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Blow-up and global existence for the non-local reaction diffusion problem with time dependent coefficient

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Abstract

Blow-up and global existence for the non-local reaction diffusion problem with time dependent coefficient under the Dirichlet boundary condition are investigated. We derive the conditions on the data of problem (1.1) sufficient to guarantee that blow-up will occur, and obtain an upper bound for t^*. Also we give the condition for global existence of the solution.

Keywords: blow-up; global existence; non-local reaction diffusion problem; Dirichlet boundary condition

1 Introduction

In this work, we consider the following non-local reaction diffusion problem with time dependent coefficient under the Dirichlet boundary condition

$$\begin{cases} u_t = \Delta u + \int_\Omega u^p\,dx - k(t)u^q, & x = (x_1, x_2, \dots, x_N) \in \Omega, t \in (0, t^*), \\ u(x,t) = 0, & x \in \partial\Omega, t \in (0, t^*), \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^N$ is a bounded domain with a smooth boundary $\partial\Omega$, Δ is the Laplace operator, and t^* is the possible blow-up time. By the maximum principle, it follows that $u(x,t) \geq 0$ in the time interval of existence. The coefficient $k(t)$ is assumed to be non-negative. The particular case of $k =$ const of problem (1.1) has already been investigated by many authors, in [1, 2], they studied the question of blow-up for the solution, and in [3–5], they derived lower bounds for blow-up time under different boundary conditions. To deal with problem (1.1) with time dependent coefficient, we make the assumption on the parameters p and q, that is, $p = q > 1$.

The motivation of this article comes from the work of Payne and Philippin in [6], where they investigated the blow-up phenomena of the solution of the following problem

$$\begin{cases} u_t = \Delta u + k(t)f(u), & x = (x_1, \dots, x_N) \in \Omega, t \in (0, t^*), \\ u(x,t) = 0, & x \in \partial\Omega, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded sufficiently smooth domain in R^N, $N \geq 2$, and the coefficient $k(t)$ is assumed nonnegative or strictly positive depending on the situation.

In next, we employ a method used by Kaplan in [7] to obtain a condition, which leads to blow-up at some finite time and also leads to an upper bound for the blow-up time. In Section 3, we derive the condition on the data of problem (1.1) sufficient to guarantee the global existence of $u(x,t)$.

2 Conditions for blow-up in finite time t^*

Let λ_1 be the first eigenvalue, and let ϕ_1 be the associated eigenfunction of the Dirichlet-Laplace operator defined as

$$\Delta\phi_1 = -\lambda_1\phi_1, \quad \phi_1 > 0, x \in \Omega; \qquad \phi_1 = 0, x \in \partial\Omega, \tag{2.1}$$

$$\int_\Omega \phi_1 \, dx = 1. \tag{2.2}$$

Let the auxiliary function $\eta(t)$

$$\eta(t) := \bigl(|\Omega| - k(t)\bigr)^{\frac{1}{q-1}} \int_\Omega u\phi_1 \, dx \tag{2.3}$$

defined in $(0, t^*)$, where $u(x,t)$ is the solution of (1.1) and $q > 1$.

We assume that for all $t \in (0, t^*)$,

$$|\Omega| > k(t) > 0, \qquad \frac{-k'(t)}{|\Omega| - k(t)} \geq \beta, \qquad \max_{x\in\Omega} \phi_1 |\Omega| \leq 1, \tag{2.4}$$

for some constant β, and

$$\gamma := \lambda_1 - \frac{\beta}{q-1}. \tag{2.5}$$

We deduce from (2.4) and (2.5) that

$$\begin{aligned}
\eta'(t) &\geq \frac{\beta}{q-1}\eta(t) + \bigl(|\Omega| - k(t)\bigr)^{\frac{1}{q-1}} \int_\Omega \phi_1 \left[\Delta u + \int_\Omega u^q \, dx - k(t)u^q\right] dx \\
&= -\gamma\eta(t) + \bigl(|\Omega| - k(t)\bigr)^{\frac{1}{q-1}} \left(\int_\Omega u^q \, dx - k(t)\int_\Omega \phi_1 u^q \, dx\right) \\
&\geq -\gamma\eta(t) + \bigl(|\Omega| - k(t)\bigr)^{\frac{1}{q-1}} \left(\left(\frac{1}{\max_{x\in\Omega}\phi_1} - k(t)\right)\int_\Omega \phi_1 u^q \, dx\right) \\
&\geq -\gamma\eta(t) + \bigl(|\Omega| - k(t)\bigr)^{\frac{q}{q-1}} \int_\Omega \phi_1 u^q \, dx.
\end{aligned} \tag{2.6}$$

Furthermore, using (2.2) and Hölder's inequality, we get

$$\int_\Omega \phi_1 u \, dx \leq \left(\int_\Omega \phi_1 u^q \, dx\right)^{\frac{1}{q}}. \tag{2.7}$$

Combining (2.6) and (2.7), we get

$$\eta'(t) \geq -\gamma\eta(t) + \eta^q(t), \quad t \in \bigl(0, t^*\bigr). \tag{2.8}$$

By integrating (2.8), we get

$$\big(\eta(t)\big)^{1-q} \le \Theta(t) := \begin{cases} ((\eta(0))^{1-q} - \frac{1}{\gamma})e^{\gamma(q-1)t} + \frac{1}{\gamma}, & \gamma \neq 0, \\ (\eta(0))^{1-q} + (1-q)t, & \gamma = 0. \end{cases} \tag{2.9}$$

If $\Theta(T_1) = 0$ for some $T_1 > 0$, then $\eta(t)$ blows up at time $t^* < T_1$. This result is summarized in the following theorem.

Theorem 1 *Let $u(x,t)$ be the solution of problem* (1.1). *Then the auxiliary function $\eta(t)$ defined in* (2.3) *blows up at time $t^* < T_1$ with*

$$T_1 := \begin{cases} \frac{1}{\gamma(q-1)} \log(-\frac{1}{\gamma((\eta(0))^{1-q} - \frac{1}{\gamma})}) & \textit{if } 0 < \gamma(\eta(0))^{1-q} < 1, \\ \frac{1}{(q-1)(\eta(0))^{q-1}} & \textit{if } \gamma \le 0. \end{cases} \tag{2.10}$$

3 Condition for global existence

In this section, our argument makes use of the following Sobolev-type inequality

$$\left(\int_\Omega v^6\,dx\right)^{1/4} \le \Gamma\left(\int_\Omega |\nabla v|^2\,dx\right)^{3/4}, \quad \Gamma = \frac{2.3^{-3/4}}{\pi}, \tag{3.1}$$

valid in R^3 for a nonnegative function v that vanishes on $\partial\Omega$. In this section, our results restricted to R^3 for proof of (3.1), see [8].

We consider the auxiliary function $\sigma(t)$ defined as

$$\sigma(t) := M^{-1}\big(|\Omega| - k(t)\big)^{2n} \int_\Omega u^{2n(p-1)}\,dx, \quad t \in \big(0, t^*\big), \tag{3.2}$$

with

$$M := \big(|\Omega| - k(0)\big)^{2n} \int_\Omega u_0^{2n(q-1)}\,dx, \tag{3.3}$$

we assume that for all $t \in (0, t^*)$,

$$|\Omega| > k(t) > 0, \qquad \frac{-k'(t)}{|\Omega| - k(t)} < \beta, \tag{3.4}$$

for some constant β. In (3.2)-(3.3), n is subjected to restrictions

$$n(q-1) \ge 1, \qquad n > \frac{3}{4}. \tag{3.5}$$

For convenience, we set

$$v(x,t) = u^{n(q-1)}, \tag{3.6}$$

and compute

$$\begin{aligned} \sigma'(t) = {} & 2n\frac{-k'(t)}{|\Omega| - k(t)}\sigma(t) + 2n(q-1)M^{-1}\big(|\Omega| - k(t)\big)^{2n} \\ & \times \int_\Omega u^{2n(q-1)-1}\left[\Delta u + \int_\Omega u^q\,dx - k(t)u^q\right]dx \end{aligned} \tag{3.7}$$

with

$$\int_\Omega u^{2n(q-1)-1}\Delta u\,dx = -\frac{2n(q-1)-1}{n^2(q-1)^2}\int_\Omega |\nabla v|^2\,dx, \tag{3.8}$$

due to (3.4), we obtain

$$\begin{aligned}\sigma'(t) &\le 2n\beta\sigma(t) - \frac{2[2n(q-1)-1]}{n(q-1)}\big(|\Omega|-k(t)\big)^{2n}M^{-1}\int_\Omega |\nabla v|^2\,dx \\ &\quad + 2n(q-1)M^{-1}\big(|\Omega|-k(t)\big)^{2n}\left[|\Omega|\int_\Omega v^{2+\frac{1}{n}}\,dx - k(t)\int_\Omega v^{2+\frac{1}{n}}\,dx\right] \\ &= 2n\beta\sigma(t) - \frac{2[2n(q-1)-1]}{n(q-1)}\big(|\Omega|-k(t)\big)^{2n}M^{-1}\int_\Omega |\nabla v|^2\,dx \\ &\quad + 2n(q-1)M^{-1}\big(|\Omega|-k(t)\big)^{2n+1}\int_\Omega v^{2+\frac{1}{n}}\,dx. \end{aligned}\tag{3.9}$$

By using Hölder's inequality,

$$\int_\Omega v^{2+\frac{1}{n}}\,dx \le \left(\int_\Omega v^2\,dx\right)^{(4n-1)/4n}\left(\int_\Omega v^6\,dx\right)^{1/4n}, \tag{3.10}$$

and Sobolev-type inequality (3.1), we obtain

$$\begin{aligned}&\big(|\Omega|-k(t)\big)^{2n+1}\int_\Omega v^{2+\frac{1}{n}}\,dx \\ &\quad\le \big(|\Omega|-k(t)\big)^{2n+1}\left(\int_\Omega v^2\,dx\right)^{(4n-1)/4n}\left(\int_\Omega |\nabla v|^2\,dx\right)^{3/4n}\Gamma^{1/n} \\ &\quad= \Gamma^{1/n}M^{(4n-1)/4n}\sigma^{(4n-1)/4n}\left(\big(|\Omega|-k(t)\big)^{2n+1}\int_\Omega |\nabla v|^2\,dx\right)^{3/4n}, \end{aligned}\tag{3.11}$$

where Γ is defined in (3.1). Joining (3.11) and (3.9), we obtain

$$\begin{aligned}&\sigma'(t) \\ &\quad\le 2n\beta\sigma(t) - \frac{2[2n(q-1)-1]}{n(q-1)}M^{-1}\big(|\Omega|-k(t)\big)^{2n}\int_\Omega |\nabla v|^2\,dx \\ &\qquad + 2n(q-1)\Gamma^{1/n}M^{1/2n}\sigma^{(4n-1)/4n}\left(M^{-1}\big(|\Omega|-k(t)\big)^{2n}\int_\Omega |\nabla v|^2\,dx\right)^{3/4n} \\ &\quad= 2n\beta\sigma(t) + 2n\left(\lambda^{-1}M^{-1}\big(|\Omega|-k(t)\big)^{2n}\int_\Omega |\nabla v|^2\,dx\right)^{3/4n} \\ &\qquad\times\left\{\lambda^{3/4n}(q-1)\Gamma^{1/n}\sigma^{(4n-1)/4n}M^{1/2n} - \frac{2[2n(q-1)-1]}{n^2(q-1)}\right. \\ &\qquad\left.\times\lambda\left(M^{-1}\big(|\Omega|-k(t)\big)^{2n}\lambda^{-1}\int_\Omega |\nabla v|^2\,dx\right)^{(4n-3)/4n}\right\} \end{aligned}\tag{3.12}$$

with arbitrary $\lambda \neq 0$. Choosing $\lambda := \lambda_1$, the first eigenvalue of problem (2.1), we have

$$\int_\Omega |\nabla v|^2\,dx \ge \lambda_1\int_\Omega v^2\,dx, \tag{3.13}$$

by the Rayleigh principle. By using (3.13) in the last factor of (3.12), we obtain

$$\begin{aligned}\sigma'(t) &\le 2n\beta\sigma(t) + 2n\left(\lambda_1{}^{-1}M^{-1}\big(|\Omega| - k(t)\big)^{2n}\int_\Omega |\nabla v|^2\,dx\right)^{3/4n} \\ &\quad\times\left\{\lambda_1{}^{3/4n}(q-1)\Gamma^{1/n}\sigma(t)^{(4n-1)/4n}M^{1/2n} - \frac{2[2n(q-1)-1]}{n^2(q-1)}\lambda_1\sigma(t)^{(4n-3)/4n}\right\} \\ &= 2n\beta\sigma(t) + 2n\sigma(t)^{3/4n}\times\sigma(t)^{(4n-3)/4n}\big\{\omega\sigma(t)^{1/2n} - (\mu+\beta)\big\},\end{aligned} \tag{3.14}$$

with

$$\omega = \lambda_1{}^{3/4n}(q-1)\Gamma^{1/n}M^{1/2n}, \qquad \mu = \frac{2[2n(q-1)-1]}{n^2(q-1)}\lambda_1 - \beta. \tag{3.15}$$

Suppose that β is small enough to satisfy the condition

$$\mu > 0, \tag{3.16}$$

and that initial data is small enough to satisfy the condition

$$\omega - \mu < 0. \tag{3.17}$$

Then either $\omega(\sigma(t))^{1/2n} - \mu$ remains negative for all time, or there exists a first time t_0 such that

$$\omega\big(\sigma(t_0)\big)^{1/2n} - \mu = 0. \tag{3.18}$$

Then we obtain the differential inequality

$$\sigma'(t) \le 2n\sigma(t)\big\{\omega\big(\sigma(t)\big)^{1/2n} - \mu\big\} \le 0, \quad t \in (0, t_0). \tag{3.19}$$

Integrating this differential inequality, we obtain

$$\sigma(t) \le \left\{\left(1 - \frac{\omega}{\mu}\right)e^{\mu t} + \frac{\omega}{\mu}\right\}^{-2n}, \quad t > 0. \tag{3.20}$$

This result is summarized in the next theorem.

Theorem 2 *Let Ω be a bounded domain in R^3, and assume that the data of problem* (1.1) *satisfy conditions* (3.4), (3.16), (3.17). *Then the auxiliary function $\sigma(t)$ defined in* (3.2) *satisfies* (3.20), *and $u(x,t)$ exists for all time $t > 0$.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to each part of this study equally and read and approved the final version of the manuscript.

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