The proved results generalize and extend the corresponding results of Chen and Li [10], Al-Thagafi and Shahzad [16], Akbar *et al.* [22], Chandok and Narang [24], Al-Thagafi [25] and Jungck and Sessa [26], Shahzad [28] to the class of generalized almost (f, g)-contractions.

2 Preliminaries

First, we introduce some well-known notations and definitions that will be needed in the sequel.

Let (X, d) be a metric space, M be a subset of X and f, T be self-maps of M. A point $x \in M$ is a coincidence point (common fixed point) of f and T if $fx = Tx$ ($fx = Tx = x$). The set of coincidence points of f and T is denoted by $C(f, T)$ and the set of fixed points of f is denoted by $F(f)$. The pair $\{f, T\}$ is called

(1) commuting if $Tfx = fTx$ for all $x \in M$,
(2) compatible [8] if $\lim_{n \to \infty} d(Tfx_n, fTx_n) = 0$ whenever $\{x_n\}$ is a sequence in M such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} Tx_n = t$ for some $t \in M$,
(3) weakly compatible [9] if $Tfx = fTx$ for all $x \in C(f, T)$,
(4) a Banach operator pair [10] if the set $F(f)$ is T-invariant, namely $T(F(f)) \subseteq F(f)$.

Obviously, a commuting pair (T, f) is a Banach operator pair but not conversely. If (T, f) is a Banach operator pair, then (f, T) need not be Banach operator pair (see [10]).

Let M be a subset of a normed space $(X, \| \cdot \|)$. The set $B_M(p) = \{x \in M : \|x - p\| = \mathrm{dist}(p, M)\}$ is called the set of best approximants to $p \in X$ out of M, where $\mathrm{dist}(p, M) = \inf\{\|y - p\| : p \in M\}$. We denote by $\mathbb{N}$ and $\mathrm{cl}(M)$ ($\mathrm{wcl}(M)$) the set of positive integers and the closure (weak closure) of a set M in X, respectively.

The set M is said to be (a) q-starshaped if there exists $q \in M$ such that the line segment $[q, x] = \{(1 - k)q + kx : 0 \le k \le 1\}$ joining q to x is contained in M for all $x \in M$; (b) convex if $kx + (1 - k)y \in M$ for all $x, y \in M$. The map f defined on a set M is called

(1) *affine* [11] if M is convex and $f((1 - k)y + kx) = (1 - k)fy + kfx$, for all $x, y \in M$,
(2) *q-affine* [11] if M is q-starshaped and $f((1 - k)q + kx) = (1 - k)q + kfx$, for all $x \in M$.

Suppose that M is q-starshaped with $q \in F(f)$ and is both T- and f-invariant. Then T and f are called

(1) C_q-commuting [11] if $fTx = Tfx$ for all $x \in C_q(f, T)$, where
$C_q(f, T) = \bigcup\{C(f, T_k) : 0 \le k \le 1\}$ where $T_k(x) = (1 - k)q + kTx$,
(2) R-subcommuting on M [12] if, for all $x \in M$, there exists a real number $R > 0$ such that $\|Tfx - fTx\| \le \frac{R}{k}\|kTx + (1 - k)q - fx\|$, $0 < k \le 1$,
(3) R-sub-weakly commuting on M [13] if, for all $x \in M$, there exists a real number $R > 0$ such that $\|Tfx - fTx\| \le R\, \mathrm{dist}(fx, [q, Tx])$.

A Banach space X is said to satisfy Opial's condition if, whenever $\{x_n\}$ is a sequence in X such that $\{x_n\}$ converges weakly to $x \in X$, the inequality

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|$$

holds for all $y \ne x$. A Hilbert space and the space l_p ($1 < p < \infty$) satisfy Opial's condition. The map $T : M \to X$ is said to be demiclosed at zero if, whenever $\{x_n\}$ is a sequence in M such that $\{x_n\}$ converges weakly to $x \in M$ and $\{Tx_n\}$ converges to 0, then $Tx = 0$.

The following important extension of the concept of starshapedness was defined by Naimpally *et al.* [14] and has been studied by many authors.

Definition 2.1 A subset M of a linear space X is said to have property (N) with respect to T if

(1) $T : M \to M$,
(2) $(1 - k_n)q + k_n Tx \in M$, for some $q \in M$ and a fixed sequence of real numbers k_n ($0 < k_n < 1$) converging to 1 and for each $x \in M$.

It is to be noted that each T-invariant q-starshaped set has property (N) but converse does not hold in general. This is shown by the following example.

Example 2.2 Let $X = \Re$ be the set of real numbers and $M = \{1/n,\text{where } n \text{ is a natural number}\}$ be endowed with the usual norm. Define $Tx = 1$ for each $x \in M$. Then clearly M is not q-starshaped but has property (N) with respect to T, for $q = 1$, $k_n = 1 - 1/n$.

3 Main results

First we introduce the notion of a generalized almost (f,g)-contraction.

Definition 3.1 Let (X,d) be a metric space and f, g be self-maps of X. A mapping $T : X \to X$ is said to be a generalized almost (f,g)-contraction if there exist $\delta \in (0,1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta M_1(x,y) + LN_1(x,y) \quad \text{for all } x, y \in X, \tag{3.1}$$

where

$$M_1(x,y) = \max\left\{d(fx,gy), d(fx,Tx), d(gy,Ty), \frac{1}{2}\big[d(fx,Ty) + d(gy,Tx)\big]\right\}$$

and

$$N_1(x,y) = \min\big\{d(fx,Tx), d(gy,Ty), d(fx,Ty), d(gy,Tx)\big\}.$$

If $g = f$, then Definition 1.4 is a particular case of Definition 3.1. If $g = f = I$ (identity operator), then equation (1.3) can be obtained as a special case of equation (3.1).

Here we observe that if T satisfies 'condition (B)' then T is a generalized almost contraction but its converse need not be true. This is shown by the following example.

Example 3.2 Let $X = [0,\infty)$ be endowed with the Euclidean metric $d(x,y) = |x - y|$. We define a mapping $T : X \to X$ by

$$T(x) = \begin{cases} \frac{3}{4} & \text{if } 0 \leq x \leq 1, \\ \frac{1}{2} & \text{if } 0 \leq x < \infty. \end{cases}$$

Then T is a generalized almost contraction with $\delta = \frac{2}{3}$ and $L = 0$. But T does not satisfy 'condition (B)' at $x = \frac{3}{4}$, $y = 1$ for any $\delta \in (0,1)$ and $L \geq 0$.

In (3.1) if $L = 0$, then T is called a generalized (f,g)-contraction. Obviously, a generalized (f,g)-contraction implies a generalized almost (f,g)-contraction, but the converse is not true in general.

Example 3.3 Let $X = \{0,1,2\}$ with the usual metric and $f, g : X \to X$ be given by $f(x) = g(x) = 1$ for all $x \in X$. Also define a mapping $T : X \to X$ as

$$T(x) = \begin{cases} 0, & x \in \{0,2\}, \\ 2, & x = 1. \end{cases}$$

Then T is a generalized almost (f, g)-contraction with any $\delta \in (0, 1)$ and $L \geq 2$. But T is not a generalized (f, g)-contraction at $x = 0$, $y = 1$ or $x = 1$, $y = 2$ for any $\delta \in (0, 1)$.

The following lemma is a particular case of the main theorem of Abbas and Ilić [15].

Lemma 3.4 *Let M be a nonempty subset of a metric space (X, d), and T be a self-map of M. Assume that $\mathrm{cl}(T(M)) \subseteq M$, $\mathrm{cl}(T(M))$ is complete, and T is a generalized almost contraction. Then $M \cap F(T)$ is singleton.*

Now, we start with the following common fixed-point result, which will be used in sequel.

Theorem 3.5 *Let M be a nonempty subset of a metric space (X, d), and T, f and g be self-maps of M. Assume that $F(f) \cap F(g)$ is nonempty, $\mathrm{cl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$, $\mathrm{cl}(T(M))$ is complete, and T is a generalized almost (f, g)-contraction. Then $M \cap F(T) \cap F(f) \cap F(g)$ is singleton.*

Proof The completeness of $\mathrm{cl}(T(M))$ implies that of $\mathrm{cl}(T(F(f) \cap F(g)))$. Further, by a generalized almost (f, g)-contraction of T, for all $x, y \in F(f) \cap F(g)$, we have

$$d(Tx, Ty) \leq \delta M_1(x, y) + L N_1(x, y)$$

$$= \delta \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2} \big[d(x, Ty) + d(y, Tx) \big] \right\}$$

$$+ L \min \big\{ d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx) \big\}.$$

Hence T is a generalized almost contraction mapping on $F(f) \cap F(g)$ and $\mathrm{cl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$. By Lemma 3.4, T has a unique fixed point z in $F(f) \cap F(g)$ and consequently $M \cap F(T) \cap F(f) \cap F(g)$ is singleton. □

Corollary 3.6 *Let M be a nonempty subset of a metric space (X, d), and T, f and g be self-maps of M such that (T, f) and (T, g) are Banach operator pairs on M. Assume that $\mathrm{cl}(T(M))$ is complete, T is a generalized almost (f, g)-contraction and $F(f) \cap F(g)$ is nonempty and closed. Then $M \cap F(T) \cap F(f) \cap F(g)$ is singleton.*

In Theorem 3.5 if we take $L = 0$, then we easily obtain the following result, which improves and extends Lemma 3.1 of Chen and Li [10] and Theorem 2.2 of Al-Thagafi and Shahzad [16].

Corollary 3.7 *Let M be a nonempty subset of a metric space (X, d), and T, f, and g be self-maps on M. Assume that $F(f) \cap F(g)$ is nonempty, $\mathrm{cl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$, $\mathrm{cl}(T(M))$ is complete, and T is a generalized (f, g)-contraction. Then $M \cap F(T) \cap F(f) \cap F(g)$ is singleton.*

Remark 3.8 By comparing Theorem 2.1 of Shahzad [17] with Corollary 3.7 (when $g = f$), their assumptions that M is closed, $T(M) \subseteq f(M)$, T is continuous and (T, f) is R-weakly commuting pair on M are replaced with '$F(f)$ is nonempty, $\mathrm{cl}(T(F(f))) \subseteq F(f)$'.

Theorem 3.9 *Let M be a nonempty subset of a normed* (*respectively, Banach*) *space X and T, f, and g be self-maps of M. If $F(f) \cap F(g)$ has the property* (N) *with respect to T, $\mathrm{cl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$* (*respectively, $\mathrm{wcl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$*), *and there exists a constant $L \geq 0$ such that*

$$\|Tx - Ty\| \leq m(x, y) + Ln(x, y) \quad \text{for all } x, y \in M, \tag{3.2}$$

where

$$m(x, y) = \max\Big\{ \|fx - gy\|, \mathrm{dist}(fx, [q, Tx]), \mathrm{dist}(gy, [q, Ty]),$$
$$\frac{1}{2}\big[\mathrm{dist}(gy, [q, Tx]) + \mathrm{dist}(fx, [q, Ty])\big] \Big\}$$

and

$$n(x, y) = \min\big\{ \mathrm{dist}(fx, [q, Tx]), \mathrm{dist}(gy, [q, Ty]), \mathrm{dist}(gy, [q, Tx]), \mathrm{dist}(fx, [q, Ty]) \big\}$$

then $M \cap F(T) \cap F(f) \cap F(g) \neq \phi$, provided $\mathrm{cl}(T(M))$ is compact (*respectively, $\mathrm{wcl}(T(M))$ is weakly compact*) *and T is continuous* (*respectively, $I - T$ is demiclosed at 0, where I stands for identity map*).

Proof As $T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$ and $F(f) \cap F(g)$ has the property (N) with respect to T, for each $n \in \mathbb{N}$, we can define $T_n : F(f) \cap F(g) \to F(f) \cap F(g)$ by $T_n x = (1 - k_n)q + k_n Tx$ for all $x \in F(f) \cap F(g)$ and a fixed sequence of real numbers k_n ($0 < k_n < 1$) converging to 1. Since $F(f) \cap F(g)$ has the property (N) with respect to T, and $\mathrm{cl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$ (respectively, $\mathrm{wcl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$), we have $\mathrm{cl}(T_n(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$ (respectively, $\mathrm{wcl}(T_n(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$) for each $n \in \mathbb{N}$. Also, by the inequality (3.2),

$$\|T_n x - T_n y\| = k_n \|Tx - Ty\|$$
$$\leq k_n\big[m(x, y) + Ln(x, y)\big]$$
$$= k_n m(x, y) + L_n n(x, y),$$

where

$$m(x, y) = \max\Big\{ \|fx - gy\|, \mathrm{dist}(fx, [q, Tx]), \mathrm{dist}(gy, [q, Ty]),$$
$$\frac{1}{2}\big[\mathrm{dist}(gy, [q, Tx]) + \mathrm{dist}(fx, [q, Ty])\big] \Big\}$$
$$\leq \max\Big\{ \|fx - gy\|, \|fx - T_n x\|, \|gy - T_n y\|,$$
$$\frac{1}{2}\big[\|fx - T_n y\| + \|gy - T_n x\|\big] \Big\}$$

and

$$n(x, y) = \min\big\{ \mathrm{dist}(fx, [q, Tx]), \mathrm{dist}(gy, [q, Ty]), \mathrm{dist}(gy, [q, Tx]), \mathrm{dist}(fx, [q, Ty]) \big\}$$
$$\leq \min\big\{ \|fx - T_n x\|, \|gy - T_n y\|, \|fx - T_n y\|, \|gy - T_n x\| \big\}$$

for all $x, y \in F(f) \cap F(g)$, $L_n := k_n L$, and $0 < k_n < 1$. Thus, for each $n \in \mathbb{N}$, T_n is a generalized (f, g)-almost contraction.

If $\mathrm{cl}(T(M))$ is compact, then, for each $n \in \mathbb{N}$, $\mathrm{cl}(T_n(M))$ is compact and hence complete. By Theorem 3.5, for each $n \geq 1$, there is a unique x_n in M such that $x_n = f(x_n) = g(x_n) = T_n(x_n)$. The compactness of $\mathrm{cl}(T(M))$ implies that there exists a subsequence $\{T x_m\}$ of $\{T x_n\}$ such that $T x_m \to z \in \mathrm{cl}(T(M))$. Since $\{T x_m\}$ is a sequence in $T(F(f) \cap F(g))$ and $\mathrm{cl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$, we have $z \in F(f) \cap F(g)$. Moreover,

$$x_m = T_m(x_m) = (1 - k_m)q + k_m T x_m \to z.$$

As T is continuous on M, we have $Tz = z$. Thus $M \cap F(T) \cap F(f) \cap F(g) \neq \phi$.

Next, the weak compactness of $\mathrm{wcl}(T(M))$ implies that $\mathrm{wcl}(T_n(M))$ is weakly compact and hence complete due to completeness of X. From Theorem 3.5, for each $n \geq 1$, there is a unique x_n in M such that $x_n = f(x_n) = g(x_n) = T_n(x_n)$. The weak compactness of $\mathrm{wcl}(T(M))$ implies that there is a subsequence $\{T x_m\}$ of $\{T x_n\}$ such that $T x_m$ converges weakly to $z \in \mathrm{wcl}(T(M))$. Since $\{T x_m\}$ is a sequence in $T(F(f) \cap F(g))$ and $\mathrm{wcl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$, therefore $z \in F(f) \cap F(g)$. Also we have $(I - T)x_m \to 0$ as $m \to \infty$. Further, demiclosedness of $I - T$ at 0 implies $z = Tz$, thus $M \cap F(T) \cap F(f) \cap F(g) \neq \phi$. $\square$

Corollary 3.10 *Let M be a nonempty subset of a normed (respectively, Banach) space X and $T, f,$ and g be self-maps of M. If $F(f) \cap F(g)$ has the property* (N) *with respect to T and is closed (respectively, weakly closed), (T, f) and (T, g) are Banach operator pairs and satisfy* (3.2) *for all $x, y \in M$. Then $M \cap F(T) \cap F(f) \cap F(g) \neq \phi$, provided $\mathrm{cl}(T(M))$ is compact (respectively, $\mathrm{wcl}(T(M))$ is weakly compact) and T is continuous (respectively, $I - T$ is demiclosed at 0, where I stands for the identity map).*

Remark 3.11 (1) By comparing Theorem 2.2 of Shahzad [17] with the first case of Theorem 3.9 (when $g = f$, $L = 0$), their assumptions '$q \in F(f)$, M is closed and q-starshaped, f is linear and continuous on M, $T(M) \subseteq f(M)$ and (T, f) is R-sub-weakly commuting pair on M' are replaced with 'M is a nonempty subset, $F(f)$ has the property (N) with respect to T, $\mathrm{cl}(T(F(f))) \subseteq F(f)$'.

(2) By comparing Theorem 2.2(i) of Hussain and Jungck [18] with the first case of Theorem 3.9 (when $L = 0$), their assumptions 'M is complete and q-starshaped, f and g are continuous and affine on M, $T(M) \subseteq f(M) \cap g(M)$, $q \in F(f) \cap F(g)$, and (T, f) and (T, g) are R-sub-weakly commuting pair on M' are replaced with '$F(f) \cap F(g)$ has the property (N) with respect to T, $\mathrm{cl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$'.

(3) By comparing Theorem 2.2(ii) of Hussain and Jungck [18] with the second case of Theorem 3.9 (when $L = 0$), their assumptions 'M is weakly compact and q-starshaped, f and g are affine and continuous on M, $T(M) \subseteq f(M) \cap g(M)$, $q \in F(f) \cap F(g)$, and (T, f) and (T, g) are R-sub-weakly commuting pair on M, and $f - T$ is demiclosed at 0' are replaced with '$\mathrm{wcl}(T(M))$ is weakly compact, $F(f) \cap F(g)$ has the property (N) with respect to T, $\mathrm{wcl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$ and $I - T$ is demiclosed at 0'.

Remark 3.12 If the contractive condition (3.2) in Theorem 3.9 is replaced with the stronger contractive condition

$$\|Tx - Ty\| \leq m(x, y) + Ln(x, y) \tag{3.3}$$

for all $x, y \in M$ and some $L \geq 0$, where

$$m(x,y) = \max\left\{\|fx - gy\|, \frac{1}{2}\big[\text{dist}(fx, [q, Tx]) + \text{dist}(gy, [q, Ty])\big],\right.$$
$$\left.\frac{1}{2}\big[\text{dist}(gy, [q, Tx]) + \text{dist}(fx, [q, Ty])\big]\right\}$$

and

$$n(x,y) = \min\big\{\text{dist}(fx, [q, Tx]), \text{dist}(gy, [q, Ty]), \text{dist}(gy, [q, Tx]), \text{dist}(fx, [q, Ty])\big\},$$

then continuity of T can be relaxed in the first case of Theorem 3.9.

Proof The proof will be similar to the first case of Theorem 3.9. To prove $Tz = z$, instead of continuity of T, using (3.3) we have

$$\|Tx_m - Tz\| \leq m(x_m, z) + Ln(x_m, z), \tag{3.4}$$

where

$$m(x_m, z) = \max\left\{\|fx_m - gz\|, \frac{1}{2}\big[\text{dist}(fx_m, [q, Tx_m]) + \text{dist}(gz, [q, Tz])\big],\right.$$
$$\left.\frac{1}{2}\big[\text{dist}(gz, [q, Tx_m]) + \text{dist}(fx_m, [q, Tz])\big]\right\}$$
$$\leq \max\left\{\|x_m - z\|, \frac{1}{2}\big[\|x_m - Tx_m\| + \|z - Tz\|\big],\right.$$
$$\left.\frac{1}{2}\big[\|z - Tx_m\| + \|x_m - Tz\|\big]\right\}$$
$$\leq \max\left\{\|x_m - z\|, \frac{1}{2}\big[\|x_m - Tx_m\| + \|z - Tx_m\| + \|Tx_m - Tz\|\big],\right.$$
$$\left.\frac{1}{2}\big[\|z - Tx_m\| + \|x_m - Tx_m\| + \|Tx_m - Tz\|\big]\right\}$$

and

$$n(x_m, z) = \min\big\{\text{dist}(fx_m, [q, Tx_m]), \text{dist}(gz, [q, Tz]),$$
$$\text{dist}(gz, [q, Tx_m]), \text{dist}(fx_m, [q, Tz])\big\}$$
$$\leq \min\big\{\|x_m - Tx_m\|, \|z - Tz\|, \|z - Tx_m\|, \|x_m - Tz\|\big\}.$$

Now taking $m \to \infty$ in (3.4) we can write

$$\lim_{m\to\infty} \|Tx_m - Tz\| \leq \frac{1}{2} \lim_{m\to\infty} \|Tx_m - Tz\|.$$

This is possible only if $Tx_m \to Tz$ as $m \to \infty$, which implies $Tz = z$. □

Let $C = B_M(p) \cap C_M^{f,g}(p)$, where $C_M^{f,g}(p) = \{x \in M : fx \in B_M(p), gx \in B_M(p)\}$.

Corollary 3.13 *Let X be a normed (respectively, Banach) space and let T, f, and g be self-maps of X. If $p \in X$ and $D \subseteq C$, $D_0 := D \cap F(f) \cap F(g)$ has the property* (N) *with respect to T,* $\mathrm{cl}(T(D_0)) \subseteq D_0$ *(respectively,* $\mathrm{wcl}(T(D_0)) \subseteq D_0$*),* $\mathrm{cl}(T(D))$ *is compact (respectively,* $\mathrm{wcl}(T(D))$ *is weakly compact),* T *is continuous on D (respectively, $I - T$ is demiclosed at 0, where I stands for identity map) and* (3.2) *holds for all $x, y \in D$, then $B_M(p) \cap F(T) \cap F(f) \cap F(g) \neq \phi$.*

Corollary 3.14 *Let X be a normed (respectively, Banach) space and let T, f, and g be self-maps of X. If $p \in X$ and $D \subseteq B_M(p)$, $D_0 := D \cap F(f) \cap F(g)$ has the property* (N) *with respect to T,* $\mathrm{cl}(T(D_0)) \subseteq D_0$ *(respectively,* $\mathrm{wcl}(T(D_0)) \subseteq D_0$*),* $\mathrm{cl}(T(D))$ *is compact (respectively,* $\mathrm{wcl}(T(D))$ *is weakly compact),* T *is continuous on D (respectively, $I - T$ is demiclosed at 0, where I stands for the identity map) and* (3.2) *holds for all $x, y \in D$, then $B_M(p) \cap F(T) \cap F(f) \cap F(g) \neq \phi$.*

Remark 3.15 Corollaries 3.13 and 3.14 improve and develop Theorems 2.8-2.11 of Hussain and Jungck [18] and Theorems 3.1-3.4 of Song [19] to the non-starshaped domain.

Denote by $\mathcal{L}_0$ the class of closed convex subsets of X containing 0. For $M \in \mathcal{L}_0$, we define $M_p = \{x \in M : \|x\| \leq 2\|p\|\}$. Clearly $B_M(p) \subseteq M_p \in \mathcal{L}_0$.

The following invariant approximation result constitutes an extension of Theorem 2.6 of Al-Thagafi and Shahzad [16] and Corollary 2.10 of [29] to a non-starshaped domain.

Theorem 3.16 *Let X be a normed (respectively, Banach) space and $T, f, g : X \to X$. If $p \in X$ and $M \in \mathcal{L}_0$ such that $T(M_p) \subseteq M$, $\mathrm{cl}(T(M_p))$ is compact (respectively, $\mathrm{wcl}(T(M_p))$ is weakly compact), and $\|Tx - p\| \leq \|x - p\|$ for all $x \in M_p$, then $B_M(p)$ is nonempty, closed, and convex with $T(B_M(p)) \subseteq B_M(p)$. If, in addition, D is a subset of $B_M(p)$, $D_0 := D \cap F(f) \cap F(g)$ has the property* (N) *with respect to T,* $\mathrm{cl}(T(D_0)) \subseteq D_0$ *(respectively,* $\mathrm{wcl}(T(D_0)) \subseteq D_0$*),* T *is continuous on D (respectively, $I - T$ is demiclosed at 0, where I stands for the identity map) and* (3.2) *holds for all $x, y \in D$, then $B_M(p) \cap F(T) \cap F(f) \cap F(g) \neq \phi$.*

Proof We may assume that $p \notin M$. If $y \in M \backslash M_p$, then $\|y\| > 2\|p\|$ and, so

$$\|y - p\| \geq \|y\| - \|p\| > \|p\| \geq \mathrm{dist}(p, M).$$

Thus $\mathrm{dist}(p, M_p) = \mathrm{dist}(p, M)$. Assume that $\mathrm{cl}(T(M_p))$ is compact, then by the continuity of the norm there exists $z \in \mathrm{cl}(T(M_p))$ such that $\|z - p\| = \mathrm{dist}(p, clT(M_p))$.

If we assume that $\mathrm{wcl}(T(M_p))$ is weakly compact, then by using Lemma 5.5 of [20, p.192] we can show the existence of $z \in \mathrm{wcl}(T(M_p))$ such that $\|z - p\| = \mathrm{dist}(p, \mathrm{wcl}\, T(M_p))$. Thus in both cases, we have

$$\mathrm{dist}(p, M_p) \leq \mathrm{dist}\big(p, \mathrm{cl}\, T(M_p)\big) \leq \mathrm{dist}\big(p, T(M_p)\big) \leq \|Tx - p\| \leq \|x - p\|$$

for all $x \in M_p$. It follows that $\|z - p\| = \mathrm{dist}(p, M)$. Thus $B_M(p)$ is nonempty, closed, and convex with $T(B_M(p)) \subseteq B_M(p)$. The compactness of $\mathrm{cl}(T(M_p))$ (respectively, weak compactness of $\mathrm{wcl}(T(M_p))$) implies that $\mathrm{cl}(T(D))$ is compact (respectively, $\mathrm{wcl}(T(D))$ is weakly compact). Then by Corollary 3.14, $B_M(p) \cap F(T) \cap F(f) \cap F(g) \neq \phi$. □

Now, we present some non-trivial examples in support of Theorem 3.9.

Example 3.17 Let $X = \Re$ be the set of real numbers with the usual norm and $M = [0,1)$. We define mappings $f, g, T : M \to M$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{2}{3}, \\ \frac{4}{3} - x & \text{if } \frac{2}{3} \leq x < 1, \end{cases} \qquad g(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x < \frac{2}{3}, \\ \frac{2}{3} & \text{if } \frac{2}{3} \leq x < 1, \end{cases}$$

and $T(x) = \frac{2}{3}$, for $0 \leq x < 1$.

Here we observe that $F(f) \cap F(g) = \{0, \frac{2}{3}\}$, $\text{cl}(T(F(f) \cap F(g))) = \{\frac{2}{3}\} \subseteq F(f) \cap F(g)$ and $\text{cl}(T(M)) = \{\frac{2}{3}\}$ is compact. Clearly $F(f) \cap F(g)$ is not starshaped but has property (N) with respect to T, for $q = \frac{2}{3}$ and $k_n = 1 - 1/n$. Further, the mappings T, f, and g satisfy the contractive condition (3.2) and also T is continuous. Hence all the conditions of the first case of Theorem 3.9 are satisfied and consequently T, f, and g have a common fixed point, $x = \frac{2}{3}$.

Remark 3.18 In Example 3.17, it is interesting to note that Theorem 2.19 of Hussain and Cho [21], and Corollary 3.10 of Akbar *et al.* [22] cannot apply, since $F(f) \cap F(g)$ is not q-starshaped.

Example 3.19 Let $X = \Re$ be the set of real numbers with the usual norm and $M = [0,1]$. Define $f, g, T : M \to M$ by

$$f(x) = \begin{cases} x, & x \text{ is rational in } M, \\ 0, & \text{otherwise,} \end{cases} \qquad g(x) = x \quad \text{for all } x \in M$$

and

$$T(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

Clearly $F(f) \cap F(g) = \{x, x \text{ is rational in } M\}$ has property (N) with respect to T, for $q = 0$, $k_n = 1 - 1/n$. Further, $\text{cl}(T(F(f) \cap F(g))) = \{0,1\} \subseteq F(f) \cap F(g)$, $\text{cl}(T(M)) = \{0,1\}$ is compact and T, f, and g satisfy the contractive condition (3.2). Hence all the conditions of the first case of Theorem 3.9 are satisfied except the continuity of T. Note that $F(T) \cap F(f) \cap F(g) = \phi$.

Remark 3.20 It is to be noted that the maps T, f, and g given in Example 3.19 do not satisfy the contractive condition (3.3) at the point $x = \frac{1}{2}, y = 1$.

4 Results with joint contractive family

Dotson [23] proved some results concerning the existence of fixed points of nonexpansive mappings on a certain class of non-convex sets. For proving these results, he extends the concept of starshapedness by introducing the following class of non-convex set.

Let M be a subset of a normed space X and $\Gamma = \{h_x : x \in M\}$ be a family of functions from $[0,1]$ to M such that $h_x(1) = x$ for each $x \in M$. The family Γ is said to be contractive if there exists a function $\varphi : (0,1) \to (0,1)$ such that for all $x, y \in M$ and all $t \in (0,1)$, we have

$$\|h_x(t) - h_y(t)\| \leq \varphi(t)\|x - y\|.$$

Such a family Γ is said to be jointly continuous (jointly weakly continuous) if $t \to t_0$ in $[0,1]$ and $x \to x_0$ ($x \to x_0$ weakly) in M; then $h_x(t) \to h_{x_0}(t)$ ($h_x(t) \to h_{x_0}(t)$ weakly) in M.

We observe that if M is q-starshaped subset of a normed linear space X and $h_x(t) = (1-t)q + tx$, for each $x \in M$, $q \in M$ and $t \in [0,1]$, then Γ is a contractive jointly continuous and jointly weakly continuous family with $\varphi(t) = t$. Thus the class of subsets of X with the property of contractiveness and joint continuity contains the class of starshaped sets which in turns contains the class of convex sets.

We shall denote $Y_q^{Tx} = \{h_{Tx}(k) : 0 \le k \le 1\}$ where $q = h_{Tx}(0)$.

The following results properly contain Theorems 3.2 and 3.3 of [10], Theorems 1 and 2 of [24] and improves Theorem 2.2 of [25], Theorem 6 of [26].

Theorem 4.1 *Let M be a nonempty subset of a normed (respectively, Banach) space X and T, f and g be self-maps of M. Suppose $F(f) \cap F(g)$ is nonempty and has a contractive, jointly continuous (respectively, jointly weakly continuous) family of functions $\Gamma = \{h_x : x \in F(f) \cap F(g)\}$, $\mathrm{cl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$ (respectively, $\mathrm{wcl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$), and there exists a constant $L \ge 0$ such that*

$$\|Tx - Ty\| \le m(x,y) + Ln(x,y) \tag{4.1}$$

for all $x, y \in M$, where

$$m(x,y) = \max\left\{\|fx - gy\|, \mathrm{dist}(fx, Y_q^{Tx}), \mathrm{dist}(gy, Y_q^{Ty}), \frac{1}{2}\left[\mathrm{dist}(gy, Y_q^{Tx}) + \mathrm{dist}(fx, Y_q^{Ty})\right]\right\}$$

and

$$n(x,y) = \min\left\{\mathrm{dist}(fx, Y_q^{Tx}), \mathrm{dist}(gy, Y_q^{Ty}), \mathrm{dist}(gy, Y_q^{Tx}), \mathrm{dist}(fx, Y_q^{Ty})\right\}.$$

Then $M \cap F(T) \cap F(f) \cap F(g) \neq \phi$, provided $\mathrm{cl}(T(M))$ is compact (respectively, $\mathrm{wcl}(T(M))$ is weakly compact) and T is continuous (respectively, T is weakly continuous).

Proof For each natural number n, let $k_n = \frac{n}{n+1}$. Define $T_n : F(f) \cap F(g) \to F(f) \cap F(g)$ by $T_n(x) = h_{Tx}(k_n)$ for all $x \in F(f) \cap F(g)$. Since $F(f) \cap F(g)$ has a contractive family and $\mathrm{cl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$ (respectively, $\mathrm{wcl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$), so for each $n \in \mathbb{N}$, $\mathrm{cl}(T_n(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$ (respectively, $\mathrm{wcl}(T_n(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$).

We have

$$\|T_n x - T_n y\| = \left\|h_{Tx}(k_n) - h_{Ty}(k_n)\right\|$$

$$\le \Psi(k_n)\|Tx - Ty\| \quad \text{since } \Gamma \text{ is a contractive family}$$

$$\le \Psi(k_n)\left\{m(x,y) + Ln(x,y)\right\} \quad \text{using (4.1)}$$

$$= \Psi(k_n)m(x,y) + L_n n(x,y)$$

for each $x, y \in F(f) \cap F(g)$, where $L_n = L\Psi(k_n)$, $\Psi(k_n) \in (0,1)$,

$$m(x,y) = \max\left\{\|fx - gy\|, \mathrm{dist}(fx, Y_q^{Tx}), \mathrm{dist}(gy, Y_q^{Ty}), \frac{1}{2}\left[\mathrm{dist}(gy, Y_q^{Tx}) + \mathrm{dist}(fx, Y_q^{Ty})\right]\right\}$$

$$\le \max\left\{\|fx - gy\|, \|fx - T_n x\|, \|gy - T_n y\|, \frac{1}{2}\left[\|gy - T_n x\| + \|fx - T_n y\|\right]\right\}$$

and

$$n(x,y) = \min\left\{\text{dist}\big(fx, Y_q^{Tx}\big), \text{dist}\big(gy, Y_q^{Ty}\big), \text{dist}\big(gy, Y_q^{Tx}\big), \text{dist}\big(fx, Y_q^{Ty}\big)\right\}$$

$$\leq \min\left\{\|fx - T_n x\|, \|gy - T_n y\|, \|gy - T_n x\|, \|fx - T_n y\|\right\}.$$

Thus, for each $n \in \mathbb{N}$, T_n is a generalized almost (f, g)-contraction.

If $\text{cl}(T(M))$ is compact, then, for each $n \in \mathbb{N}$, $\text{cl}(T_n(M))$ is compact and hence complete. By Theorem 3.5, for each $n \geq 1$, there exists a unique $x_n \in F(f) \cap F(g)$ such that $x_n = f(x_n) = g(x_n) = T_n(x_n)$. Again the compactness of $\text{cl}(T(M))$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to z \in \text{cl}(T(M))$. Since $\{Tx_m\}$ is a sequence in $T(F(f) \cap F(g))$ and $\text{cl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$, we have $z \in F(f) \cap F(g)$. Further, the joint continuity of family Γ implies that

$$x_m = T_m x_m = h_{Tx_m}(k_m) \to h_z(1) = z \quad \text{as } m \to \infty.$$

By the continuity of T, we obtain $z = T(z)$. Thus, $M \cap F(T) \cap F(f) \cap F(g) \neq \phi$.

The weak compactness of $\text{wcl}(T(M))$ implies that $\text{wcl}(T_n(M))$ is weakly compact and hence complete due to completeness of X. From Theorem 3.5 for each $n \geq 1$, there exists a unique $x_n \in F(f) \cap F(g)$ such that $x_n = f(x_n) = g(x_n) = T_n(x_n)$. The weak compactness of $\text{wcl}(T(M))$ implies that there is a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that Tx_m converges weakly to $z \in \text{wcl}(T(M))$ as $m \to \infty$. Since $\{Tx_m\}$ is a sequence in $T(F(f) \cap F(g))$ and $\text{wcl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$, we have $z \in F(f) \cap F(g)$. By the joint weak continuity of the family we obtain

$$x_m = T_m x_m = h_{Tx_m}(k_m) \to h_z(1) = z \quad \text{(weakly) as } m \to \infty.$$

Since the weak topology is Hausdorff, by weak continuity of T, we have $z = T(z)$. Thus, $M \cap F(T) \cap F(f) \cap F(g) \neq \phi$. □

Remark 4.2 By comparing Theorem 2.2(i) of Chandok and Narang [27] with the first case of Theorem 4.1 (when $L = 0$), their assumptions 'M is complete and has a contractive jointly continuous family Γ with $g(h_x(k)) = h_{gx}(k)$ and $f(h_x(k)) = h_{fx}(k)$ for $k \in (0, 1)$, $\text{cl}(T(M)) \subseteq f(M) \cap g(M)$, the pairs (T, f) and (T, g) are C_q-commuting and f, g are continuous on M' are replaced with 'M is nonempty subset, $F(f) \cap F(g)$ is nonempty and has a contractive jointly continuous family Γ, and $\text{cl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$'.

Corollary 4.3 *Let M be a nonempty subset of a normed (respectively, Banach) space X and T, f, and g be self-maps of M. Suppose $F(f) \cap F(g)$ is q-starshaped, $\text{cl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$ (respectively, $\text{wcl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$), and there exists a constant $L \geq 0$ such that*

$$\|Tx - Ty\| \leq m(x, y) + Ln(x, y) \tag{4.2}$$

for all $x, y \in M$, where

$$m(x,y) = \max\Big\{\|fx - gy\|, \text{dist}\big(fx, [q, Tx]\big), \text{dist}\big(gy, [q, Ty]\big),$$

$$\frac{1}{2}\big[\text{dist}\big(gy, [q, Tx]\big) + \text{dist}\big(fx, [q, Ty]\big)\big]\Big\}$$

and

$$n(x, y) = \min\{\operatorname{dist}(fx, [q, Tx]), \operatorname{dist}(gy, [q, Ty]), \operatorname{dist}(gy, [q, Tx]), \operatorname{dist}(fx, [q, Ty])\}.$$

Then $M \cap F(T) \cap F(f) \cap F(g) \neq \phi$, provided $\operatorname{cl}(T(M))$ is compact (respectively, $\operatorname{wcl}(T(M))$ is weakly compact) and T is continuous (respectively, T is weakly continuous).

Remark 4.4 (1) By comparing Theorem 2.3(i) of Abbas and Ilić [15] with the first case of Corollary 4.3 (when $g = f$), their assumptions 'M is q-starshaped, $\operatorname{cl}(T(M)) \subseteq f(M), f$ and T are weakly compatible on M' are replaced with '$F(f)$ is q-starshaped, $\operatorname{cl}(T(F(f))) \subseteq F(f)$'.

(2) By comparing Theorem 2.3(ii) of Abbas and Ilić [15] with the second case of Corollary 4.3 (when $g = f$), their assumptions 'M is q-starshaped, $\operatorname{cl}(T(M)) \subseteq f(M), f$ and T are weakly compatible on M, f is weakly continuous and $f - T$ is demiclosed at 0' are replaced with '$F(f)$ is q-starshaped, $\operatorname{cl}(T(F(f))) \subseteq F(f)$ and T is weakly continuous'.

(3) By comparing Theorem 2.4 of Song [19] with the first case of Corollary 4.3 (when $L = 0$), their assumptions 'M is q-starshaped, $\operatorname{cl}(T(M)) \subseteq f(M) \cap g(M)$, the pairs (T, f) and (T, g) are C_q-commuting, f and g are q-affine and continuous on M' are replaced with '$F(f) \cap F(g)$ is q-starshaped, $\operatorname{cl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$'.

Corollary 4.5 *Let M be a nonempty subset of a normed (respectively, Banach) space X and T, f, and g be self-maps of M. If M has a contractive jointly continuous (respectively, jointly weakly continuous) family $\Gamma = \{h_x : x \in M\}$ such that $g(h_x(k)) = h_{gx}(k)$ and $f(h_x(k)) = h_{fx}(k)$ for all $x \in M, k \in [0, 1]$. Suppose $F(f) \cap F(g)$ is nonempty, closed (respectively, weakly closed), $\operatorname{cl}(T(M))$ is compact (respectively, $\operatorname{wcl}(T(M))$ is weakly compact), T is continuous (respectively, weakly continuous), (T, f) and (T, g) are Banach operator pair on M and satisfy (4.1). Then $M \cap F(T) \cap F(f) \cap F(g) \neq \phi$.*

Proof For each natural number n, define $T_n : M \to M$ by $T_n(x) = h_{Tx}(k_n)$, for all $x \in M$. Clearly, for each $n \geq 1$, T_n is a self-map on M. Since (T, f) is Banach operator pair on M, for each $x \in F(f)$, we have $Tx \in F(f)$. Consider

$$f(T_n x) = f(h_{Tx}(k_n)) = h_{fTx}(k_n) = h_{Tx}(k_n) = T_n x.$$

This implies that $T_n x \in F(f)$ for each $x \in F(f)$. Thus for each $n \in \mathbb{N}$, (T_n, f) is a Banach operator pair on M. Similarly, for each $n \in \mathbb{N}$, (T_n, g) is a Banach operator on M. Now the result follows from Theorem 4.1. □

Corollary 4.6 *Let X be a normed (respectively, Banach) space and let T, f, and g be self-maps of X. If $p \in X$ and $D \subseteq C$, $D_0 := D \cap F(f) \cap F(g)$ is nonempty, has a contractive jointly continuous (respectively, jointly weakly continuous) family of functions $\Gamma = \{h_x : x \in D_0\}$, $\operatorname{cl}(T(D_0)) \subseteq D_0$ (respectively, $\operatorname{wcl}(T(D_0)) \subseteq D_0$), $\operatorname{cl}(T(D))$ is compact (respectively, $\operatorname{wcl}(T(D))$ is weakly compact), T is continuous on D (respectively, T is weakly continuous) and (4.1) holds for all $x, y \in D$, then $B_M(p) \cap F(T) \cap F(f) \cap F(g) \neq \phi$.*

Corollary 4.7 *Let X be a normed (respectively, Banach) space and T, f, and g be self-maps of X. If $p \in X$ and $D \subseteq B_M(p)$, $D_0 := D \cap F(f) \cap F(g)$ is nonempty, has a contractive jointly continuous (respectively, jointly weakly continuous) family of $\Gamma = \{h_x : x \in D_0\}$, $\operatorname{cl}(T(D_0)) \subseteq$*

D_0 (*respectively,* $\mathrm{wcl}(T(D_0)) \subseteq D_0$), $\mathrm{cl}(T(D))$ *is compact* (*respectively,* $\mathrm{wcl}(T(D))$ *is weakly compact*), T *is continuous on* D (*respectively,* T *is weakly continuous*) *and* (4.1) *holds for all* $x, y \in D$, *then* $B_M(p) \cap F(T) \cap F(f) \cap F(g) \neq \phi$.

Remark 4.8 (1) Theorems 4.1 and 4.2 of Chen and Li [10], Theorems 3 and 4 of Chandok and Narang [24] are particular cases of Corollaries 4.6 and 4.7.

(2) By Proposition 2.2 of Chen and Li [10], it can be concluded that Corollary 4.5 extends and generalizes Corollary 2.1 of Shahzad [28].

Now we present two examples in support of Theorem 4.1 and Theorem 3.5, respectively.

Example 4.9 Let $X = \Re$ be the set of real numbers with the usual norm and $M = [0, 1]$. Assume $T(x) = \frac{1}{2}$, for every x in M and define $f, g : M \to M$ by

$$f(x) = \begin{cases} x, & x \text{ is rational,} \\ 1 - x, & x \text{ is irrational,} \end{cases} \qquad g(x) = x \quad \text{for all } x \in M.$$

Then $F(f) \cap F(g) = \{x, x \text{ is rational in } M\}$, $\mathrm{cl}(T(F(f) \cap F(g))) = \{\frac{1}{2}\} \subseteq F(f) \cap F(g)$ and $\mathrm{cl}(T(M)) = \{\frac{1}{2}\}$ is compact. Suppose that $\Gamma = \{h_x : x \in F(f) \cap F(g)\}$ is a family of functions from $[0, 1]$ into $F(f) \cap F(g)$, defined by

$$h_x(t) = \begin{cases} 1, & x \in F(f) \cap F(g), t \in M \backslash F(f) \cap F(g), \\ t^2 x, & x, t \in F(f) \cap F(g). \end{cases}$$

We observe that the family Γ is contractive jointly continuous for $\varphi(t) = t^2$, $t \in (0, 1)$. Thus all the conditions of Theorem 4.1 are satisfied. Consequently T, f, and g have a common fixed point. Here it is seen that $x = \frac{1}{2}$ is the common fixed point of T, f, and g.

Remark 4.10 (1) Theorem 2.2(i) of Chandok and Narang [27] cannot apply to Example 4.9, since f is not continuous.

(2) It is interesting to note that the results of Akbar *et al.* [22] cannot apply to Example 4.9, since $F(f) \cap F(g)$ is not q-starshaped.

Example 4.11 Let $X = M = \{\alpha, \beta, \gamma, \delta\}$ and let $d : X \times X \to \Re$ be given as

$$d(\alpha, \beta) = d(\beta, \alpha) = 0.5, \qquad d(\alpha, \gamma) = d(\gamma, \alpha) = 2.5,$$

$$d(\alpha, \delta) = d(\delta, \alpha) = 1.6, \qquad d(\beta, \gamma) = d(\gamma, \beta) = 2.5,$$

$$d(\beta, \delta) = d(\delta, \beta) = 1.5, \qquad d(\gamma, \delta) = d(\delta, \gamma) = 2 \quad \text{and}$$

$$d(\alpha, \alpha) = d(\beta, \beta) = d(\gamma, \gamma) = d(\delta, \delta) = (0, 0).$$

Then (X, d) is a metric space. Let $T, f, g : M \to M$ is defined, respectively, as follows:

$$T(x) = \begin{cases} \beta, & x \neq \gamma, \\ \delta, & x = \gamma \end{cases}$$

and

$$f\alpha = \beta, \qquad f\beta = \beta, \qquad f\gamma = \alpha, \qquad f\delta = \beta,$$

$$g\alpha = \delta, \qquad g\beta = \beta, \qquad g\gamma = \gamma, \qquad g\delta = \alpha.$$

Clearly $F(f) \cap F(g) = \{\beta\}$ and $\text{cl}(T(F(f) \cap F(g))) = \{\beta\} \subseteq F(f) \cap F(g)$. Further T is a generalized almost (f, g)-contraction for $\delta = \frac{19}{20}$ and $L = 0$. Hence, all the conditions of Theorem 3.5 are satisfied. Consequently T, f, and g have a unique common fixed point. Here it is seen that $x = \beta$ is the unique common fixed point of T, f, and g.

Remark 4.12 (1) In Example 4.11, $f(M) = \{\alpha, \beta\}$, $g(M) = \{\alpha, \beta, \gamma, \delta\}$ and $T(M) = \{\beta, \delta\}$, therefore $\text{cl}\,T(M)$ is not contained in $f(M) \cap g(M)$. Hence Theorem 2.1 of Song [19] cannot apply to Example 4.11.

(2) In Example 4.11, if we take $g(x) = f(x) = \begin{cases} \beta, & x \neq \gamma, \\ \alpha, & x = \gamma, \end{cases}$ then T and f does not satisfy the contractive condition of Lemma 3.1 of [10] and Theorem 2.2 of [16] at $x = \gamma$, $y = \alpha$. Hence Lemma 3.1 of [10] and Theorem 2.2 of [16] cannot apply to Example 4.11.

Remark 4.13 (1) Example 3.3 satisfies all the conditions of Theorem 3.5 except the condition $\text{cl}(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$. Note that $F(T) \cap F(f) \cap F(g) \neq \phi$.

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