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Chaos in a single-species discrete population model with stage structure and birth pulses

Hui Fang*

*Correspondence:
kmustfanghui@hotmail.com
Department of Mathematics, Kunming University of Science and Technology, Kunming, Yunnan 650500, China

Abstract

This paper gives an analytical proof of the existence of chaotic dynamics for a single-species discrete population model with stage structure and birth pulses. The approach is based on a general existence criterion for chaotic dynamics of *n*-dimensional maps and inequality techniques. An example is given to illustrate the effectiveness of the result.

1 Introduction

Many papers have been published on chaos in discrete models (see [1–11] and references cited therein). However, in most cases, chaotic behaviors they observed were obtained only by numerical simulations and have not been proved rigorously. In 2005, Gao and Chen [10] proposed a single-species discrete population model with stage structure and birth pulses:

$$\begin{cases} u_{n+1} = ru_n + be^{-(r+p)u_n - qv_n}(pu_n + qv_n), \\ v_{n+1} = pu_n + qv_n, \end{cases} \tag{1.1}$$

where $0 < r < 1$, $b > 0$, $p > 0$, $0 < q < 1$. System (1.1) describes the numbers of immature population and mature population at a pulse in terms of values at the previous pulse. They proved numerically that system (1.1) can be chaotic.

Since numerical simulations may lead to erroneous conclusions, numerical evidence of the existence of chaotic behaviors still needs to be confirmed analytically. Some researchers proved analytically the existence of chaotic behavior of discrete systems under different definitions of chaos (for example, see [12–17]). Recently, Liz and Ruiz-Herrera [12] established a general existence criterion for chaotic dynamics of n-dimensional maps under a new definition of chaos, and they applied it to prove analytically the existence of chaotic dynamics in some classical discrete-time age-structured population models. This novel analytical approach is very effective in detecting chaos of discrete-time dynamical systems.

The main purpose of this paper is to give an analytical proof of the existence of chaotic dynamics of (1.1). To this end, we use the analytical approach for detecting chaos developed by Liz and Ruiz-Herrera [12].

The rest of the paper is organized as follows. In Section 2, some basic definitions and tools are introduced. In Section 3, it is rigorously proved that there exists chaotic behavior in the discrete population model (1.1). Finally, our conclusions are presented in Section 4.

2 Preliminaries

For the reader's convenience, we give a brief introduction to the main tools and definitions that we use in this paper. For more details, we refer the reader to [12].

In this paper, we denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ the set of all positive integers, integers, and real numbers, respectively.

Definition 2.1 [12] Consider (X, d) a metric space. We say that a continuous map $\psi : X \to X$ induces chaotic dynamics on two symbols if there exist two disjoint compact sets $K_0, K_1 \subset X$ such that, for each two-sided sequence $(s_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$, there exists a corresponding sequence $(\omega_i)_{i\in\mathbb{Z}} \in (K_0 \cup K_1)^{\mathbb{Z}}$ such that

$$\omega_i \in K_{s_i} \quad \text{and} \quad \omega_{i+1} = \psi(\omega_i) \quad \text{for all } i \in \mathbb{Z}, \tag{2.1}$$

and, whenever $(s_i)_{i\in\mathbb{Z}}$ is a k-periodic sequence (that is, $s_{i+k} = s_i$, $\forall i \in \mathbb{Z}$) for some $k \geq 1$, there exists a k-periodic sequence $(\omega_i)_{i\in\mathbb{Z}} \in (K_0 \cup K_1)^{\mathbb{Z}}$ satisfying (2.1).

The following basic facts are listed in [12]:

1. Definition 2.1 guarantees natural properties of complex dynamics, such as sensitive dependence on the initial conditions or the presence of an invariant set Λ being transitive and semi-conjugate with the Bernoulli shift, the existence of periodic points of any period $n \in \mathbb{N}$.
2. A map that is chaotic according to Definition 2.1 is also chaotic in the sense of Block and Coppel [18] and in the sense of coin-tossing [19, 20].

We understand chaos in the sense of Definition 2.1. A map that is chaotic according to Definition 2.1 is called chaotic in the sense of Liz and Ruiz-Herrera.

We employ the usual maximum norm in $\mathbb{R}^n$,

$$\left\|(x_1, x_2, \ldots, x_n)\right\| = \max\left\{|x_i| : i = 1, 2, \ldots, n\right\},$$

and use the notation $J_n = [-1, 1]^n$ for the closed cube centered at $0 \in \mathbb{R}^n$.

Definition 2.2 [12] An h-set is a quadruple consisting of

- a compact subset N of $\mathbb{R}^n$,
- a pair of numbers $u = u(N), s = s(N) \in \{0, 1, 2, , \ldots\}$, with $u + s = n$,
- a homeomorphism $c_N : \mathbb{R}^n \to \mathbb{R}^n$, such that $c_N(N) = J_n$.

In this setting, we employ the notation

$$N_c^- = \partial J_u \times J_s,$$
$$N_c^+ = J_u \times \partial J_s.$$

Definition 2.3 [12] Assume that N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f : N \to \mathbb{R}^n$ be a continuous map, and define $f_c = c_M \circ f \circ c_N^{-1} : J_n \to \mathbb{R}^n$. We say that N f-covers M, and we write it as

$$N \overset{f}{\Longrightarrow} M,$$

if the following conditions are satisfied:

1. There exists a continuous homotopy $H : [0,1] \times J_n \to \mathbb{R}^n$, such that the following conditions hold true:

$$\begin{aligned}
&H(0,\cdot) = f_c(\cdot),\\
&H\bigl([0,1], N_c^-\bigr) \cap J_n = \emptyset,\\
&H\bigl([0,1], J_n\bigr) \cap M_c^+ = \emptyset.
\end{aligned}$$

2. There exists a linear map $A : \mathbb{R}^u \to \mathbb{R}^u$, such that $H(1,(p,q)) = (Ap,0)$ for $p \in J_u$ and $q \in J_s$, and $A(\partial J_u) \subset \mathbb{R}^u \backslash J_u$.

Lemma 2.4 [12] *Let $F : D \to \mathbb{R}^n$ be a continuous map and assume that there exist two disjoint h-sets N_0 and N_1 such that*

$$N_i \quad \overset{f}{\Longrightarrow} \quad N_j$$

for all $i,j = 0,1$. Then F induces chaotic dynamics on two symbols (*with compact sets $\mathcal{K}_0 = N_0$ and $\mathcal{K}_1 = N_1$*).

Definition 2.5 [12] Let I be a real interval and $g : I \to I$ a continuous map. We say that g is δ-strictly turbulent if there exist four constants $\alpha_0 < \alpha_1 < \beta_0 < \beta_1$, and $\delta > 0$ so that

$$\begin{aligned}
&g(\alpha_0) < \alpha_0 - \delta < \beta_1 + \delta < g(\alpha_1),\\
&g(\beta_1) < \alpha_0 - \delta < \beta_1 + \delta < g(\beta_0).
\end{aligned}$$

3 Chaotic dynamics in the model (1.1)

Associated to (1.1), we define the map in $\mathbb{R}^2$

$$F(x,y) = \bigl(F_1(x,y), F_2(x,y)\bigr) = \bigl(rx + b(px+qy)e^{-(r+p)x-qy}, px+qy\bigr),$$

where $0 < r < 1$, $b > 0$, $p > 0$, $0 < q < 1$.

Denote

$$F^2(x,y) = F\bigl(F_1(x,y), F_2(x,y)\bigr) = \bigl(F_1^2(x,y), F_2^2(x,y)\bigr).$$

Set $f(x) = bpxe^{-(r+p)x}$, then one has

$$f^2(x) = f\bigl(f(x)\bigr) = b^2p^2xe^{-(r+p)x}e^{[-(r+p)bpxe^{-(r+p)x}]}.$$

First, we provide a technical lemma, which will play a key role in the proof of the existence of chaotic dynamics.

Lemma 3.1 *The first component $F_1^2(x,y)$ and the second component $F_2^2(x,y)$ of $F^2(x,y)$ satisfy the following inequalities*: *for $x > 0$, $y \geq 0$,*

(a) $F_1^2(x,y) > r^2x + f^2(x) \cdot e^{-[(r+p)r+pq]x} \cdot e^{-[q^2+(r+p)bq+q]y}$;

(b) $F_1^2(x,y) \leq [r^2 + rbp + bp(r+q)]x + [rqb + bq^2 + pqb^2]y + f^2(x)[e^{-\frac{bp}{e}}]^{[e^{-qy}-1]}$;

(c) $0 < F_2^2(x,y) < \frac{pF_1^2(x,y)}{r} + pqx + q^2y$.

Proof The first component of F^2 has the following expression:

$$\begin{aligned} F_1^2(x,y) &= rF_1(x,y) + b\bigl(pF_1(x,y) + qF_2(x,y)\bigr)e^{[-(r+p)F_1(x,y)-qF_2(x,y)]} \\ &= r^2x + rb(px+qy)e^{-(r+p)x-qy} \\ &\quad + b\bigl[prx + pb(px+qy)e^{-(r+p)x-qy} + pqx + q^2y\bigr] \\ &\quad \cdot e^{[-(r+p)[rx+b(px+qy)e^{-(r+p)x-qy}]-q(px+qy)]}. \end{aligned}$$

We easily deduce that, for $x > 0$, $y \geq 0$,

$$\begin{aligned} F_1^2(x,y) &> r^2x + \bigl[b^2p^2xe^{-(r+p)x-qy}\bigr] \cdot e^{[-(r+p)[rx+b(px+qy)e^{-(r+p)x-qy}]-q(px+qy)]} \\ &= r^2x + \bigl[b^2p^2xe^{-(r+p)x-qy}\bigr] \cdot e^{[-(r+p)rx-q(px+qy)]} \\ &\quad \cdot e^{[-(r+p)bpxe^{-(r+p)x-qy}]} \cdot e^{[-(r+p)bqye^{-(r+p)x-qy}]} \\ &\geq r^2x + \bigl[b^2p^2xe^{-(r+p)x-qy}\bigr] \cdot e^{[-(r+p)rx-q(px+qy)]} \\ &\quad \cdot e^{[-(r+p)bpxe^{-(r+p)x}]} \cdot e^{[-(r+p)bqy]} \\ &= r^2x + e^{[-(r+p)rx-q(px+qy)]} \cdot \bigl[b^2p^2xe^{-(r+p)x}e^{[-(r+p)bpxe^{-(r+p)x}]}\bigr] \\ &\quad \cdot e^{[-(r+p)bqy-qy]} \\ &= r^2x + f^2(x) \cdot e^{-[(r+p)r+pq]x} \cdot e^{-[q^2+(r+p)bq+q]y}, \end{aligned}$$

which implies assertion (a) holds.

On the other hand, for $x > 0$ and $y \geq 0$,

$$\begin{aligned} F_1^2(x,y) &\leq r^2x + rbpx + rbqy + be^{[-(r+p)bpxe^{-(r+p)x-qy}]} \\ &\quad \cdot \bigl[p(r+q)x + q^2y + p^2bxe^{-(r+p)x} + pbqy\bigr] \\ &\leq \bigl[r^2 + rbp + bp(r+q)\bigr]x + \bigl[rqb + bq^2 + pqb^2\bigr]y \\ &\quad + b^2p^2xe^{-(r+p)x} \cdot e^{[-(r+p)bpxe^{-(r+p)x} \cdot e^{-qy}]} \\ &= \bigl[r^2 + rbp + bp(r+q)\bigr]x + \bigl[rqb + bq^2 + pqb^2\bigr]y \\ &\quad + f^2(x)\bigl[e^{-(r+p)bpxe^{-(r+p)x}}\bigr]^{[e^{-qy}-1]}. \end{aligned}$$

Now using

$$f(x) = bpxe^{-(r+p)x} \leq \frac{bp}{(r+p)e},$$

we arrive at

$$F_1^2(x,y) \leq \bigl[r^2 + rbp + bp(r+q)\bigr]x + \bigl[rqb + bq^2 + pqb^2\bigr]y + f^2(x)\bigl[e^{-\frac{bp}{e}}\bigr]^{[e^{-qy}-1]},$$

which implies assertion (b) holds.

For the second component $F_2^2(x,y)$, noticing that $axe^{-cx} \leq \frac{a}{ce}$ and

$$F_1^2(x,y) > r^2x + rb(px+qy)e^{-(r+p)x-qy},$$

we obtain, for $x > 0, y \geq 0$,

$$0 < F_2^2(x,y) = prx + pb(px + qy)e^{-(r+p)x-qy} + pqx + q^2y < \frac{pF_1^2(x,y)}{r} + pqx + q^2y,$$

which implies assertion (c) holds. The proof is complete. □

Next, we prove the following result by following the idea of the proof of Theorem 5.1 in [12] with appropriate modifications.

Theorem 3.2 *Assume that* $f(x) = bpxe^{-(r+p)x}$ *satisfies the requirement that* f^2 *is* δ*-strictly turbulent with parameters* $0 < \alpha_0 < \alpha_1 < \beta_0 < \beta_1$ *and* $\delta > 0$. *Suppose that* $r > q$, *and the following inequalities are fulfilled*:

$$-\frac{bp}{e}\bigl[e^{-\frac{pq\beta_1}{r-q}} - 1\bigr] < \ln\left(\frac{\alpha_0 - [r^2 + \frac{2r^2bq+p^2qb^2}{r-q}]\beta_1}{\alpha_0 - \delta}\right), \tag{3.1}$$

$$\left[(r+p)r + \frac{pqr + (r+p)bpq + pq}{r-q}\right]\beta_1 < \ln\left(\frac{\beta_1 + \delta}{\beta_1 - r^2\alpha_1}\right). \tag{3.2}$$

Then F^2 *induces chaotic dynamics on two symbols relative to*

$$N_0 = \left\{(x,y) : \alpha_0 \leq x \leq \alpha_1, 0 \leq y \leq \frac{p}{r-q}x\right\},$$

$$N_1 = \left\{(x,y) : \beta_0 \leq x \leq \beta_1, 0 \leq y \leq \frac{p}{r-q}x\right\}.$$

Proof Set

$$g_0(x,y) = \left(x, \frac{\alpha_1 y}{x}\right), \qquad g_1(x,y) = \left(x, \frac{\beta_1 y}{x}\right).$$

Then we have

$$g_0(N_0) = \left\{(x,y) : \alpha_0 \leq x \leq \alpha_1, 0 \leq y \leq \frac{p\alpha_1}{r-q}\right\},$$

$$g_1(N_1) = \left\{(x,y) : \beta_0 \leq x \leq \beta_1, 0 \leq y \leq \frac{p\beta_1}{r-q}\right\}.$$

From this, it is easy to check that N_0 and N_1 are h-sets, with

$$u(N_0) = u(N_1) = 1 \quad (x\text{-direction}), \qquad s(N_0) = s(N_1) = 1 \quad (y\text{-direction})$$

and

$$c_{N_0} = h_0 \circ t_v \circ g_0, \qquad c_{N_1} = h_1 \circ t_w \circ g_1,$$

where t_v and t_w are the translations according to the vectors

$$v = \left(-\frac{\alpha_0 + \alpha_1}{2}, -\frac{\alpha_1 p}{2(r-q)}\right), \qquad w = \left(-\frac{\beta_0 + \beta_1}{2}, -\frac{\beta_1 p}{2(r-q)}\right),$$

respectively, and

$$h_0(x,y) = \left(\frac{2x}{\alpha_1-\alpha_0}, \frac{2(r-q)y}{\alpha_1 p}\right), \qquad h_1(x,y) = \left(\frac{2x}{\beta_1-\beta_0}, \frac{2(r-q)y}{\beta_1 p}\right).$$

In order to apply Lemma 2.4, it suffices to demonstrate that

$$N_i \quad \overset{F^2}{\Longrightarrow} \quad N_j$$

for $i, j = 0, 1$.

We give the proof only for the case $i = 0$. Indeed, consider the homotopy

$$H_j\bigl(t,(x,y)\bigr) = (1-t)\bigl(c_{N_j} \circ F^2 \circ c_{N_0}^{-1}\bigr)(x,y) + tA(x,y) \quad (j = 0,1),$$

where $A(x,y) = (2x, 0)$.

Define $f(x) = bpxe^{-(r+p)x}$. Then it follows from (a) and (c) of Lemma 3.1 that, for all $(x,y) \in N_i$ $(i = 0,1)$,

$$\begin{aligned} 0 &< F_2^2(x,y) < \frac{pF_1^2(x,y)}{r} + pqx + q^2 y \\ &\leq \frac{pF_1^2(x,y)}{r} + pqx + \frac{pq^2}{r-q}x \\ &= \frac{pF_1^2(x,y)}{r} + \frac{rpq}{r-q}x \\ &\leq \frac{pF_1^2(x,y)}{r} + \frac{pqF_1^2(x,y)}{r(r-q)} \\ &= \frac{p}{r-q}F_1^2(x,y). \end{aligned} \tag{3.3}$$

As $f^2(\alpha_0) < \alpha_0 - \delta$, from (b) of Lemma 3.1, we obtain, for all $y \in [0, \frac{p\beta_1}{r-q}]$,

$$\begin{aligned} F_1^2(\alpha_0,y) &\leq \bigl[r^2 + rbp + bp(r+q)\bigr]\alpha_0 + \frac{[rqb + bq^2 + pqb^2]p\beta_1}{r-q} + (\alpha_0-\delta)\bigl[e^{-\frac{bp}{e}}\bigr]^{e^{-[\frac{pq\beta_1}{r-q}]}-1} \\ &< \bigl[r^2 + rbp + bp(r+q)\bigr]\beta_1 + \frac{[rqb + bq^2 + pqb^2]p\beta_1}{r-q} + (\alpha_0-\delta)\bigl[e^{-\frac{bp}{e}}\bigr]^{e^{-[\frac{pq\beta_1}{r-q}]}-1} \\ &= \left[r^2 + \frac{2r^2bq + p^2qb^2}{r-q}\right]\beta_1 + (\alpha_0-\delta)\bigl[e^{-\frac{bp}{e}}\bigr]^{e^{-[\frac{pq\beta_1}{r-q}]}-1}. \end{aligned} \tag{3.4}$$

By using (3.1) together with the inequality $\alpha_0 < \beta_0$, we get from (3.4), for all $y \in [0, \frac{p\beta_1}{r-q}]$,

$$F_1^2(\alpha_0,y) < \alpha_0 < \beta_0. \tag{3.5}$$

Analogously, as $f^2(\alpha_1) > \beta_1 + \delta$, from (a) of Lemma 3.1, we obtain, for all $y \in [0, \frac{p\beta_1}{r-q}]$,

$$\begin{aligned} F_1^2(\alpha_1,y) &> r^2\alpha_1 + f^2(\alpha_1) \cdot e^{-[(r+p)r+pq]\alpha_1} \cdot e^{-[q^2+(r+p)bq+q]\frac{p\beta_1}{r-q}} \\ &> r^2\alpha_1 + (\beta_1+\delta)e^{-[(r+p)r+\frac{pqr+(r+p)bpq+pq}{r-q}]\beta_1}. \end{aligned} \tag{3.6}$$

By using (3.2) together with the inequality $\beta_1 > \alpha_1$, we get from (3.6), for all $y \in [0, \frac{p\beta_1}{r-q}]$,

$$F_1^2(\alpha_1, y) > \beta_1 > \alpha_1. \tag{3.7}$$

From inequalities (3.3), (3.5), and (3.7), it follows that, for $j = 0, 1$,

$$\begin{aligned}
&c_{N_j} \circ F^2 \circ c_{N_0}^{-1}\big(\{-1\} \times [-1,1]\big) \subset \big\{(x,y) : x < -1\big\}, \\
&c_{N_j} \circ F^2 \circ c_{N_0}^{-1}\big(\{1\} \times [-1,1]\big) \subset \big\{(x,y) : x > 1\big\}, \\
&c_{N_j} \circ F^2 \circ c_{N_0}^{-1}\big([-1,1]^2\big) \subset \big\{(x,y) : -1 < y < 1\big\}.
\end{aligned}$$

These properties, together with the expression of A, lead to the conclusion

$$\begin{aligned}
&H_j\big([0,1], \{-1,1\} \times [-1,1]\big) \cap [-1,1]^2 = \emptyset \quad (j = 0, 1), \\
&H_j\big([0,1], [-1,1]^2\big) \cap \big([-1,1] \times \{-1,1\}\big) = \emptyset \quad (j = 0, 1).
\end{aligned}$$

This leads to the covering relations

$$N_0 \quad \overset{F^2}{\Longrightarrow} \quad N_j \quad (j = 0, 1).$$

It can be similarly verified for the covering relations

$$N_1 \quad \overset{F^2}{\Longrightarrow} \quad N_j \quad (j = 0, 1),$$

taking the linear map $A(x, y) = (-2x, 0)$. The proof is complete. □

Now we apply Theorem 3.2 in a particular example.

Example 3.3 Take $f(x) = bpxe^{-(r+p)x}$ with $bp = \exp(3.2)$ and $r + p = 0.6$, then f^2 is 0.545-strictly turbulent with parameters $2.5 < 7.5 < 8.5 < 13.9$. Straightforward computations show that conditions (3.1), (3.2) in Theorem 3.2 hold for

$$r = 0.003, \qquad p = 0.597, \qquad q = 1.5 \times 10^{-7}, \qquad b = \exp(3.2)/p.$$

4 Conclusions

This paper rigorously proves the existence of chaotic dynamics for a single-species discrete population model with stage structure and birth pulses. The result shows that the second composition map of a two-dimensional map associated to this model is chaotic in the sense of Liz and Ruiz-Herrera.

Competing interests

The author declares that he has no competing interests.

Author's contributions

The author contributed to the manuscript and read and approved the final manuscript.

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