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# The $C^0$ solutions of the Feigenbaum-like functional equation

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**Abstract**

By using the Schauder's fixed point theorem, and constructing the special functional space and the construction operator, the existence, uniqueness, quasi-convexity (or quasi-concavity), symmetry and stability of the  $C^0$  solutions of the Feigenbaum-like functional equations are discussed.

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**Keywords:** Feigenbaum-like functional equations; quasi-convex (or quasi-concave); symmetry; stability

## 1 Introduction

As early as 1978, Feigenbaum found the period-doubling bifurcation phenomenon by researching the iteration of a single-peak function class [1]. To reveal the mechanism of the Feigenbaum phenomenon, many years ago, the Feigenbaum functional equation had been researched extensively. McCarthy [2] obtained the general continuous exact bijective solutions. Epstein [3] gave a new proof of the existence of analytic, unimodal solutions by taking advantage of the normality properties of Herglotz functions and the Schauder-Tikhonov theorem. Eokmann and Wittwer [4] studied the Feigenbaum fixed point by using the computer. Thompson [5] investigated an essentially singular solution by expressing Feigenbaum's equation as a singular Schroder functional equation whose solution was obtained using a scaling ansatz, and so on. Thus, some solutions in specific cases were found.

Specifically, in 1985, to give a feasible method, the second kind of the Feigenbaum functional equation,

$$\begin{cases} f(x) = \frac{1}{\lambda}f(f(\lambda x)), & 0 < \lambda < 1, \\ f(0) = 1, \\ 0 \leq f(x) \leq 1, & x \in [0, 1] \end{cases}$$

a kind of the equivalent equation, was given by Yang and Zhang [6]. The continuous valley-unimodal solutions were shown by using the constructive method. Recently, there have been a lots of results about the polynomial-like iterative equation. In 1987, by using Schauder's fixed point theorem to an operator defined by a linear combination of iterates of the unknown mapping  $f$ , a result on the existence of continuous solutions of the polynomial-like iterative equation was given in [7]. Furthermore, the results were given

for its differentiable solutions [8]. Then the convex solutions and concave solutions [9, 10], the analytic solutions [11–13], the symmetric solutions [14], the higher-dimensional solutions [15], and the results on the unit circle [16] were obtained. In order to understand the dynamics of a second order delay differential equation with a piecewise constant argument, the derived planar mappings and their invariant curves were studied [17]. Based on the iterative root theory for monotone functions, an algorithm for computing polygonal iterative roots of increasing polygonal functions was given [18]. Else, a problem about the Hyers-Ulam stability was raised first by Ulam in 1940 and solved for Cauchy equation by Hyers [19]. Later, many papers on the Hyers-Ulam stability have been published, especially, for the polynomial-like iterative equation [20–22].

In this paper, by using Schauder’s fixed point theorem, and constructing the special functional space and the construction operator, we consider the properties of the solutions of the Feigenbaum-like functional equation, which is a non-extended iterative equation,

$$\begin{cases} f(x) = \frac{1}{\lambda+1}f^2(\lambda x) + \frac{\lambda}{\lambda+1}g(x), & 0 < \lambda < 1, \\ a \leq f(x) \leq b, & x \in I, \end{cases} \quad (1.1)$$

where  $g(x)$  is a given disturbance function,  $f(x)$  is an unknown function, and  $f^2(x) = f(f(x))$ ,  $I = [a, b]$ . It is clear that  $a \leq 0 \leq b$ , since  $\lambda x \in I$  for all  $x \in I$ . We give not only the existence of continuous solutions of (1.1) but also their uniqueness, stability (the continuous dependence and the Hyers-Ulam stability), quasi-convexity (or quasi-concavity), symmetry by applying fixed point theorems. Finally, we give an example to verify those conditions given in theorems.

## 2 Preliminary

In this section, we give several important definitions, lemmas and notions.

Let  $C^0(I, \mathbf{R}) = \{f : I \rightarrow \mathbf{R}, f \text{ is continuous}\}$ . Obviously,  $C^0(I, \mathbf{R})$  is a Banach space with the norm  $\|\cdot\|_{C^0}$ , where the norm  $\|f\|_{C^0} = \max_{x \in I} |f(x)|$  for  $f \in C^0(I, \mathbf{R})$ .

Let  $C^0(I) = \{f \in C^0(I, \mathbf{R}) : a \leq f(x) \leq b, a \leq f(\lambda x) \leq b, f \text{ is continuous}\}$ . Then  $C^0(I)$  is a complete metric space.

Let  $X(I; M) = \{f \in C^0(I) : |f(x) - f(y)| \leq M|y - x|, \forall x, y \in I\}$ , where  $M$  is a positive constant.

Let  $X(I; m, M) = \{f \in X(I; M) : |f(x) - f(y)| \geq m|x_2 - x_1|, \forall x \in I, 0 < m \leq M\}$ , where  $m$  is a positive constant.

Let  $f(\lambda x) := f^{(\lambda)}(x), f^2(\lambda x) := f(f(\lambda x)) := f^{2(\lambda)}(x), f^k(\lambda x) := f^{k(\lambda)}(x)$ .

**Definition 2.1**  $f : I \rightarrow R$  is a quasi-convex (or quasi-concave) function [23] if for  $\forall x, y \in I$  and  $\lambda \in [0, 1]$ , we have

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\} \quad (\text{or } f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}).$$

Let  $X_\sigma(I; M)$  denote the families consisting of all quasi-convex functions or quasi-concave ones in  $X(I; M)$ , where  $\sigma = qcv$  or  $\sigma = qcc$ .

The following Lemma 2.1 and Lemma 2.2 are useful, and the methods of their proofs are similar to ones in the paper [24].

**Lemma 2.1**  $X(I; M)$ ,  $X_{qcv}(I; M)$ , and  $X_{qcc}(I; M)$  are compact convex subsets of  $C^0(I, R)$ .

**Lemma 2.2** The composition  $f \circ g$  is quasi-convex (or quasi-concave) if  $f$  is increasing and  $g$  is quasi-convex (or quasi-concave). In particular, for an increasing quasi-convex (or quasi-concave) function  $f$ ,  $f^k$  is also quasi-convex (or quasi-concave).

**Lemma 2.3** If  $f, h \in X(I; M)$ , then

$$\|f^{2(\lambda)} - h^{2(\lambda)}\|_{c^0} \leq (M + 1)\|f - h\|_{c^0}. \tag{2.1}$$

*Proof* Note that

$$\begin{aligned} \|f^{2(\lambda)} - h^{2(\lambda)}\|_{c^0} &= \max_{x \in I} |f^2(\lambda x) - h^2(\lambda x)| \\ &\leq \max_{x \in I} |f(f(\lambda x)) - f(h(\lambda x))| + \max_{x \in I} |f(h(\lambda x)) - h(h(\lambda x))| \\ &\leq M\|f^{(\lambda)} - h^{(\lambda)}\|_{c^0} + \|f - h\|_{c^0}. \end{aligned}$$

Let  $y = \lambda x$ . Then  $y \in \lambda I$ , and

$$\begin{aligned} \|f^{(\lambda)} - h^{(\lambda)}\|_{c^0} &= \max_{y \in \lambda I} |f(y) - h(y)| \\ &\leq \max_{y \in I} |f(y) - h(y)| \\ &\leq \|f - h\|_{c^0}. \end{aligned}$$

Then

$$\begin{aligned} \|f^{2(\lambda)} - h^{2(\lambda)}\|_{c^0} &\leq M\|f - h\|_{c^0} + \|f - h\|_{c^0} \\ &= (M + 1)\|f - h\|_{c^0}. \end{aligned}$$

Thus, (2.1) holds. □

**Lemma 2.4** Suppose that  $\varphi \in X(I; m, M)$ . If the positive constants  $m, M$  and  $\lambda$  satisfy

$$\frac{1}{\lambda} + 1 > \frac{M^2}{m}, \tag{2.2}$$

then  $L\varphi$ , defined by

$$L\varphi(x) = \left(1 + \frac{1}{\lambda}\right)x - \frac{1}{\lambda}\varphi^2(\lambda\varphi^{-1}(x)), \tag{2.3}$$

is an orientation-preserving homeomorphism from  $I$  onto itself, and

$$(L\varphi)^{-1} \in X\left(I; \frac{1}{\xi_2}, \frac{1}{\xi_1}\right), \tag{2.4}$$

where

$$\xi_1 = 1 + \frac{1}{\lambda} + \frac{M^2}{m}, \quad \xi_2 = 1 + \frac{1}{\lambda} - \frac{M^2}{m} > 0. \tag{2.5}$$

*Proof* Because  $\varphi \in X(I; m, M)$ , by the paper [7],  $\varphi^{-1} \in X(I; \frac{1}{M}, \frac{1}{m})$ . Thus, for any  $x_1 \neq x_2 \in I$ , by (2.3) and (2.5)

$$\begin{aligned} |L\varphi(x_2) - L\varphi(x_1)| &= \left| \left(1 + \frac{1}{\lambda}\right)(x_2 - x_1) + \frac{1}{\lambda}(\varphi^2(\lambda\varphi^{-1}(x_2)) - \varphi^2(\lambda\varphi^{-1}(x_1))) \right| \\ &\geq \left(1 + \frac{1}{\lambda}\right)|x_2 - x_1| - M^2|\varphi^{-1}(x_2) - \varphi^{-1}(x_1)| \\ &\geq \left(1 + \frac{1}{\lambda}\right)|x_2 - x_1| - \frac{M^2}{m}|x_2 - x_1| \\ &\geq \xi_2|x_2 - x_1| > 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} |L\varphi(x_2) - L\varphi(x_1)| &= \left| \left(1 + \frac{1}{\lambda}\right)(x_2 - x_1) + \frac{1}{\lambda}(\varphi^2(\lambda\varphi^{-1}(x_2)) - \varphi^2(\lambda\varphi^{-1}(x_1))) \right| \\ &\leq \left(1 + \frac{1}{\lambda}\right)|x_2 - x_1| + \frac{M^2}{m}|x_2 - x_1| \\ &\leq \xi_1|x_2 - x_1|. \end{aligned}$$

Therefore,  $L\varphi \in X(I; \xi_1, \xi_2)$ . This implies that  $L\varphi$  is strictly increasing and invertible on  $I$ , and  $(L\varphi)^{-1} \in X(I; \frac{1}{\xi_2}, \frac{1}{\xi_1})$ . □

**Lemma 2.5** Suppose that  $g \in X(I; m_1, M_1)$  and  $\varphi_0 \in X(I; m, M)$ . If

$$1 + \frac{1}{\lambda} > \max \left\{ \frac{M^2}{m}, \frac{1}{2} \left( \frac{M_1}{M} + \frac{m_1}{m} \right) \right\}, \tag{2.6}$$

then

$$\varphi_k := (L\varphi_{k-1})^{-1} \circ g \tag{2.7}$$

and

$$L\varphi_{k-1}(x) := \left(1 + \frac{1}{\lambda}\right)x - \frac{1}{\lambda}\varphi_{k-1}^2(\lambda\varphi_{k-1}^{-1}(x)) \tag{2.8}$$

are well defined, and  $\varphi_k \in X(I; m, M)$ ,  $k = 1, 2, \dots$

*Proof* Let

$$L\varphi_0 := \left(1 + \frac{1}{\lambda}\right)x - \frac{1}{\lambda}\varphi_0^2(\lambda\varphi_0^{-1}(x)). \tag{2.9}$$

From Lemma 2.4,  $L\varphi_0$  is well defined and is an orientation-preserving homeomorphism from  $I$  onto itself, and  $(L\varphi_0)^{-1} \in X(I; \frac{1}{\xi_2}, \frac{1}{\xi_1})$ . Then  $\varphi_1(x) := (L\varphi_0)^{-1} \circ g(x)$  is well defined and  $\varphi_1 \in X(I; \frac{m_1}{\xi_2}, \frac{M_1}{\xi_1}) \subset X(I; m, M)$  by (2.6) and Lemma 2.4. If

$$L\varphi_k := \left(1 + \frac{1}{\lambda}\right)x - \frac{1}{\lambda}\varphi_k^2(\lambda\varphi_k^{-1}(x)) \tag{2.10}$$

is well defined and is an orientation-preserving homeomorphism from  $I$  onto itself, then  $(L\varphi_k)^{-1} \in X(I; \frac{1}{\xi_2}, \frac{1}{\xi_1})$ . We similarly see that

$$\varphi_{k+1}(x) := (L\varphi_k)^{-1} \circ g(x) \tag{2.11}$$

is well defined, and

$$\varphi_{k+1} \in X\left(I; \frac{m_k}{\xi_2}, \frac{M_k}{\xi_1}\right) \subset X(I; m, M). \tag{2.12}$$

This implies that the results in Lemma 2.5 are also true for  $k + 1$ , which completes the proof.  $\square$

### 3 Main results

In this section, we give several important theorems on the existence, uniqueness, quasi-convex (quasi-concave), symmetry and stability of equation (1.1).

**Theorem 3.1** (Existence) *Given a positive constant  $M_1$  and  $g(x) \in X(I; M_1)$ . If there exist constants  $M$  and  $\lambda$  such that*

$$\frac{\lambda}{\lambda + 1}(M^2 + M_1) \leq M, \tag{3.1}$$

*then equation (1.1) has a solution  $f$  in  $X(I; M)$ .*

*Proof* Define  $T : X(I; M) \rightarrow C^0(I)$  by

$$Tf(x) = \frac{1}{\lambda + 1}f^2(\lambda x) + \frac{\lambda}{\lambda + 1}g(x), \quad \forall x \in I. \tag{3.2}$$

Because  $f, f(\lambda x)$  and  $g$  are continuous for all  $x \in I$ , we obtain that  $Tf$  is continuous for all  $x \in I$ , and  $Tf \in C^0(I)$ . By (3.1), for any  $x, y$  in  $I$ ,

$$\begin{aligned} |Tf(x) - Tf(y)| &= \left| \frac{1}{\lambda + 1}f^2(\lambda x) + \frac{\lambda}{\lambda + 1}g(x) - \frac{1}{\lambda + 1}f^2(\lambda y) - \frac{\lambda}{\lambda + 1}g(y) \right| \\ &\leq \frac{1}{\lambda + 1}|f^2(\lambda x) - f^2(\lambda y)| + \frac{\lambda}{\lambda + 1}|g(x) - g(y)| \\ &\leq \frac{M}{\lambda + 1}|f(\lambda x) - f(\lambda y)| + \frac{\lambda M_1}{\lambda + 1}|x - y| \\ &\leq \frac{\lambda M^2}{\lambda + 1}|x - y| + \frac{\lambda M_1}{\lambda + 1}|x - y| \\ &= \frac{\lambda}{\lambda + 1}(M^2 + M_1)|x - y| \\ &\leq M|x - y|. \end{aligned}$$

Thus,  $Tf \in X(I, M)$ . Now we prove the continuity of  $T$  under the norm  $\|\cdot\|_{C^0}$ . For arbitrary  $f_1, f_2 \in X(I, M)$ ,

$$\begin{aligned} \|Tf_1 - Tf_2\|_{C^0} &= \max_{x \in I} |Tf_1(x) - Tf_2(x)| \\ &= \max_{x \in I} \left| \frac{1}{\lambda + 1} f_1^2(\lambda x) - \frac{1}{\lambda + 1} f_2^2(\lambda x) \right| \\ &= \frac{1}{\lambda + 1} \max_{x \in I} |f_1^2(\lambda x) - f_2^2(\lambda x)| \\ &= \frac{1}{\lambda + 1} \|f_1^{2(\lambda)} - f_2^{2(\lambda)}\| \\ &\leq E \|f_1 - f_2\|_{C^0}, \end{aligned}$$

where Lemma 2.3 is used and

$$E = \frac{M + 1}{\lambda + 1}. \tag{3.3}$$

Thus,  $T$  is continuous under the norm  $\|\cdot\|_{C^0}$ . Summarizing all the above, we see that  $T$  is a continuous mapping from the compact convex subset  $X(I, M)$  of the Banach space  $C^0(I, R)$  into itself. By Schauder's fixed point theorem, we assert that there is a mapping  $f \in X(I, M)$  such that

$$f(x) = Tf(x) = \frac{1}{\lambda + 1} f^2(\lambda x) + \frac{\lambda}{\lambda + 1} g(x), \quad \forall x \in I. \tag{3.4}$$

This completes the proof. □

**Theorem 3.2** (Uniqueness) *Suppose that (3.1) is satisfied and*

$$M < \lambda. \tag{3.5}$$

*For any function  $g \in X(I, M_1)$ , equation (1.1) has a unique solution  $f \in X(I, M)$ .*

*Proof* The existence of equation (1.1) in  $X(I; M)$  is given by Theorem 3.1. Note that  $X(I; M)$  is a closed subset of  $C^0(I)$ . By (3.3) and (3.5), we see that  $T : X(I; M) \rightarrow X(I; M)$  is a contraction mapping. Therefore  $T$  has a unique fixed point  $f(x)$  in  $X(I; M)$ , that is, equation (1.1) has a unique solution  $f(x)$  in  $X(I; M)$ . □

Below, we discuss the quasi-convex (or quasi-concave) solutions of equation (1.1).

**Definition 3.1** Suppose that  $\Gamma$  is a Lie group of all linear transformations on  $R$ . A mapping  $f : R \rightarrow R$ , is said to be  $\Gamma$ -equivariant [25] if  $f(\gamma x) = \gamma f(x)$ ,  $\forall \gamma \in \Gamma, \forall x \in R$ .

This implies that  $f^i$  is the  $\Gamma$ -equivariant. Let  $G_\Gamma(I) = \{g \in C^0(I) \mid g(\gamma x) = \gamma g(x), \forall \gamma \in \Gamma, \forall x \in I\}$  and  $G_\Gamma(I; M) = G_\Gamma(I) \cap X(I; M)$ .

**Lemma 3.1**  $G_\Gamma(I)$  is a closed convex subset of  $C^0(I)$ , and  $G_\Gamma(I; M)$  is a compact convex subset of  $C^0(I)$ .

The methods of the proofs are similar to the paper [24].

**Theorem 3.3** (Quasi-convexity (Quasi-concavity)) *If  $g \in X_\sigma(I; M_1)$ , (3.1) and (3.5) are satisfied, then (1.1) has a solution  $f \in X_\sigma(I; M)$ .*

*Proof* Define a mapping  $T : X_{qcv}(I; M) \rightarrow C^0(I)$  as in (3.2). Note that each  $f \in X_{qcv}(I; M)$  is an increasing function. In fact, if  $x < y$  in  $I$ , there exists  $t_0 \in (0, 1)$  such that  $x = t_0a + (1 - t_0)y$ , and by the quasi-convexity, we get

$$f(x) \leq \max\{f(a), f(y)\} = f(y). \tag{3.6}$$

Thus, for  $f \in X_{qcv}(I, M)$ ,  $x, y \in I$ , and  $t \in [0, 1]$ , we get

$$\begin{aligned} Tf(tx + (1 - t)y) &= \frac{1}{\lambda + 1}f^2(\lambda(tx + (1 - t)y)) + \frac{\lambda}{\lambda + 1}g(tx + (1 - t)y) \\ &\leq \frac{1}{\lambda + 1}f(\max\{f(\lambda x), f(\lambda y)\}) + \frac{\lambda}{\lambda + 1}\max\{g(x), g(y)\} \\ &\leq \frac{1}{\lambda + 1}\max\{f^2(\lambda x), f^2(\lambda y)\} + \frac{\lambda}{\lambda + 1}\max\{g(x), g(y)\} \\ &\leq \max\{Tf(x), Tf(y)\}. \end{aligned}$$

Thus,  $T$  maps  $X_{qcv}(I, M)$  into itself. Similarly, we can prove that  $T$  is continuous. By Lemma 2.1,  $X_{qcv}(I, M)$  is a compact convex subset of the Banach space  $C^0(I)$ . Then Schauder’s fixed point theorem guarantees the existence of a fixed point  $f$  of  $T$  in  $X_{qcv}(I; M)$ . In the same way, the proof of the quasi-concave solution of equation (1.1) is similarly obtained.  $\square$

Now, we study the symmetric solutions of (1.1).

**Theorem 3.4** (Symmetry) *If (3.1) and (3.5) are satisfied, and  $g \in G_\Gamma(I; M_1)$ , then equation (1.1) has a unique  $\Gamma$ -equivariant solution  $f \in G_\Gamma(I; M_2)$ .*

*Proof* By Lemma 3.1 and (3.2), we have

$$\begin{aligned} Tf(\gamma x) &= \frac{1}{\lambda + 1}f^2(\gamma \lambda x) + \frac{\lambda}{\lambda + 1}g(\gamma x) \\ &= \gamma \frac{1}{\lambda + 1}f^2(\lambda x) + \gamma \frac{\lambda}{\lambda + 1}g(x) \\ &= \gamma Tf(x). \end{aligned}$$

From Theorem 3.1, we can find that  $T$  is a contraction mapping in  $G_\Gamma(I; M_2)$ . Since  $G_\Gamma(I; M_2)$  is a compact convex subset of  $C^0(I)$ , by Banach’s fixed point theorem, we assert that there is a unique fixed point  $f \in G_\Gamma(I; M_2)$ .  $\square$

In the following, we give the conditions to guarantee two kinds of stability: the continuous dependence and the Hyers-Ulam stability [21].

**Theorem 3.5** (Continuous dependence) *If (3.1) and (3.5) are satisfied, the solutions of (1.1) in  $X(I; M)$  is continuously dependent on the given function  $g(x)$  in  $X(I; M_1)$ .*

*Proof* For  $g_1, g_2 \in X(I; M_1)$ , Theorem 3.1 implies that there are functions  $f_1, f_2 \in X(I; M_1)$  such that

$$f_1(x) = \frac{1}{\lambda + 1} f_1^2(\lambda x) + \frac{\lambda}{\lambda + 1} g_1(x), \tag{3.7}$$

$$f_2(x) = \frac{1}{\lambda + 1} f_2^2(\lambda x) + \frac{\lambda}{\lambda + 1} g_2(x). \tag{3.8}$$

Thus, by Lemma 2.3

$$\begin{aligned} \|f_1 - f_2\|_{c^0} &= \max_{x \in I} |f_1(x) - f_2(x)| \\ &\leq \frac{1}{\lambda + 1} \max_{x \in I} |f_1^2(\lambda x) - f_2^2(\lambda x)| + \frac{\lambda}{\lambda + 1} \max_{x \in I} |g_1(x) - g_2(x)| \\ &= \frac{1}{\lambda + 1} \|f_1^{2(\lambda)} - f_2^{2(\lambda)}\|_{c^0} + \frac{\lambda}{\lambda + 1} \|g_1 - g_2\|_{c^0} \\ &\leq \frac{M + 1}{\lambda + 1} \|f_1 - f_2\|_{c^0} + \frac{\lambda}{\lambda + 1} \|g_1 - g_2\|_{c^0}, \end{aligned}$$

which implies

$$\left(1 - \frac{M + 1}{\lambda + 1}\right) \|f_1 - f_2\|_{c^0} \leq \frac{\lambda}{\lambda + 1} \|g_1 - g_2\|_{c^0},$$

so

$$\|f_1 - f_2\|_{c^0} \leq \frac{\lambda}{\lambda - M} \|g_1 - g_2\|_{c^0}. \tag{3.9}$$

Inequality (3.5) yields that the solution of (1.1) in  $X(I; M)$  is continuously dependent on the given function  $g$  in  $X(I; M_1)$ .  $\square$

**Definition 3.2** The functional equation

$$E_1(\varphi) = E_2(\varphi) \tag{3.10}$$

has the Hyers-Ulam stability [26] if for any approximate solution  $\varphi_s$  such as  $\|E_1(\varphi_s) - E_2(\varphi_s)\| \leq \delta$ , there exist a solution  $\varphi$  of equation (3.10) such as  $\|\varphi - \varphi_s\| \leq \varepsilon$ , where  $\delta \geq 0$ ,  $\varepsilon > 0$ , and a constant  $\varepsilon$  dependent only on  $\delta$ .

**Theorem 3.6** (Hyers-Ulam stability) *Suppose that  $g \in X(I; m_1, M_1)$ , and  $\varphi_s \in X(I; m, M)$  satisfy*

$$\left|g(x) - \left(1 + \frac{1}{\lambda}\right)\varphi_s(x) + \frac{1}{\lambda}\varphi_s^2(\lambda x)\right| \leq \delta, \quad \forall x \in I, \tag{3.11}$$

where  $\delta > 0$  is a positive constant. If (2.6), (3.1) and (3.5) are satisfied, there exists a unique continuous solution  $\varphi \in X(I; m, M)$  of (1.1) such that

$$|\varphi(x) - \varphi_s(x)| \leq \zeta \delta, \quad \forall x \in I, \tag{3.12}$$

where  $\zeta = \frac{1}{\xi_1(1-\eta)}$ ,  $\eta = \frac{M+1}{\lambda\xi_1} + \frac{M^2}{m_1} < 1$ .



*Proof* Construct a sequence  $\{\varphi_k\}$  of functions as follows. Take  $\varphi_0 = \varphi_s$ , and then define  $\varphi_k$  as in (2.7) and  $L\varphi_k$  as in (2.8). By Lemma 2.4, both  $\varphi_k$  and  $L\varphi_k$  are well defined for any integer  $k \geq 1$ . Lemma 2.4 and Lemma 2.5 also imply that  $\varphi_k$  or  $L\varphi_k$  is an orientation-preserving homeomorphism from  $I$  into itself with  $(L\varphi_k)^{-1} \in X(I; \frac{1}{\xi_2}, \frac{1}{\xi_1})$ , where  $\xi_1$  and  $\xi_2$  are given in (2.5).

Now we claim that

$$\|g - L\varphi_{k-1} \circ \varphi_{k-1}\| \leq \eta^{k-1}\delta, \tag{3.13}$$

$$\|\varphi_k - \varphi_{k-1}\| \leq \eta^{k-1} \frac{\delta}{\xi_1} \tag{3.14}$$

for all  $x \in I$  and  $k = 1, 2, \dots$

The case  $k = 1$  is trivial. Assume that (3.13) and (3.14) hold for  $k$ . Since

$$\begin{aligned} \|g(x) - L\varphi_k \circ \varphi_k\| &\leq \|L\varphi_{k-1} \circ \varphi_k - L\varphi_k \circ \varphi_k\| \\ &\leq \frac{1}{\lambda} \|- \varphi_{k-1}^2(\lambda\varphi_{k-1}^{-1}(\varphi_k)) + \varphi_k^2(\lambda\varphi_k^{-1}(\varphi_k))\| \\ &\leq \frac{1}{\lambda} \|\varphi_k^2(\lambda\varphi_k^{-1}) - \varphi_{k-1}^2(\lambda\varphi_{k-1}^{-1})\| \\ &\leq \frac{1}{\lambda} (\|\varphi_k^2(\lambda\varphi_k^{-1}) - \varphi_k^2(\lambda\varphi_{k-1}^{-1})\| + \|\varphi_k^2(\lambda\varphi_{k-1}^{-1}) - \varphi_{k-1}^2(\lambda\varphi_{k-1}^{-1})\|) \\ &\leq \frac{1}{\lambda} (\lambda M^2 \|\varphi_k^{-1} - \varphi_{k-1}^{-1}\| + (M+1)\|\varphi_k - \varphi_{k-1}\|), \end{aligned}$$

where Lemma 2.3 is applied. From the hypothesis of induction, it follows that

$$\begin{aligned} \|g(x) - L\varphi_k \circ \varphi_k\| &\leq M^2 \|\varphi_k^{-1} - \varphi_{k-1}^{-1}\| + \frac{(M+1)}{\lambda} \|\varphi_k - \varphi_{k-1}\| \\ &\leq \left( \frac{M^2}{m_1} + \frac{M+1}{\lambda\xi_1} \right)^k \delta = \eta^k \delta. \end{aligned}$$

Moreover,

$$\begin{aligned} \|\varphi_{k+1} - \varphi_k\| &= \|(L\varphi_k)^{-1} \circ g - (L\varphi_k)^{-1} \circ L\varphi_k \circ \varphi_k\| \\ &\leq \frac{1}{\xi_1} \|g - L\varphi_k \circ \varphi_k\| \\ &\leq \eta^k \frac{\delta}{\xi_1}. \end{aligned}$$

Thus, (3.13) and (3.14) hold.

On the other hand, for any positive integers  $k$  and  $l$  with  $k > l$ , by (3.14)

$$\begin{aligned} \|\varphi_k - \varphi_l\| &\leq \|\varphi_k - \varphi_{k-1}\| + \|\varphi_{k-1} - \varphi_{k-2}\| + \dots + \|\varphi_{l+1} - \varphi_l\| \\ &\leq \eta^{k-1} \frac{\delta}{\xi_1} + \eta^{k-2} \frac{\delta}{\xi_1} + \dots + \eta^l \frac{\delta}{\xi_1} \\ &\leq \frac{\eta^k - \eta^l}{1 - \eta} \frac{\delta}{\xi_1}. \end{aligned} \tag{3.15}$$

Note that  $\eta < 1$ , so from (3.15), it follows that

$$\|\varphi_k - \varphi_l\| \rightarrow 0 \quad \text{as } k, l \rightarrow +\infty.$$

This implies that  $\{\varphi_k(x)\}$  is a Cauchy sequence. Hence,  $\{\varphi_k(x)\}$  uniformly converges in the Banach space  $C^0(I)$ . Let  $\varphi = \lim_{k \rightarrow +\infty} \varphi_k(x)$ . Clearly,  $\varphi \in X(I; m, M)$ . From (3.13),

$$\|g - L\varphi \circ \varphi\| = \lim_{k \rightarrow +\infty} \|g - L\varphi_k \circ \varphi_k\| \leq \lim_{k \rightarrow +\infty} \eta^k \delta = 0, \tag{3.16}$$

i.e.,  $\varphi$  is a solution of (1.1). Furthermore, from (3.14)

$$\begin{aligned} \|\varphi - \varphi_s\| &= \lim_{k \rightarrow +\infty} \|\varphi_k - \varphi_0\| \\ &\leq \lim_{k \rightarrow +\infty} (\|\varphi_k - \varphi_{k-1}\| + \|\varphi_{k-1} - \varphi_{k-2}\| + \dots + \|\varphi_1 - \varphi_0\|) \\ &\leq \lim_{k \rightarrow +\infty} \left( \eta^{k-1} \frac{\delta}{\xi_1} + \eta^{k-2} \frac{\delta}{\xi_1} + \dots + \frac{\delta}{\xi_1} \right) \\ &\leq \frac{\frac{\delta}{\xi_1}}{1 - \eta} = \frac{1}{\xi_1(1 - \eta)} \delta. \end{aligned}$$

Thus,  $\|\varphi - \varphi_s\| < \zeta \delta$ . Then (3.12) holds.

Concerning the uniqueness, we assume that there is another continuous solution  $\phi \in X(I; m, M)$  ( $\phi \neq \varphi$ ), such that

$$|\phi(x) - \varphi_s(x)| \leq \varepsilon,$$

where  $\varepsilon > 0$  only depends on  $\delta$ . Then

$$\begin{aligned} \|\varphi - \phi\| &= \|(L\varphi)^{-1} \circ g - (L\phi)^{-1} \circ g\| \\ &\leq \|(L\varphi)^{-1} - (L\phi)^{-1}\| \\ &\leq \|(L\varphi)^{-1} - (L\varphi)^{-1} \circ (L\varphi) \circ (L\phi)^{-1}\| \\ &\leq \frac{1}{\xi_1} \|(L\phi) \circ (L\phi)^{-1} - (L\varphi) \circ (L\phi)^{-1}\| \\ &\leq \frac{1}{\xi_1} \|L\varphi - L\phi\| \\ &\leq \frac{1}{\lambda \xi_1} (\|\varphi^2(\lambda\varphi^{-1}) - \phi^2(\lambda\varphi^{-1})\| + \|\phi^2(\lambda\varphi^{-1}) - \phi^2(\lambda\phi^{-1})\|) \\ &\leq \frac{M+1}{\lambda \xi_1} \|\varphi - \phi\| + \frac{M^2}{\xi_1} \|\varphi^{-1} - \phi^{-1}\| \\ &\leq \frac{M+1}{\lambda \xi_1} \|\varphi - \phi\| + \frac{M^2}{\xi_1} \|\varphi^{-1} - \varphi^{-1} \circ \varphi \circ \phi^{-1}\| \\ &\leq \frac{M+1}{\lambda \xi_1} \|\varphi - \phi\| + \frac{M^2}{m \xi_1} \|\phi \circ \phi^{-1} - \varphi \circ \phi^{-1}\| \\ &\leq \frac{M+1}{\lambda \xi_1} \|\varphi - \phi\| + \frac{M^2}{m \xi_1} \|\phi - \varphi\|, \end{aligned}$$

that is,

$$\left(1 - \frac{M+1}{\lambda\xi_1} - \frac{M^2}{m\xi_1}\right) \|\varphi - \phi\| \leq 0. \tag{3.17}$$

The assumption of Theorem 3.6 yields  $\|\varphi - \phi\| = 0$ , *i.e.*,  $\varphi = \phi$ , which contradicts with the assumption. The proof is completed.  $\square$

#### 4 Example

**Example 1** Consider the equation

$$f(x) = \frac{2}{3}f^2\left(\frac{1}{2}x\right) + \frac{1}{3}g(x), \tag{4.1}$$

where  $x \in [0, 1]$  and  $\lambda = \frac{1}{2}$ . Let

$$g(x) = \begin{cases} \frac{1}{8}x, & 0 \leq x \leq \frac{1}{4}, \\ \frac{1}{4}x - \frac{1}{32}, & \frac{1}{4} < x \leq \frac{1}{2}, \\ \frac{1}{8}x + \frac{1}{32}, & \frac{1}{2} < x \leq \frac{3}{4}, \\ \frac{1}{2}x - \frac{1}{4}, & \frac{3}{4} < x \leq 1. \end{cases} \tag{4.2}$$

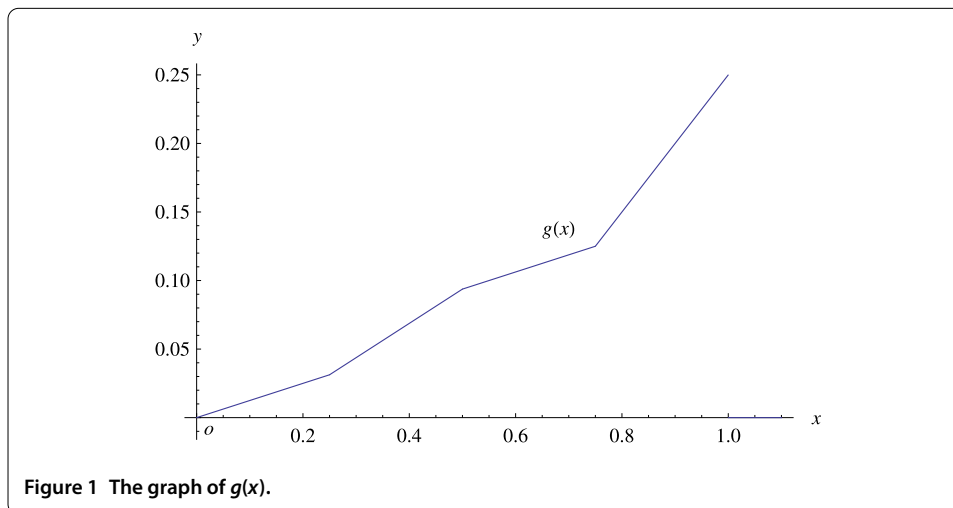
Then  $g(x)$  is quasi-convex and nonconvex (see Figure 1). Note that, for  $x \in [0, \frac{1}{4}]$  and  $y \in [\frac{3}{4}, 1]$

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{2}y - \frac{1}{4} - \frac{1}{8}x \right| = \left| \frac{1}{8}(y-x) + \frac{3}{8}\left(y - \frac{2}{3}\right) \right| \\ &\leq \frac{1}{8}|y-x| + \frac{3}{8}|y-x| = \frac{1}{2}|y-x|. \end{aligned}$$

Similarly, we can show that for any  $x, y \in [0, 1]$ ,  $|f(x) - f(y)| \leq \frac{1}{2}|y-x|$ . Thus,  $M_1 = \frac{1}{2}$ .

For  $x \in [\frac{1}{4}, \frac{1}{2}]$  and  $y \in [\frac{3}{4}, 1]$ , we have

$$|f(x) - f(y)| = \left| \frac{1}{2}y - \frac{1}{4} - \frac{1}{4}x + \frac{1}{32} \right| \geq \frac{1}{4}|y-x| - \frac{1}{4}\left|y - \frac{7}{8}\right|.$$



**Figure 1** The graph of  $g(x)$ .

If  $1 \geq y \geq \frac{7}{8}$ , then  $|f(x) - f(y)| \geq \frac{1}{4}|y - x| \geq \frac{1}{8}|y - x|$ ; if  $\frac{7}{8} \geq y \geq \frac{3}{4}$ , then

$$|f(x) - f(y)| \geq \frac{1}{4}|y - x| - \frac{1}{32} \geq \frac{3}{16}|y - x| \geq \frac{1}{8}|y - x|,$$

since  $\frac{1}{2} \leq |y - x| \leq \frac{3}{4}$ . Similarly, we can show that for any  $x, y \in [0, 1]$ ,  $|f(x) - f(y)| \geq \frac{1}{8}|y - x|$ . Thus,  $m_1 = \frac{1}{8}$ . Therefore, we can get a quasi-convex solution  $f(x)$  of equation (4.1) by Theorem 3.3, which is continuously dependent on the given function  $g(x) \in X(I; M_1)$  with  $M = \frac{3-\sqrt{7}}{2}$  by Theorem 3.5. Moreover, equation (4.1) satisfies the Hyers-Ulam stability in  $X(I; \frac{131-49\sqrt{7}}{152-48\sqrt{7}}, \frac{3-\sqrt{7}}{2})$  by Theorem 3.6.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All the authors have contributed in all the parts, and they have read and approved the final manuscript.

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