Park *Advances in Difference Equations* 2014, 2014:304
http://www.advancesindifferenceequations.com/content/2014/1/304

Advances in Difference Equations
a SpringerOpen Journal

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On the twisted Daehee polynomials with q-parameter

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Abstract

The nth twisted Daehee numbers with q-parameter are closely related to higher-order Bernoulli numbers and Bernoulli numbers of the second kind. In this paper, we give a p-adic integral representation of the twisted Daehee polynomials with q-parameter, and we derive some interesting properties related to the nth twisted Daehee polynomials with q-parameter.

Keywords: Bernoulli polynomials; Daehee polynomials with q-parameter; p-adic invariant integral

1 Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completions of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is defined $|p|_p = \frac{1}{p}$.

When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation

$$[x]_q = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to 1}[x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the *p-adic invariant integral on $\mathbb{Z}_p$* is defined by Kim as follows:

$$I(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_0(x) = \lim_{n\to\infty} \frac{1}{p^n} \sum_{x=0}^{p^n-1} f(x) \quad \text{(see [1–3])}. \tag{1.1}$$

Let f_1 be the translation of f with $f_1(x) = f(x+1)$. Then, by (1.1), we get

$$I(f_1) = I(f) + f'(0), \quad \text{where } f'(0) = \left.\frac{df(x)}{dx}\right|_{x=0}. \tag{1.2}$$

As is well known, the *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \tag{1.3}$$

and the *Stirling number of the second kind* is given by the generating function to be

$$\left(e^t - 1\right)^m = m! \sum_{l=m}^{\infty} S_2(l,m)\frac{t^l}{l!} \tag{1.4}$$

(see [4–6]).

Unsigned Stirling numbers of the first kind are given by

$$x^{\underline{n}} = x(x+1)\cdots(x+n-1) = \sum_{l=0}^{n} \left|S_1(n,l)\right| x^l. \tag{1.5}$$

Note that if we replace x to $-x$ in (1.3), then

$$\begin{aligned} (-x)_n = (-1)^n x^{\underline{n}} &= \sum_{l=0}^{n} S_1(n,l)(-1)^l x^l \\ &= (-1)^n \sum_{l=0}^{n} \left|S_1(n,l)\right| x^l. \end{aligned} \tag{1.6}$$

Hence $S_1(n,l) = |S_1(n,l)|(-1)^{n-l}$.

For $r \in \mathbb{N}$, the *Bernoulli polynomials of order r* are defined by the generating function to be

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!} \quad \text{(see [1, 4, 7–18])}. \tag{1.7}$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the *Bernoulli numbers of order r*, and in the special case, $r = 1$, $B_n^{(1)}(x) = B_n(x)$ are called the *ordinary Bernoulli polynomials*.

For $n \in \mathbb{N}$, let T_p be the p-adic locally constant space defined by

$$T_p = \bigcup_{n\geq 1} C_{p^n} = \lim_{n\to\infty} C_{p^n},$$

where $C_{p^n} = \{\omega | \omega^{p^n} = 1\}$ is the cyclic group of order p^n.

We assume that q is an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. Then we define the q-analog of a falling factorial sequence as follows:

$$(x)_{n,q} = x(x-q)(x-2q)\cdots\left(x-(n-1)q\right) \quad (n \geq 1), \qquad (x)_{0,q} = 1.$$

Note that

$$\lim_{q\to 1}(x)_{n,q} = (x)_n = \sum_{l=0}^{n} S_1(n,l)x^l.$$

Recently, DS Kim and T Kim introduced the *Daehee polynomials* as follows:

$$D_n(x) = \int_{\mathbb{Z}_p} (x+y)_n \, d\mu_0(y) \quad (n \geq 0) \text{ (see [2, 9, 19]).} \tag{1.8}$$

When $x = 0$, $D_n = D_n(0)$ are called the *nth Daehee numbers*. From (1.8), we can derive the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!} \quad \text{(see [9]).} \tag{1.9}$$

In addition, DS Kim *et al.* consider the *Daehee polynomials with q-parameter*, which are defined by the generating function to be

$$\sum_{n=0}^{\infty} D_{n,q}\frac{t^n}{n!} = (1+qt)^{\frac{x}{q}} \frac{\log(1+qt)}{q((1+qt)^{\frac{1}{q}}-1)} \quad \text{(see [20, 21]).} \tag{1.10}$$

When $x = 0$, $D_{n,q} = D_{n,q}(0)$ are called the *Daehee numbers with q-parameter*.

From the viewpoint of a generalization of the Daehee polynomials with q-parameter, we consider the *twisted Daehee polynomials with q-parameter*, defined to be

$$\sum_{n=0}^{\infty} D_{n,\xi,q}\frac{t^n}{n!} = (1+q\xi t)^{\frac{x}{q}} \frac{\log(1+q\xi t)}{q((1+q\xi t)^{\frac{1}{q}}-1)}, \tag{1.11}$$

where $t, q \in \mathbb{C}_p$ with $|t|_p < |q|_p p^{-\frac{1}{p-1}}$ and $\xi \in T_p$.

In this paper, we give a p-adic integral representation of the twisted Daehee polynomials with q-parameter, which is called the Witt-type formula for the twisted Daehee polynomials with q-parameter. We can derive some interesting properties related to the nth twisted Daehee polynomials with q-parameter.

2 Witt-type formula for the *n*th twisted Daehee polynomials with *q*-parameter

First, we consider the following integral representation associated with falling factorial sequences:

$$\xi^n \int_{\mathbb{Z}_p} (x+y)_{n,q} \, d\mu_0(y), \quad \text{where } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \text{ and } \xi \in T_p. \tag{2.1}$$

By (2.1),

$$\begin{aligned}\sum_{n=0}^{\infty} \xi^n \int_{\mathbb{Z}_p} (x+y)_{n,q} \, d\mu_0(y)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} \xi^n q^n \int_{\mathbb{Z}_p} \left(\frac{x+y}{q}\right)_n d\mu_0(y)\frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} (1+q\xi t)^{\frac{x+y}{q}} \, d\mu_0(y),\end{aligned} \tag{2.2}$$

where $t,q \in \mathbb{C}_p$ with $|t|_p < |q|_p p^{-\frac{1}{p-1}}$. For $t \in \mathbb{C}_p$ with $|t|_p < |q|_p p^{-\frac{1}{p-1}}$, put $f(x) = (1+q\xi t)^{\frac{x+y}{q}}$. By (1.1), we get

$$\begin{aligned}\int_{\mathbb{Z}_p} (1+q\xi t)^{\frac{x+y}{q}} \, d\mu_0(y) &= (1+q\xi t)^{\frac{x}{q}} \frac{\log(1+q\xi t)}{q((1+q\xi t)^{\frac{1}{q}}-1)} \\ &= \sum_{n=0}^{\infty} D_{n,\xi,q}(x)\frac{t^n}{n!}. \end{aligned} \tag{2.3}$$

By (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1 *For $n \geq 0$, we have*

$$D_{n,\xi,q}(x) = \xi^n \int_{\mathbb{Z}_p} (x+y)_{n,q} \, d\mu_0(y).$$

In (2.3), by replacing t by $\frac{1}{\xi q}(e^{\xi t}-1)$, we have

$$\sum_{n=0}^{\infty} D_{n,\xi,q}(x) \frac{1}{\xi^n q^n} \frac{(e^{\xi t}-1)^n}{n!} = e^{\frac{\xi t x}{q}} \frac{\frac{\xi t}{q}}{e^{\frac{\xi t}{q}}-1} = \sum_{n=0}^{\infty} B_n(x) \frac{\xi^n}{q^n}\frac{t^n}{n!} \tag{2.4}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{D_{n,\xi,q}(x)}{\xi^n q^n} \frac{1}{n!}\left(e^{\xi t}-1\right)^n &= \sum_{n=0}^{\infty} \frac{D_{n,\xi,q}(x)}{\xi^n q^n} \sum_{m=n}^{\infty} \xi^m S_2(m,n)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{D_{n,\xi,q}(x)}{\xi^n q^n} \xi^m S_2(m,n)\frac{t^m}{m!}. \end{aligned} \tag{2.5}$$

By (2.4) and (2.5), we obtain the following corollary.

Corollary 2.2 *For $n \geq 0$, we have*

$$B_n(x) = \sum_{m=0}^{n} D_{m,\xi,q}(x) \xi^{-m} q^{n-m} S_2(n,m).$$

By Theorem 2.1,

$$\begin{aligned} D_{n,\xi,q}(x) &= \xi^n \int_{\mathbb{Z}_p} (x+y)_{n,q} \, d\mu_0(y) \\ &= \xi^n q^n \sum_{l=0}^{n} \frac{1}{q^l} S_1(n,l) \int_{\mathbb{Z}_p} (x+y)^l \, d\mu_0(y). \end{aligned} \tag{2.6}$$

By (1.2), we can derive easily that

$$\begin{aligned}\int_{\mathbb{Z}_p} e^{(x+y)t} \, d\mu_0(y) &= \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \\ &= \sum_{l=0}^{\infty} \int_{\mathbb{Z}_p} (x+y)^l \, d\mu_0(y) \frac{t^l}{l!}, \end{aligned} \tag{2.7}$$

and so

$$B_n(x) = \int_{\mathbb{Z}_p} (x+y)^n \, d\mu_0(y). \tag{2.8}$$

By (1.6), (2.7), and (2.8), we obtain the following corollary.

Corollary 2.3 *For $n \geq 0$, we have*

$$D_{n,\xi,q}(x) = \xi^n \sum_{l=0}^{n} q^{n-l} S_1(n,l) B_l(x) = \xi^n \sum_{l=0}^{n} \left|S_1(n,l)\right| (-q)^{n-l} B_l(x).$$

From now on, we consider *twisted Daehee polynomials of order $k \in \mathbb{N}$ with q-parameter*. Twisted Daehee polynomials of order $k \in \mathbb{N}$ with q-parameter are defined by the multivariant p-adic invariant integral on $\mathbb{Z}_p$:

$$D_{n,\xi,q}^{(k)}(x) = \xi^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)_{n,q} \, d\mu_0(x_1) \cdots d\mu_0(x_k), \tag{2.9}$$

where n is a nonnegative integer and $k \in \mathbb{N}$. In the special case, $x = 0$, $D_{n,\xi,q}^{(k)} = D_{n,\xi,q}^{(k)}(0)$ are called the *Daehee numbers of order k with q-parameter*.

From (2.9), we can derive the generating function of $D_{n,\xi,q}^{(k)}(x)$ as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} D_{n,\xi,q}^{(k)}(x) \frac{t^n}{n!} \\
&\quad = \sum_{n=0}^{\infty} \xi^n q^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{\frac{x_1+\cdots+x_k+x}{q}}{n} d\mu_0(x_1) \cdots d\mu_0(x_k) t^n \\
&\quad = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+q\xi t)^{\frac{x_1+\cdots+x_k+x}{q}} \, d\mu_0(x_1) \cdots d\mu_0(x_k) \\
&\quad = (1+q\xi t)^{\frac{x}{q}} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+q\xi t)^{\frac{x_1+\cdots+x_k}{q}} \, d\mu_0(x_1) \cdots d\mu_0(x_k) \\
&\quad = (1+q\xi t)^{\frac{x}{q}} \left(\frac{\log(1+q\xi t)}{q((1+q\xi t)^{\frac{1}{q}} - 1)} \right)^k.
\end{aligned} \tag{2.10}$$

Note that, by (2.9),

$$D_{n,\xi,q}^{(k)}(x) = \xi^n q^n \sum_{m=0}^{n} \frac{S_1(n,m)}{q^m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)^m \, d\mu_0(x_1) \cdots d\mu_0(x_k). \tag{2.11}$$

Since

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_k+x)t} \, d\mu_0(x_1) \cdots d\mu_0(x_k) = \left(\frac{t}{e^t - 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!},$$

we can derive easily

$$B_n^{(k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)^n \, d\mu_0(x_1) \cdots d\mu_0(x_k). \tag{2.12}$$

Thus, by (2.11) and (2.12), we have

$$\begin{aligned} D_{n,\xi,q}^{(k)}(x) &= \xi^n q^n \sum_{m=0}^{n} \frac{S_1(n,m)}{q^m} B_m^{(k)}(x) \\ &= \xi^n \sum_{m=0}^{n} q^{n-m} S_1(n,m) B_m^{(k)}(x) \\ &= \xi^n \sum_{m=0}^{n} \left|S_1(n,m)\right| (-q)^{n-m} B_m^{(k)}(x). \end{aligned} \tag{2.13}$$

In (2.10), by replacing t by $\frac{1}{q\xi}(e^{\xi t}-1)$, we get

$$\sum_{n=0}^{\infty} D_{n,\xi,q}^{(k)}(x) \frac{(e^{\xi t}-1)^n}{\xi^n q^n n!} = e^{\frac{\xi t x}{q}} \left(\frac{\frac{\xi t}{q}}{e^{\frac{\xi t}{q}}-1} \right)^k = \sum_{n=0}^{\infty} \frac{\xi^n B_n^{(k)}(x)}{q^n} \frac{t^n}{n!} \tag{2.14}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{D_{n,\xi,q}^{(k)}(x)}{\xi^n q^n} \frac{1}{n!} \left(e^{\xi t}-1\right)^n &= \sum_{n=0}^{\infty} \frac{D_{n,\xi,q}^{(k)}(x)}{\xi^n q^n} \sum_{l=n}^{\infty} S_2(l,n) \xi^l \frac{t^l}{l!} \\ &= \sum_{m=0}^{\infty} \left(\xi^m \sum_{n=0}^{m} \frac{D_{n,\xi,q}^{(k)}(x)}{\xi^n q^n} S_2(m,n) \right) \frac{t^m}{m!}. \end{aligned} \tag{2.15}$$

By (2.13), (2.14), and (2.15), we obtain the following theorem.

Theorem 2.4 *For $n \geq 0$ and $k \in \mathbb{N}$, we have*

$$D_{n,\xi,q}^{(k)}(x) = \xi^n \sum_{m=0}^{n} q^{n-m} S_1(n,m) B_m^{(k)}(x) = \xi^n \sum_{m=0}^{n} \left|S_1(n,m)\right| (-q)^{n-m} B_m^{(k)}(x)$$

and

$$B_n^{(k)}(x) = \sum_{m=0}^{n} D_{m,\xi,q}^{(k)}(x) \xi^{-m} q^{n-m} S_2(n,m).$$

Now, we consider the *twisted Daehee polynomials of the second kind with q-parameter* as follows:

$$\hat{D}_{n,\xi,q}(x) = \xi^n \int_{\mathbb{Z}_p} (-y+x)_{n,q} \, d\mu_0(y) \quad (n \geq 0). \tag{2.16}$$

In the special case $x = 0$, $\hat{D}_{n,\xi,q}(0) = \hat{D}_{n,\xi,q}$ are called the *twisted Daehee numbers of the second kind with q-parameter*.

By (2.16), we have

$$\hat{D}_{n,\xi,q}(x) = \xi^n q^n \int_{\mathbb{Z}_p} \left(\frac{-y+x}{q} \right)_n d\mu_0(y), \tag{2.17}$$

and so we can derive the generating function of $\hat{D}_{n,\xi,q}(x)$ by (1.1) as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\hat{D}_{n,\xi,q}(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}q^n\xi^n\int_{\mathbb{Z}_p}\left(\frac{-y+x}{q}\right)_n d\mu_0(y)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}q^n\xi^n\int_{\mathbb{Z}_p}\binom{\frac{-y+x}{q}}{n}d\mu_0(y)t^n \\ &= \int_{\mathbb{Z}_p}(1+q\xi t)^{\frac{-y+x}{q}}\,d\mu_0(y) \\ &= (1+q\xi t)^{\frac{x}{q}}\frac{\log(1+q\xi t)}{q((1+q\xi t)^{\frac{1}{q}}-1)}(1+q\xi t)^{\frac{1}{q}}. \end{aligned}\tag{2.18}$$

From (1.3), (1.6), and (2.17), we get

$$\begin{aligned}\hat{D}_{n,\xi,q}(x) &= q^n\xi^n\int_{\mathbb{Z}_p}\left(\frac{-y+x}{q}\right)_n d\mu_0(y) \\ &= q^n\xi^n\int_{\mathbb{Z}_p}\sum_{l=0}^{n}\frac{S_1(n,l)}{q^l}(-y+x)^l\,d\mu_0(y) \\ &= \xi^n\sum_{l=0}^{n}S_1(n,l)(-1)^l\int_{\mathbb{Z}_p}(y-x)^l\,d\mu_0(y)q^{n-l} \\ &= \xi^n\sum_{l=0}^{n}S_1(n,l)(-1)^lB_l(-x)q^{n-l} \\ &= (-\xi)^n\sum_{l=0}^{n}\left|S_1(n,l)\right|B_l(-x)q^{n-l}. \end{aligned}\tag{2.19}$$

By (1.10), it is easy to show that $B_n(-x) = (-1)^nB_n(x+1)$. Thus, from (2.19), we have the following theorem.

Theorem 2.5 *For $n \geq 0$, we have*

$$\hat{D}_{n,\xi,q}(x) = \xi^n\sum_{l=0}^{n}S_1(n,l)(-1)^lB_l(-x)q^{n-l} = \xi^n\sum_{l=0}^{n}\left|S_1(n,l)\right|B_l(x+1)(-q)^{n-l}.$$

By replacing t by $\frac{1}{q\xi}(e^{\xi t}-1)$ in (2.18), we have

$$\sum_{n=0}^{\infty}\hat{D}_{n,\xi,q}(x)\frac{1}{q^n\xi^n}\frac{(e^{\xi t}-1)^n}{n!} = e^{\frac{\xi t}{q}(x+1)}\frac{\frac{\xi t}{q}}{e^{\frac{\xi t}{q}}-1} = \sum_{n=0}^{\infty}\frac{\xi^nB_n(x+1)}{q^n}\frac{t^n}{n!}\tag{2.20}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\hat{D}_{n,\xi,q}(x)\frac{1}{q^n\xi^n}\frac{(e^{\xi t}-1)^n}{n!} &= \sum_{n=0}^{\infty}\frac{\hat{D}_{n,\xi,q}(x)}{q^n\xi^n}\sum_{m=n}^{\infty}S_2(m,n)\frac{(\xi t)^m}{m!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\hat{D}_{m,\xi,q}(x)S_2(n,m)q^{-m}\xi^{n-m}\right)\frac{t^n}{n!}. \end{aligned}\tag{2.21}$$

By (2.20) and (2.21), we obtain the following theorem.

Theorem 2.6 *For $n \geq 0$, we have*

$$B_n(x+1) = \sum_{m=0}^{n} q^{n-m}\xi^{-m}\hat{D}_{m,\xi,q}(x)S_2(n,m).$$

Now, we consider *higher-order twisted Daehee polynomials of the second kind with q-parameter*. Higher-order twisted Daehee polynomials of the second kind with q-parameter are defined by the multivariant p-adic invariant integral on $\mathbb{Z}_p$:

$$\hat{D}^{(k)}_{n,\xi,q}(x) = \xi^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - \cdots - x_k + x)_{n,q}\, d\mu_0(x_1)\cdots d\mu_0(x_k), \tag{2.22}$$

where n is a nonnegative integer and $k \in \mathbb{N}$. In the special case, $x = 0$, $\hat{D}^{(k)}_{n,\xi,q} = \hat{D}^{(k)}_{n,\xi,q}(0)$ are called the *higher-order twisted Daehee numbers of the second kind with q-parameter*.

From (2.22), we can derive the generating function of $\hat{D}^{(k)}_{n,\xi,q}(x)$ as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \hat{D}^{(k)}_{n,\xi,q}(x)\frac{t^n}{n!} \\
&\quad= \sum_{n=0}^{\infty} \xi^n q^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{\frac{-x_1-\cdots-x_k+x}{q}}{n} d\mu_0(x_1)\cdots d\mu_0(x_k)t^n \\
&\quad= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+q\xi t)^{\frac{-x_1-\cdots-x_k+x}{q}}\, d\mu_0(x_1)\cdots d\mu_0(x_k) \\
&\quad= (1+q\xi t)^{\frac{x+k}{q}} \left(\frac{\log(1+q\xi t)}{q((1+q\xi t)^{\frac{1}{q}}-1)}\right)^k.
\end{aligned} \tag{2.23}$$

By (2.22),

$$\begin{aligned}
&\hat{D}^{(k)}_{n,\xi,q}(x) \\
&\quad= \xi^n q^n \sum_{m=0}^{n} \frac{S_1(n,m)}{q^m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - \cdots - x_k + x)^m\, d\mu_0(x_1)\cdots d\mu_0(x_k) \\
&\quad= \xi^n q^n \sum_{m=0}^{n} \frac{S_1(n,m)}{(-q)^m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k - x)^m\, d\mu_0(x_1)\cdots d\mu_0(x_k) \\
&\quad= \xi^n q^n \sum_{m=0}^{n} \frac{S_1(n,m)}{(-q)^m} B^{(k)}_m(-x) \\
&\quad= \xi^n \sum_{m=0}^{n} q^{n-m}\bigl|S_1(n,m)\bigr|B^{(k)}_m(-x).
\end{aligned} \tag{2.24}$$

From (1.10), we know that $B^{(k)}_n(-x) = (-1)^n B^{(k)}_n(k+x)$. Hence, by (2.24), we obtain the following theorem.

Theorem 2.7 *For $n \geq 0$, we have*

$$\hat{D}^{(k)}_{n,\xi,q}(x) = \xi^n \sum_{m=0}^{n} (-1)^m q^{n-m} S_1(n,m) B^{(k)}_m(-x) = \xi^n \sum_{m=0}^{n} (-1)^m q^{n-m}\bigl|S_1(n,m)\bigr|B^{(k)}_m(x+k).$$

In (2.23), by replacing t by $\frac{1}{q\xi}(e^{\xi t}-1)$, we get

$$\sum_{n=0}^{\infty}\hat{D}_{n,\xi,q}^{(k)}(x)\frac{(e^{\xi t}-1)^n}{\xi^n q^n n!}=e^{\frac{\xi t}{q}(x+k)}\left(\frac{\frac{\xi t}{q}}{e^{\frac{\xi t}{q}}-1}\right)^k=\sum_{n=0}^{\infty}\frac{\xi^n B_n^{(k)}(x+k)}{q^n}\frac{t^n}{n!} \tag{2.25}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\hat{D}_{n,\xi,q}^{(k)}(x)}{\xi^n q^n}\frac{1}{n!}\left(e^{\xi t}-1\right)^n&=\sum_{n=0}^{\infty}\frac{\hat{D}_{n,\xi,q}^{(k)}(x)}{\xi^n q^n}\sum_{l=n}^{\infty}S_2(l,n)\xi^l\frac{t^l}{l!}\\&=\sum_{n=0}^{\infty}\left(\xi^n\sum_{m=0}^{n}\frac{\hat{D}_{m,\xi,q}^{(k)}(x)}{\xi^m q^m}S_2(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.26}$$

By (2.25) and (2.26), we obtain the following theorem.

Theorem 2.8 *For $n \geq 0$ and $k \in \mathbb{N}$, we have*

$$B_n^{(k)}(x+k)=\sum_{m=0}^{n}\hat{D}_{m,\xi,q}^{(k)}(x)\xi^{-m}q^{n-m}S_2(n,m).$$

Competing interests
The author declares that they have no competing interests.

Author's contributions
The author contributed to the manuscript and typed, read, and approved the final manuscript.

Acknowledgements
The author is grateful for the valuable comments and suggestions of the referees. This paper was supported by the Sehan University Research Fund in 2014.

Received: 16 September 2014 Accepted: 17 November 2014 Published: 02 Dec 2014

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10.1186/1687-1847-2014-304
Cite this article as: Park: **On the twisted Daehee polynomials with q-parameter.** *Advances in Difference Equations* 2014, 2014:304