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Boundary value problems for nonlinear fractional integro-differential equations: theoretical and numerical results

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Abstract

This article is devoted to both the theoretical and numerical study of boundary-value problems for nonlinear fractional integro-differential equations. Positivity and uniqueness results for the problem are provided and proved. Two monotone sequences of upper and lower solutions which converge uniformly to the unique solution of the problem are constructed using the method of lower and upper solutions. Sufficient numerical examples are discussed to corroborate the theory presented herein.

Keywords: nonlinear fractional integro-differential equations, monotone iterative method, lower and upper solutions.

1 Introduction

In the past few years, there has been a growing interest in the theory and applications of fractional integro-differential equations (FIDEs) due to their importance in many scientific areas such as: viscoelasticity and damping, diffusion and wave propagation, heat conduction in materials, biology, signal processing, telecommunications, physics, and finance (for more details see [1,2], and the references therein).

It is well-known that it is extremely difficult to find exact solutions of FIDEs. Therefore, several numerical methods have been proposed to approximate exact solutions for such problems. Examples of such methods are the Adomian decomposition method [3,4], collocation spline method [5], Variational iteration method and homotopy perturbation method [6], fractional differential transform method [7,8], CAS wavelets [2] and Taylor expansion method [9]. However, for recent work on existence and uniqueness of solutions of different classes of FIDEs, we may refer to [10-13], and the references therein.

In this article we consider a class of boundary value problems for nonlinear FIDEs of the form

$$\mathcal{L}y := D^\alpha y(x) + \int_0^x K(x,t) f(t,y)\, dt + h(x) = 0, \quad x \in I = [0,1], \quad 1 < \alpha < 2, \tag{1.1}$$

subject to

$$y(0) = y_0, y(1) = y_1, \tag{1.2}$$

where $f \in C[I \times \mathbb{R}, \mathbb{R}]$ is a decreasing function, $K \in C[I \times I, \mathbb{R}^+]$ is a positive kernel, $h(x) \in C[I, \mathbb{R}]$ and y_0, $y_1 \in$ •. Here, D^α denotes the fractional differential operator of order α in Caputo's sense and is given by

$$D^\alpha y(x) = \frac{1}{\Gamma(k-\alpha)} \int_0^x (x-t)^{k-\alpha-1} y^{(k)}(t)dt, \tag{1.3}$$

where $k \in$ • and satisfies the relation $k - 1 < \alpha < k$.

The purposes of this article are: (i) to prove the positivity and uniqueness results for the problem, and (ii) to employ the lower and upper solutions method (see [14]) to construct two monotone sequences which converge uniformly to the exact solution of the problem. It is worth mentioning that the present work is partially an extension to the works of [15,16].

The rest of the article is organized as follows: some definitions and preliminary results are presented in Section 2. In Section 3, some relevant theoretical results are presented. In Section 3, we describe the algorithm used to construct two uniformly convergent sequences. In Section 4, numerical examples are discussed to prove the efficiency and the rapid convergence of the present algorithm.

2 Definitions and preliminary results

This section presents some definitions and preliminary results that will be extensively used in this study. We first introduce the Riemann-Liouville definition of fractional derivative operator J_a^α.

Definition 2.1. *The Riemann-Liouville fractional integral operator of order* α *is defined by*

$$J_a^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} y(t)dt$$

where $y \in L_1[a, b]$, *and* $\alpha \in$ •$^+$.

The following lemma is important in our discussion.

Lemma 2.1. *For* $k \in$ •, $\alpha \in$ •$^+$, *if* $k - 1 < \alpha < k$, *and* $y \in L_1[a, b]$ *then*

$$D_a^\alpha J_a^\alpha y(x) = y(x)$$

and

$$J_a^\alpha D_a^\alpha y(x) = y(x) - \sum_{m=0}^{k-1} y^{(m)}(0^+) \frac{(x-a)^m}{m!}$$

where $b > a \geq 0$ *and* $x > 0$.

The definitions of lower and upper solutions for problem (1.1)-(1.2) are given by:

Definition 2.2. *A function* $w \in C^2[I, \bullet]$ *is called a lower solution of*(1.1)-(1.2) *on* I *if*

$$\mathcal{L}w := D^\alpha w + \int_0^x K(x,t)f(t,w)\,dt + h(x) \ge 0, \quad x \in I, \quad 1 < \alpha < 2,$$

with

$$w(0) \le \gamma_0, w(1) \le \gamma_1,$$

and an upper solution, if the reversed inequalities hold.

Definition 2.3. *If* $w, v \in C^2[I, \bullet]$ *are, respectively, lower and upper solutions of* (1.1)-(1.2) *on* I *with* $w(x) \le v(x)$ *for all* $x \in I$, *then we say that* w *and* v *are ordered lower and upper solutions.*

3 Analytical results

In this section we present some analytical results which end with the proof of uniqueness of the solution to (1.1)-(1.2). In the following lemma we introduce a positivity result which is the most important to establish our main results.

Lemma 3.1. *(Positivity result) Let* $Z(x) \in C^2[I, \mathbb{R}]$ *and* $R(x) < 0\ \forall x \in I$. *If* Z *satisfies the inequality*

$$D^\alpha Z(x) + \int_0^x R(t)Z(t)\,dt \le 0, \quad x \in (0,1) \tag{3.1}$$

with

$$Z(0), \ \ Z(1) \ge 0,$$

then $Z(x) \ge 0$, *for all* $x \in I$.

Proof. We use the method of proof by contradiction. Assume that Z has negative values at some points in the interval (0, 1). Since Z is a continuous function on I, then Z must attain its local and absolute minimum at some points $x_0 \in (0, 1)$; i.e., $Z(x) \ge Z(x_0)\ \forall x \in I$ with $Z(x_0) < 0$. From the result of Theorem 2.2 in [17], we have $D^\alpha Z(x_0) \ge 0$. Since $R(x) < 0\ \forall x \in I$, we may apply the weighted mean value theorem for integrals as follows

$$\begin{aligned} D^\alpha Z(x_0) + \int_0^{x_0} R(t)Z(t)\,dt &= D^\alpha Z(x_0) + Z(\mu)\int_0^{x_0} R(t)\,dt, \quad \mu \in (0, x_0) \\ &\ge D^\alpha Z(x_0) + Z(x_0)\int_0^{x_0} R(t)\,dt \\ &> 0, \end{aligned} \tag{3.2}$$

which is a contradiction. Hence, $Z(x) \ge 0\ \forall x \in I$.

Lemma 3.2. *Consider the nonlinear FIDE* (1.1)-(1.2) *with* $f(x, y)$ *be strictly decreasing with respect to* y *and* $K > 0$ *in* D. *Let* w *and* v *be, respectively, any lower and upper solutions to* (1.1)-(1.2), *then* w *and* v *are ordered.*

Proof. We shall prove that $w(x) \le v(x)$ for all $x \in I$. Since w and v are, respectively, lower and upper solutions to (1.1)-(1.2), we have

$$D^{\alpha}w(x) + \int_{0}^{x} K(x,t)f(t,w)\,dt + h(x) \geq 0, \quad x \in I, \tag{3.3}$$

$$w(0) \leq \gamma_0, \quad w(1) \leq \gamma_1, \tag{3.4}$$

$$D^{\alpha}v(x) + \int_{0}^{x} K(x,t)f(t,v)\,dt + h(x) \leq 0, \quad x \in I, \tag{3.5}$$

$$v(0) \geq \gamma_0, \quad v(1) \geq \gamma_1, \tag{3.6}$$

where $1 < \alpha < 2$. Subtracting Equation (3.3) from Equation (3.5) and then applying the mean value theorem on *f*, we obtain

$$D^{\alpha}(v - w) + \int_{0}^{x} K(x,t)\frac{\partial f}{\partial y}(\xi)(v - w)\,\,dt \leq 0, \tag{3.7}$$

where $\xi = \beta v + (1 - \beta)w$, for $\beta \in [0, 1]$. Setting $Z = v - w$ we obtain

$$D^{\alpha}Z(x) + \int_{0}^{x} K(x,t)\frac{\partial f}{\partial y}(\xi)Z\,dt \leq 0,$$

with $Z(0)$, $Z(1) \geq 0$. Since *f* is strictly decreasing with respect to *y*, $\frac{\partial f}{\partial y}(\xi)$ should be negative and, therefore, $K(x,t)\frac{\partial f}{\partial y}(\xi) < 0$. Hence, Lemma (3.1) implies that $Z(x) \geq 0$ for all $x \in I$ as desired.

Lemma 3.3. *(Uniqueness result) Let* $f(x, y)$ *be strictly decreasing with respect to y and* $K > 0$ *in D. If* Y_1 *and* Y_2 *are solutions of the problem* (1.1)-(1.2) *then* $Y_1 = Y_2$.

Proof. Since Y_1 and Y_2 are solutions of (1.1)-(1.2), we have

$$D^{\alpha}Y_1(x) + \int_{0}^{x} K(x,t)f(t,Y_1)\,dt + h(x) = 0, \quad x \in I, \tag{3.8}$$

$$Y_1(0) = \gamma_0, \quad Y_1(1) = \gamma_1, \tag{3.9}$$

$$D^{\alpha}Y_2(x) + \int_{0}^{x} K(x,t)f(t,Y_2)\,dt + h(x) = 0, \quad x \in I, \tag{3.10}$$

$$Y_2(0) = \gamma_0, \quad Y_2(1) = \gamma_1. \tag{3.11}$$

Subtracting Equation (3.8) from Equation (3.10) and then applying the mean value theorem on *f*, we obtain

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$$D^{\alpha}(Y_2 - Y_1) + \int_{0}^{x} K(x,t)\frac{\partial f}{\partial y}(\xi)(Y_2 - Y_1)\,dt = 0, \tag{3.12}$$

where $\xi = \beta Y_2 + (1 - \beta)Y_1$, $0 \le \beta \le 1$. Let $Z = Y_2 - Y_1$, then Equation (3.12) is written as

$$D^{\alpha}Z(x) + \int_0^x K(x,t)\frac{\partial f}{\partial y}(\xi)Z(t)\,dt = 0, \tag{3.13}$$

with $Z(0) = Z(1) = 0$. By applying Lemma (3.1) we conclude that $Z \ge 0$ and $-Z \ge 0$ which means $Y_1 = Y_2$ in I. Thus, the proof is complete.

4 A monotone iterative method

In the results below, we employ the concept of upper and lower solutions to construct two monotone sequences that converge uniformly to the exact solution of problem (1.1)-(1.2).

Theorem 4.1. *Consider that the nonlinear FIDE* (1.1)-(1.2) *with* $f(x, y)$ *is strictly decreasing and* $K > 0$ *in D. Let* $s_0 = w$ *and* $S_0 = v$ *be an initial ordered lower and upper solutions of* (1.1)-(1.2) *on I. Let* s_k *and* S_k *fork* ≥ 1 *be, respectively, the solutions of*

$$-D^{\alpha}s_k + \sigma\int_0^x Ks_k\,dt = \sigma\int_0^x Ks_{k-1}\,dt + \int_0^x Kf(t,s_{k-1})\,dt + h(x), \quad x \in I \tag{4.1}$$

$$s_{k-1}(0) \le s_k(0) \le \gamma_0, \quad s_{k-1}(1) \le s_k(1) \le \gamma_1, \tag{4.2}$$

$$-D^{\alpha}S_k + \sigma\int_0^x KS_k\,dt = \sigma\int_0^x KS_{k-1}\,dt + \int_0^x Kf(t,S_{k-1})\,dt + h(x), \quad x \in I \tag{4.3}$$

$$S_{k-1}(0) \ge S_k(0) \ge \gamma_0, \quad S_{k-1}(1) \ge S_k(1) \ge \gamma_1, \tag{4.4}$$

where $-\sigma \le \frac{\partial f}{\partial y} \le 0$*on* $[s_0, S_0]$. *Then we have*

(i) $\{s_k\}$ *is an increasing sequence of lower solutions to* (1.1)-(1.2) *on I.*
(ii) $\{S_k\}$ *is a decreasing sequence of upper solutions to* (1.1)-(1.2) *on I.*
(iii) $s_k \le S_k$, *for* $k \ge 1$.

Proof.

(i) Since the proof of (ii) is similar to that of (i) we prove only part (i). To show that $\{s_k\}$ is an increasing sequence, it suffices to prove

$$s_k - s_{k-1} \ge 0, \quad \forall k \ge 1. \tag{4.5}$$

To this end, we use the method of mathematical induction. For $k = 1$, Equation (4.1) gives

$$-D^{\alpha}s_1 + \sigma\int_0^x Ks_1\,dt = \sigma\int_0^x Ks_0\,dt + \int_0^x Kf(t,s_0)\,dt + h(x), \quad x \in I, \tag{4.6}$$

with

$$s_0(0) \leq s_1(0) \leq \gamma_0, \quad s_0(1) \leq s_1(1) \leq \gamma_1. \tag{4.7}$$

On the other hand, since $s_0 = w$ represents a lower solution of (1.1)-(1.2), it must satisfies

$$D^\alpha s_0 + \int_0^x K(x,t) f(t, s_0)\, dt + h(x) \geq 0, \quad x \in I, \tag{4.8}$$

with

$$s_0(0) \leq \gamma_0, \quad s_0(1) \leq \gamma_1.$$

Adding (4.6) to (4.8), we obtain

$$D^\alpha (s_1 - s_0) + \int_0^x -\sigma K(x,t)(s_1 - s_0)\, dt \leq 0. \tag{4.9}$$

If we set $Z = s_1 - s_0$ then Equation (4.9) can be written as

$$D^\alpha Z + \int_0^x -\sigma K(x,t) Z\, dt \leq 0, \tag{4.10}$$

with $Z(0)$, $Z(1) \geq 0$. According to Lemma (3.1) we conclude that $Z \geq 0$ in I which implies that $s_1 \geq s_0$. If we assume that the statement (4.5) holds for $k = n$ then we must prove that (4.5) is true for $k = n + 1$. From Equation (4.1), we have

$$\begin{aligned} D^\alpha (s_{n+1} - s_n) - \sigma \int_0^x K(s_{n+1} - s_n)\, dt &= \sigma \int_0^x K(s_{n-1} - s_n)\, dt \\ &+ \int_0^x K(f(t, s_{n-1}) - f(t, s_n))\, dt, \quad x \in I. \end{aligned} \tag{4.11}$$

Applying the mean value theorem on f and then rearranging the terms we obtain

$$D^\alpha Z - \sigma \int_0^x K(x,t) Z\, dt = \sigma \int_0^x K(x,t) \left(\sigma + \frac{\partial f}{\partial y}(t, \xi) \right) (s_{n-1} - s_n)\, dt \leq 0. \tag{4.12}$$

where $Z = s_{n+1} - s_n$ and $\xi = \beta s_n + (1 - \beta) s_{n-1}$, $0 \leq \beta \leq 1$. Since $Z(0)$, $Z(1) \geq 0$, Lemma 3.1 implies that $Z(x) \geq 0$ in I. Hence $s_{n+1} \geq s_n$ in I as desired.

To prove that s_k is a lower solution to (1.1)-(1.2) on I, it suffices to prove that $\mathcal{L} s_k \geq 0$ with $s_k(0) \leq y_0, s_k(1) \leq y_1$. Subtracting $\int_0^x K(x,t) f(t, s_k)\, dt$ from both sides of (4.1) and rearranging the terms, we obtain

$$\mathcal{L} s_k := \sigma \int_0^x K(x,t)(s_k - s_{k-1})\, dt + \int_0^x K(x,t)(f(t, s_k) - f(t, s_{k-1}))\, dt, \tag{4.13}$$

where $\mathcal{L}s_k$ is given by

$$\mathcal{L}s_k = D^{\alpha} s_k + \int_0^x K(x,t) f(t, s_k)\, dt + h(x), \quad x \in I.$$

Applying the mean value theorem on f and rearranging the terms, we obtain

$$\mathcal{L}s_k = \int_0^x K(x,t) \left(\sigma + \frac{\partial f}{\partial y}(\xi) \right) (s_k - s_{k-1})\, dt \geq 0, \tag{4.14}$$

where $\xi = \beta s_1 + (1 - \beta) s_0$, $0 \leq \beta \leq 1$. Notice that, we used the result (4.5). Now, since $\mathcal{L}s_k \geq 0$ with $s_k(0) \leq y_0$, $s_k(1) \leq y_1$, then s_k is a lower solution of (1.1)-(1.2) on I.

(iii) Finally, the proof of (iii) follows directly from Lemma 3.2 since s_k and S_k are, respectively, lower and upper solutions of (1.1)-(1.2) on I.

The following theorem proves the uniform convergence of the sequences $\{s_k\}$ and $\{S_k\}$ that already constructed in the Theorem 4.1.

Theorem 4.2. *Consider that the nonlinear FIDE* (1.1)-(1.2) *with $f(x, y)$ is strictly decreasing and $K > 0$ in D. Let $\{s_k\}$ and $\{S_k\}$ be, respectively, the sequences of lower and upper solutions as constructed in Theorem* 4.1. *If y is the exact solution of* (1.1)-(1.2) *then we have*

(i) $\{s_k\}$ and $\{S_k\}$ converge uniformly to s^* and S^*, respectively, with $s^* \leq y \leq S^*$.
(ii) if the conditions (4.2) *and* (4.4) *are strictly equal, i.e., $s_k(0) = S_k(0) = y_0$ and $s_k(1) = S_k(1) = y_1$ $\forall k \geq 1$ then $s^* = S^* = y$.*

Proof.

(i) The sequence $\{S_k\}$ is monotonically decreasing and bounded below by $s_0 = w$, therefore it is convergent to a continuous function S^*. Also, since the sequence $\{s_k\}$ is monotonically increasing and bounded above by $S_0 = v$, it is convergent to a continuous function s^*. On the other hand, since $\{s_k\}$ and $\{S_k\}$ are sequences of continuous real-valued functions on the compact set $I := [0, 1]$, then Dini's theorem [18] proves that these sequences should converge uniformly to s^* and S^*, respectively. To show that $s^* \leq S^*$, recall part (iii) of Theorem 4.1 then take the limit of both sides as $k \to \infty$; we arrive at

$$s^* = \lim_{k\to\infty} s_k \leq y \leq \lim_{k\to\infty} S_k = S^*,$$

as desired.

(ii) To prove part (ii), it is enough to show that s^* and S^* are solutions to (1.1)-(1.2) since Lemma 2.3 ensures the uniqueness of the solution. Applying the fractional derivative operator J^{α} on Equation (4.1), and imposing the conditions (1.2) we obtain

$$-s_k(x) + y(0) + y'(0)x + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}[\sigma(Ts_k)(s) - \sigma(Ts_{k-1})(s) - (Tf_{k-1})(s) - h(s)]\,ds = 0 \tag{4.15}$$

where

$$(Ts_k)(s) = \int_0^s K(s,t)s_k(t)\,dt \quad \text{and} \quad (Tf_{k-1})(s) = \int_0^s K(s,t)f(t,s_{k-1})dt.$$

Taking the limit of both sides of (4.15) as $k \to \infty$ and using the fact that $\{s_k\}$ converges uniformly to s^* we obtain

$$s^*(x) - y(0) - y'(0)x + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}[(Tf^*)(s) + h(s)]\,ds = 0, \tag{4.16}$$

where

$$(Tf^*)(s) = \int_0^s K(s,t)f(t,s^*)dt.$$

Applying the fractional derivative D^{α} on Equation (4.16) we obtain

$$D^{\alpha}s^*(x) + \int_0^x K(x,t)f(t,s^*)dt + h(x) = 0 \tag{4.17}$$

as desired. Following similar steps to the above, one can verify that S^* is also a solution to (1.1)-(1.2). Now, applying Lemma 2.3 implies that $s^* = S^* = y$. Thus, the proof is complete.

5 Numerical results

In this section we consider two examples to demonstrate the performance and efficiency of the present technique. Notice that, for a given s_0 and S_0 (initial ordered lower and upper solutions of (1.1)-(1.2) on I) we have to solve (4.1)-(4.4) iteratively to obtain the solutions. However, the typical equation for s_k or S_k is a linear FIDE of the form

$$D^{\alpha}G(x) + \alpha\int_0^x K(x,t)G(t)\,dt = F(x) \tag{5.1}$$

with

$$G(0) = y_0 \quad G(1) = y_1, \tag{5.2}$$

where $F(x)$ is known function. Finding exact solutions for (5.1)-(5.2) is, usually, a difficult task. Therefore, we solve them numerically using the collocation spline method, for the details about this algorithm we can refer to [5]. For comparison purposes, Example 5.1 is constructed in such a way that the exact solution is known.

Example 5.1. *Consider the nonlinear FIDE*

$$D^{5/4}y(x) + \int_0^x x(x-t)^2 \frac{1}{(1+y)^2} dt + h(x) = 0, \quad x \in I := [0,1], \tag{5.3}$$

subject to the boundary conditions

$$y(0) = 4, \; y(1) = 1, \tag{5.4}$$

where $h(x)$ *is given by*

$$h(x) = -\frac{1}{135} x(-9x^2 + 30(1 + \log(5))x + (50 - 30x)\log(5 - 3x) - 50\log(5)).$$

Note that the exact solution for this problem is $y(x) = 4 - 3x$.

Obviously, $K = x(x - t)^2$ is a positive on $I \times I$ and the functions $w(x) = 0$ and $v(x) = 4$ form, respectively, initial ordered lower and upper solutions of (5.3)-(5.4) on I.

Further, $f(y) = 1/(1 + y)^2$ is a strictly decreasing function with

$$-1 \le \frac{\partial f}{\partial y} < 0 \;\text{ on }\; [w, v].$$

Hence, we choose $\sigma = 1$. The graphs of s_k and S_k for $k = 0,1, 2, 3$ together with the exact solution y are plotted in Figure 1. Notice that the sequences $\{s_k\}$ and $\{S_k\}$ converge to the exact solution, $y(x)$. To measure the bound of the error (or the approximation error) at each iteration k, we use the L_2-norm defined as

$$E_U^{(k)} = \|S_k(x) - y(x)\|^2 = \int_0^1 (S_k(x) - y(x))^2 dx,$$

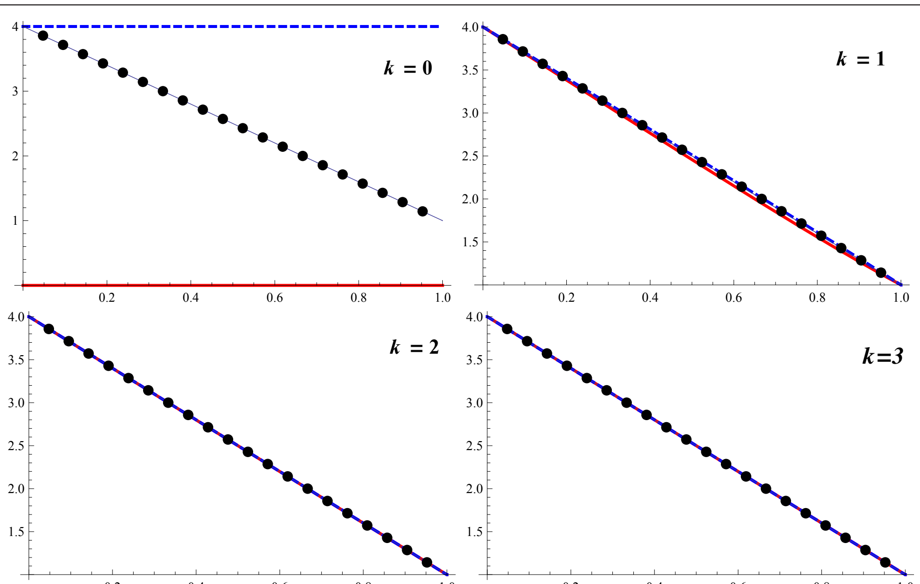

Figure 1 Graphs of s_k, S_k and y (k = 0, 1, 2, 3) for example : s_k (dashed); S_k (solid); y (dotted).

and

$$E_L^{(k)} = \left\| s_k(x) - y(x) \right\|^2 = \int_0^1 (s_k(x) - y(x))^2 dx.$$

Table 1 shows that just after three iterations the errors $E_U^{(k)}$ and $E_L^{(k)}$ are of the order 10^{-11}.

It should be noted that in the subsequent examples, the exact solutions are unknown.

Hence, we measure the bound of the error at each iteration k using the L_2-norm defined as

$$E^{(k)} = \left\| S_k(x) - s_k(x) \right\|^2 = \int_0^1 (S_k(x) - s_k(x))^2 dx. \tag{5.5}$$

This makes sense because, in view of Theorem 4.2, the exact solution is expected to be between the upper and lower solutions.

Example 5.2. *Consider the nonlinear FIDE*

$$D^{3/2}y(x) + \int_0^x K(x,t)e^{-y}dt + h(x) = 0, \quad x \in I := [0,1], \tag{5.6}$$

subject to the boundary conditions

$$y(0) = 2, \; y(1) = 0, \tag{5.7}$$

where $K(x, t) = (3 - x - t)^2$ *and* $h(x) = -\sin x$.

It can be easily verified that the functions $w(x) = 0$ and $v(x) = 3$ form, respectively, initial ordered lower and upper solutions of (5.6)-(5.7) on I. Obviously, $K > 0$ in $I \times I$ and $f(y) = e^{-y}$ is a strictly decreasing function with

$$-1 \le \frac{\partial f}{\partial y} < 0 \;\; \text{on} \;\; [w, v].$$

Therefore, by choosing $\sigma = 1$, Theorem 4.1 applies. Figure 2 clearly shows the convergence of the sequences s_k and S_k. Table 2 displays approximate error bounds for E_k as defined by (5.5).

Example 5.3. *Consider the nonlinear FIDE*

$$D^{7/4}y(x) + \int_0^x e^{x-t}(1 - y^2 \sin(y))\, dt + h(x) = 0, \quad x \in I = [0,1], \tag{5.8}$$

subject to the boundary conditions

$$y(0) = 1, \; y(1) = 2, \tag{5.9}$$

Table 1 Error bounds E_k (k = 0, 1, 2, 3) for Example 5.3

k	0	1	2	3
$E_U^{(k)}$	3	7.73889×10^{-5}	3.18565×10^{-9}	1.38462×10^{-11}
$E_L^{(k)}$	7	1.18018×10^{-3}	6.51391×10^{-8}	3.43986×10^{-11}

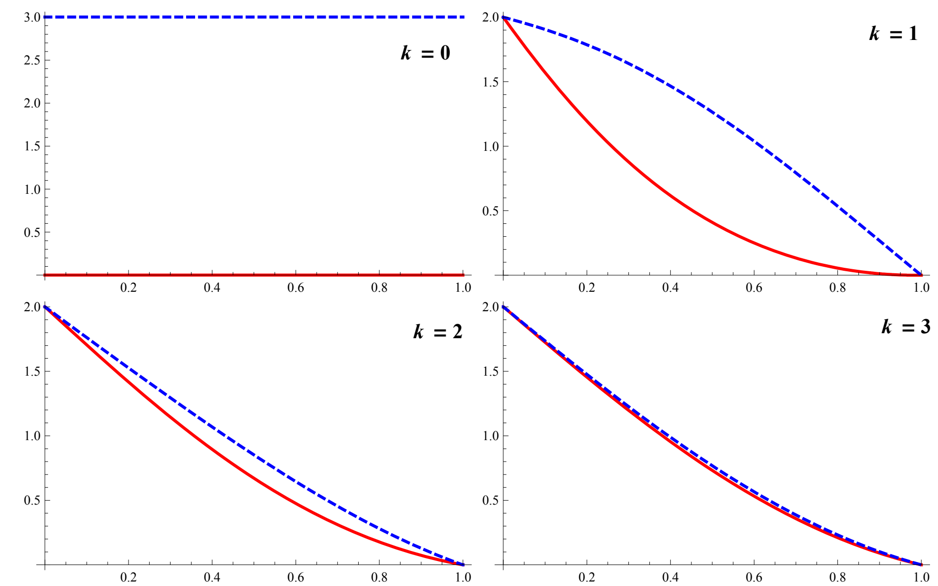

Figure 2 Graphs of s_k and S_k (k = 0, 1, 2, 3) for Example 5.2: s_k (solid); S_k (dashed).

Table 2 Error bounds E_k (k = 0, 1, 2, 3, 4) for Example 5.2

k	0	1	2	3	4
E_k	9	0.385123	0.0163607	6.47412×10^{-4}	2.51767×10^{-5}

where $K(x, t) = e^{x-t}$ *and* $h(x) = \frac{1}{4}(-1 + e^x)(-4 + \sin(\frac{1}{2}))$.

The functions $w(x) = 0.5$ and $v(x) = 2$ are, respectively, initial ordered lower and upper solutions of (5.8)-(5.9) on I. Note that K is positive in $I \times I$ and $f = 1 - y^2 \sin(y)$ is a strictly decreasing function with $\sigma = 3.2$. The graphs of s_k and S_k for $k = 0, 1, 2, 3$ are plotted in Figure 3.

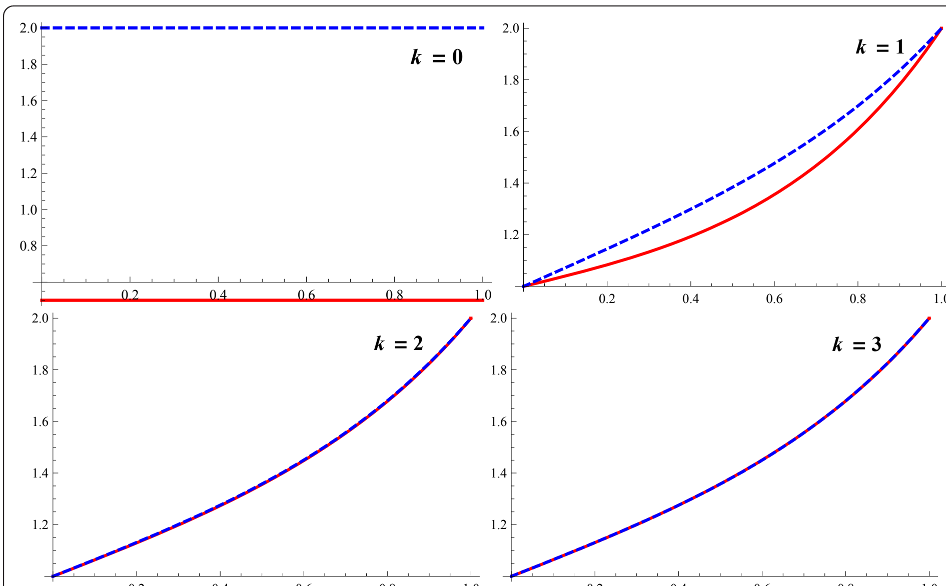

Figure 3 Graphs of s_k and S_k (k = 0, 1, 2, 3) for Example 5.3: s_k (solid); S_k (dashed).

Table 3 Error bounds E_k (k = 0, 1, 2, 3) for Example 5.3

k	0	1	2	3
E_k	2.25	7.6249×10^{-3}	3.82866×10^{-6}	1.70471×10^{-9}

The computed error bounds using (5.5) are also presented in Table 3. It is shown that the lower and upper approximations converge with an error, approximately, of order 10^{-9} just after three iterations.

6 Conclusion

In this article, the boundary value problems for nonlinear FIDEs are discussed theoretically and numerically. Theoretically, we proved the positivity and uniqueness results for the problem. On the other hand, we utilized the monotone iterative method to construct two monotone sequences of upper and lower solutions which converge uniformly to the exact solution of the problem. Numerical examples have demonstrated the efficiency of the proposed algorithm.

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Competing interests
The authors declare that they have no competing interests.

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