Alotaibi and Mohiuddine *Advances in Difference Equations* 2012, **2012**:39
http://www.advancesindifferenceequations.com/content/2012/1/39

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

On the stability of a cubic functional equation in random 2-normed spaces

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Abstract

In this article, we propose to determine some stability results for the functional equation of cubic in random 2-normed spaces which seems to be a quite new and interesting idea. Also, we define the notion of continuity, approximately and conditional cubic mapping in random 2-normed spaces and prove some interesting results.

Keywords: distribution function, *t*-norm, triangle function, random 2-normed space, cubic functional equation, Hyers-Ulam-Rassias stability

1 Introduction and preliminaries

In 1940, Ulam [1] proposed the following question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(., .)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_1$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In next year, Hyers [2] answers the problem of Ulam under the assumption that the groups are Banach spaces and then generalized by Aoki [3] and Rassias [4] for additive mappings and linear mappings, respectively. Since then several stability problems for various functional equations have been investigated in [5-12].

The stability problem for the cubic functional equation was proved by Jun and Kim [5] for mappings $f: X \to Y$, where X is a real normed space and Y is a Banach space. Later on, the problem of stability of cubic functional equation were discussed by many mathematician.

An interesting and important generalization of the notion of a metric space was introduced by Menger [13] under the name of statistical metric space, which is now called a probabilistic metric space. An important family of probabilistic metric spaces is that of probabilistic normed spaces. The theory of probabilistic normed spaces is important as a generalization of deterministic results of linear normed spaces. The theory of probabilistic normed spaces was initiated and developed in [14,15] and further it was extended to random 2-normed spaces by Goleț [16] using the concept of 2-norm of Gahler [17]. For more details of probabilistic and random/fuzzy 2-normed space, we refer to [18-22] and references therein.

In this article, we establish Hyers-Ulam stability concerning the cubic functional equations in random 2-normed spaces which is quite a new and interesting idea to study with.

In this section, we recall some notations and basic definitions used in this article.

A distribution function is an element of Δ^+, where $\Delta^+ = \{f : \mathbb{R} \to [0, 1]; f$ is left-continuous, nondecreasing, $f(0) = 0$ and $f(+\infty) = 1\}$ and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{f \in \Delta^+; l^- f(+\infty) = 1\}$. Here $l^- f(+\infty)$ denotes the left limit of the function f at the point x. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$. For any $a \in \mathbb{R}$, H_a is a distribution function defined by

$$H_a(x) = \begin{cases} 0 & \text{if } x \leq a; \\ 1 & \text{if } x > a. \end{cases}$$

The set Δ, as well as its subsets, can be partially ordered by the usual pointwise order: in this order, H_0 is the maximal element in Δ^+.

A triangle function is a binary operation on Δ^+, namely a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative nondecreasing and which has ε_0 as unit, that is, for all $f, g, h \in \Delta^+$, we have:

(i) $\tau(\tau(f, g), h) = \tau(f, \tau(g, h))$,

(ii) $\tau(f, g) = \tau(g, f)$,

(iii) $\tau(f, g) = \tau(g, f)$ whenever $f \leq g$,

(iv) $\tau(f, H_0) = f$.

A *t-norm* is a continuous mapping $* : [0, 1] \times [0, 1] \to [0, 1]$ such that $([0, 1], *)$ is abelian monoid with unit one and $c * d \geq a * b$ if $c \geq a$ and $d \geq b$ for all $a, b, c, d \in [0, 1]$.

The concept of 2-normed space was first introduced in [17] and further studied in [23-25].

Let X is a linear space of a dimension d, where $2 \leq d < \infty$. A 2-normed on X is a function $\|., .\| : X \times X \to \mathbb{R}$ satisfying the following conditions, for every $x, y \in X$ (i) $\|x, y\| = 0$ if and only if x and y are linearly dependent; (ii) $\|x, y\| = \|y, x\|$; (iii) $\|\alpha x, y\| = |\alpha|\|x, y\|$, $\alpha \in \mathbb{R}$; (iv) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$. In this case $(X, \|., .\|)$ is called a 2-norm space.

Example 1.1. Take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\|$ = the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|, \quad \text{where } x = (x_1, x_2), y = (y_1, y_2).$$

Recently, Goleț [16] introduced the notion of random 2-normed space and further studied by Mursaleen [26].

Let X be a linear space of a dimension greater than one, τ is a triangle function, and $\mathcal{F} : X \times X \to \Delta^+$. Then $\mathcal{F}$ is called a *probabilistic* 2-*norm* on X and $(X, \mathcal{F}, \tau)$ a *probabilistic* 2-*normed space* if the following conditions are satisfied:

(i) $\mathcal{F}_{x,y}(t) = H_0(t)$ if x and y are linearly dependent, where $\mathcal{F}_{x,y}(t)$ denotes the value of $\mathcal{F}_{x,y}$ at $t \in \mathbb{R}$,

(ii) $\mathcal{F}_{x,y} \neq H_0$ if x and y are linearly independent,

(iii) $\mathcal{F}_{x,y} = \mathcal{F}_{y,x}$ for every x, y in X,

(iv) $\mathcal{F}_{\alpha x,y}(t) = \mathcal{F}_{x,y}(\frac{t}{|\alpha|})$ for every $t > 0$, $\alpha \neq 0$ and $x, y \in X$,

(v) $\mathcal{F}_{x+y,z} \geq \tau(\mathcal{F}_{x,z}, \mathcal{F}_{y,z})$ whenever $x, y, z \in X$.

If (v) is replaced by

(v') $\mathcal{F}_{x+y,z}(t_1 + t_2) \geq \mathcal{F}_{x,z}(t_1) * \mathcal{F}_{y,z}(t_2)$, for all $x, y, z \in X$ and $t_1, t_2 \in \mathbb{R}_0^+$, then triple $(X, \mathcal{F}, *)$ is called a *random 2-normed space* (for short, RTN-space).

Example 1.2. Let $(X, \|.,.\|)$ be a 2-normed space with $\|x, z\| = \|x_1 z_2 - x_2 z_1\|$, $x = (x_1, x_2)$, $z = (z_1, z_2)$ and $a * b = ab$ for $a, b \in [0, 1]$. For all $x \in X$, $t > 0$ and nonzero $z \in X$, consider

$$\mathcal{F}_{x,z}(t) = \begin{cases} \frac{t}{t+\|x,z\|} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0; \end{cases}$$

Then $(X, \mathcal{F}, *)$ is a random 2-normed space.

Remark 1.3. Note that every 2-normed space $(X, \|.,.\|)$ can be made a random 2-normed space in a natural way, by setting $\mathcal{F}_{x,y}(t) = H_0(t - \|x, y\|)$, for every $x, y \in X$, $t > 0$ and $a * b = \min\{a, b\}$, $a, b \in [0, 1]$.

2 Stability of cubic functional equation

In the present section, we define the notion of convergence, Cauchy sequence and completeness in RTN-space and determine some stability results of the cubic functional equation in RTN-space.

The functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \tag{1}$$

is called the *cubic functional equation*, since the function $f(x) = cx^3$ is its solution. Every solution of the cubic functional equation is said to be a *cubic mapping*.

We shall assume throughout this article that X and Y are linear spaces; $(X, \mathcal{F}, *)$ and $(Z, \mathcal{F}', *)$ are random 2-normed spaces; and $(Y, \mathcal{F}, *)$ is a random 2-Banach space.

Let ϕ be a function from $X \times X$ to Z. A mapping $f : X \to Y$ is said to be ϕ-approximately cubic function if

$$\mathcal{F}_{E_{x,y},z}(t) \geq \mathcal{F}'_{\varphi(x,y),z}(t), \tag{2}$$

for all $x, y \in X$, $t > 0$ and nonzero $z \in X$, where

$$E_{x,y} = f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x).$$

We define:

We say that a sequence $x = (x_k)$ is *convergent* in $(X, \mathcal{F}, *)$ or simply *$\mathscr{F}$-convergent* to ℓ if for every $\epsilon > 0$ and $\theta \in (0, 1)$ there exists $k_0 \in \mathbb{N}$ such that $\mathcal{F}_{x_k - \ell, z}(\varepsilon) > 1 - \theta$ whenever $k \geq k_0$ and nonzero $z \in X$. In this case we write $\mathcal{F} - \lim_{k\to\infty} x_k = \ell$ and ℓ is called the $\mathscr{F}$-limit of $x = (x_k)$.

A sequence $x = (x_k)$ is said to be *Cauchy sequence* in $(X, \mathcal{F}, *)$ or simply *$\mathscr{F}$-Cauchy* if for every $\epsilon > 0$, $\theta > 0$ and nonzero $z \in X$ there exists a number $N = N(\epsilon, z)$ such that $\lim \mathcal{F}_{x_n - x_m, z}(\varepsilon) > 1 - \theta$ for all $n, m \geq N$. RTN-space $(X, \mathcal{F}, *)$ is said to be *complete* if every $\mathscr{F}$-Cauchy is $\mathscr{F}$-convergent. In this case $(X, \mathcal{F}, *)$ is called random 2-Banach space.

Theorem 2.1. Suppose that a function $\phi : X \times X \to Z$ satisfies $\phi(2x, 2y) = \alpha\phi(x, y)$ for all $x, y \in X$ and $\alpha \neq 0$. Let $f : X \to Y$ be a ϕ-approximately cubic function. If for some $0 < \alpha < 8$,

$$\mathcal{F}'_{\varphi(2x,2y),z}(t) \geq \mathcal{F}'_{\alpha\varphi(x,y),z}(t), \tag{3}$$

and $\lim_{n\to\infty} \mathcal{F}'_{\varphi(2^n x,2^n y),z}(8^n t) = 1$ for all $x, y \in X$, $t > 0$ and nonzero $z \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mathcal{F}_{C(x)-f(x),z}(t) \geq \mathcal{F}'_{\varphi(x,0),z}((8-\alpha)t), \tag{4}$$

for all $x \in X$, $t > 0$ and nonzero $z \in X$.

Proof. For convenience, let us fix $y = 0$ in (2). Then for all $x \in X$, $t > 0$ and nonzero $z \in X$

$$\mathcal{F}_{\frac{f(2x)}{8}-f(x),z}\left(\frac{t}{16}\right) \geq \mathcal{F}'_{\varphi(x,0),z}(t). \tag{5}$$

Replacing x by $2^n x$ in (5) and using (3), we obtain

$$\mathcal{F}_{\frac{f(2^{n+1}x)}{8^{n+1}}-\frac{f(2^n x)}{8^n},z}\left(\frac{t}{16(8^n)}\right) \geq \mathcal{F}'_{\varphi(2^n x,0),z}(t) \geq \mathcal{F}'_{\varphi(x,0),z}(t/\alpha^n),$$

for all $x \in X$, $t > 0$ and nonzero $z \in X$; and for all $n \geq 0$. By replacing t by $\alpha^n t$, we get

$$\mathcal{F}_{\frac{f(2^{n+1}x)}{8^{n+1}}-\frac{f(2^n x)}{8^n},z}\left(\frac{\alpha^n t}{16(8^n)}\right) \geq \mathcal{F}'_{\varphi(x,0),z}(t). \tag{6}$$

It follows from $\frac{f(2^n x)}{8^n} - f(x) = \sum_{k=0}^{n=1}\left(\frac{f(2^{k+1}x)}{8^{k+1}} - \frac{f(2^k x)}{8^k}\right)$ and (6) that

$$\mathcal{F}_{\frac{f(2^n x)}{8^n}-f(x),z}\left(\sum_{k=0}^{n-1}\frac{\alpha^k t}{16(8^k)}\right) \geq \prod_{k=0}^{n-1}\mathcal{F}_{\frac{f(2^{k+1}x)}{8^{k+1}}-\frac{f(2^k x)}{8^k},z}\left(\frac{\alpha^k t}{16(8^k)}\right) \geq \mathcal{F}'_{\varphi(x,0),z}(t), \tag{7}$$

for all $x \in X$, $t > 0$ and $n > 0$ where $\prod_{j=1}^n a_j = a_1 * a_2 * \cdots * a_n$. By replacing x with $2^m x$ in (7), we have

$$\mathcal{F}_{\frac{f(2^{n+m}x)}{8^{n+m}}-\frac{f(2^m x)}{8^m},z}\left(\sum_{k=0}^{n-1}\frac{\alpha^k t}{16(8)^{k+m}}\right) \geq \mathcal{F}'_{\varphi(2^m x,0),z}(t) \geq \mathcal{F}'_{\varphi(x,0),z}(t/\alpha^m).$$

Thus

$$\mathcal{F}_{\frac{f(2^{n+m}x)}{8^{n+m}}-\frac{f(2^m x)}{8^m},z}\left(\sum_{k=m}^{n+m-1}\frac{\alpha^k t}{16(8)^k}\right) \geq \mathcal{F}'_{\varphi(x,0),z}(t),$$

for all $x \in X$, $t > 0$, $m > 0$, $n \geq 0$ and nonzero $z \in X$. Hence

$$\mathcal{F}_{\frac{f(2^{n+m}x)}{8^{n+m}}-\frac{f(2^m x)}{8^m},z}(t) \geq \mathcal{F}'_{\varphi(x,0),z}\left(\frac{t}{\sum_{k=m}^{n+m-1} \frac{\alpha^k}{16(8)^k}}\right), \tag{8}$$

for all $x \in X$, $t > 0$ $m \geq 0$, $n \geq 0$ and nonzero $z \in X$. Since $0 < \alpha < 8$ and $\sum_{k=0}^{\infty} (\frac{\alpha}{8})^k < \infty$, the Cauchy criterion for convergence shows that $(\frac{f(2^n x)}{8^n})$ is a Cauchy sequence in $(Y, \mathcal{F}, *)$. Since $(Y, \mathcal{F}, *)$ is complete, this sequence converges to some point $C(x) \in Y$. Fix $x \in X$ and put $m = 0$ in (8) to obtain

$$\mathcal{F}_{\frac{f(2^n x)}{8^n}-f(x),z}(t) \geq \mathcal{F}'_{\varphi(x,0),z}\left(\frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{16(8)^k}}\right),$$

for all $t > 0$, $n > 0$ and nonzero $z \in X$. Thus we obtain

$$\mathcal{F}_{C(x)-f(x),z}(t) \geq \mathcal{F}_{C(x)-\frac{f(2^n x)}{8^n},z}(t/2) * \mathcal{F}_{\frac{f(2^n x)}{8^n}-f(x),z}(t/2) \geq \mathcal{F}'_{\varphi(x,0),z}\left(\frac{t}{\sum_{k=0}^{n-1} \frac{a^k}{8(8)^k}}\right),$$

for large n. Taking the limit as $n \to \infty$ and using the definition of RTN-space, we get

$$\mathcal{F}_{C(x)-f(x),z}(t) \geq \mathcal{F}'_{\varphi(x,0),z}((8-\alpha)t).$$

Replace x and y by $2^n x$ and $2^n y$, respectively, in (2), we have

$$\mathcal{F}_{\frac{E_{2^n x, 2^n y}}{8^n},z}(t) \geq \mathcal{F}'_{\varphi(2^n x, 2^n y),z}(8^n t),$$

for all x, $y \in X$, $t > 0$ and nonzero $z \in X$. Since

$$\lim_{n\to\infty} \mathcal{F}'_{\varphi(2^n x, 2^n y),z}(8^n t) = 1,$$

we observe that C fulfills (1). To Prove the uniqueness of the cubic function C, assume that there exists a cubic function $D : X \to Y$ which satisfies (4). For fix $x \in X$, clearly $C(2^n x) = 8^n C(x)$ and $D(2^n x) = 8^n D(x)$ for all $n \in \mathbb{N}$. It follows from (4) that

$$\begin{aligned}\mathcal{F}_{C(x)-D(x),z}(t) = \mathcal{F}_{\frac{C(2^n x)}{8^n}-\frac{D(2^n x)}{8^n},z}(t) &\geq \mathcal{F}_{\frac{C(2^n x)}{8^n}-\frac{f(2^n x)}{8^n},z}\left(\frac{t}{2}\right) * \mathcal{F}_{\frac{f(2^n x)}{8^n}-\frac{D(2^n x)}{8^n},z}\left(\frac{t}{2}\right)\\ &\geq \mathcal{F}'_{\varphi(2^n x,0),z}\left(\frac{8^n(8-\alpha)t}{2}\right) \geq \mathcal{F}'_{\varphi(x,0),z}\left(\frac{8^n(8-\alpha)t}{2\alpha^n}\right).\end{aligned}$$

Therefore

$$\mathcal{F}'_{\varphi(x,0),z}\left(\frac{8^n(8-\alpha)t}{2\alpha^n}\right) = 1.$$

Thus $\mathcal{F}_{C(x)-D(x),z}(t) = 1$ for all $x \in X$, $t > 0$ and nonzero $z \in X$. Hence $C(x) = D(x)$.

Example 2.2. Let X be a Hilbert space and Z be a normed space. By $\mathscr{F}$ and $\mathcal{F}'$, we denote the random 2-norms given as in Example 1.1 on X and Z, respectively. Let $\phi : X \times X \to Z$ be defined by $\phi(x, y) = 8(\|x\|^2 + \|y\|^2)z_o$, where z_o is a fixed unit vector in Z. Define $f : X \to X$ by $f(x) = \|x\|^2 x + \|x\|^2 x_o$ for some unit vector $x_o \in X$. Then

$$\mathcal{F}_{E_{x,y},z}(t) = \frac{t}{t + 8\|x,z\|^2 + 2\|y,z\|^2} \geq \frac{t}{t + 8\|x,z\|^2 + 8\|y,z\|^2} = \mathcal{F}'_{\varphi(x,y),z}(t).$$

Also

$$\mathcal{F}'_{\varphi(2x,0),z}(t) = \frac{t}{t + 32\|x,z\|^2} = \mathcal{F}'_{4\varphi(x,0),z}(t).$$

Thus,

$$\lim_{n\to\infty} \mathcal{F}'_{\varphi(2^n x, 2^n y),z}(8^n t) = \lim_{n\to\infty} \frac{8^n t}{8^n t + 8(4^n)(\|x,z\|^2 + \|y,z\|^2)} = 1.$$

Hence, conditions of Theorem 2.1 for $\alpha = 4$ are fulfilled. Therefore, there is a unique cubic mapping $C : X \to X$ such that $\mathcal{F}_{C(x)-f(x),z}(t) \geq \mathcal{F}'_{\varphi(x,0),z}(4t)$ for all $x \in X$, $t > 0$ and nonzero $z \in X$.

By a modification in the proof of Theorem 2.1, one can easily prove the following:

Theorem 2.3. Suppose that a function $\phi : X \times X \to Z$ satisfies $\varphi(x/2, y/2) = \frac{1}{\alpha}\varphi(x,y)$ for all $x, y \in X$ and $\alpha \neq 0$. Let $f : X \to Y$ be a ϕ-approximately cubic function. If for some $\alpha > 8$

$$\mathcal{F}'_{\varphi(x/2,y/2),z}(t) \geq \mathcal{F}'_{\varphi(x,y),z}(\alpha t)$$

and $\lim_{n\to\infty} \mathcal{F}'_{8^n\varphi(2^{-n}x, 2^{-n}y),z}(t) = 1$ for all $x, y \in X$, $t > 0$ and nonzero $z \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mathcal{F}_{C(x)-f(x),z}(t) \geq \mathcal{F}'_{\varphi(x,0),z}((\alpha - 8)t),$$

for all $x \in X$, $t > 0$ and nonzero $z \in X$.

3 Continuity in random 2-normed spaces

In this section, we establish some interesting results of continuous approximately cubic mappings.

Let $f : \mathbb{R} \to X$ be a function, where $\mathbb{R}$ is endowed with the Euclidean topology and X is an random 2-normed space equipped with random 2-norm $\mathscr{F}$. Then, f is said to be *random* 2-*continuous* or simply $\mathscr{F}$-*continuous* at a point $s_o \in \mathbb{R}$ if for all $\epsilon > 0$ and all $0 < \alpha < 1$ there exists $\delta > 0$ such that

$$\mathcal{F}_{f(sx)-f(s_\circ x),z}(\varepsilon) \geq \alpha,$$

for each s with $0 < |s - s_o| < \delta$ and nonzero $z \in X$.

A mapping $f : X \to Y$ is said to be (p, q)-approximately cubic function if, for some p, q and some $z_o \in Z$,

$$\mathcal{F}_{E_{x,y},z}(t) \geq \mathcal{F}'_{(\|x\|^p + \|y\|^q)z_\circ,z}(t),$$

for all $x, y \in X$, $t > 0$ and nonzero $z \in X$.

Theorem 3.2. Let X be a normed space and let $f : X \to Y$ be a (p, q)-approximately cubic function. If $p, q < 3$, there exists a unique cubic mapping $C : X \to Y$ such that

$$\mathcal{F}_{C(x)-f(x),z}(t) \geq \mathcal{F}'_{\|x\|^p z_\circ, z}((8-2^p)t), \tag{9}$$

for all $x \in X$, $t > 0$ and nonzero $z \in X$. Furthermore, if for some $x \in X$ and all $n \in \mathbb{N}$, the mapping $g : \mathbb{R} \to Y$ defined by $g(s) = f(2^n sx)$ is $\mathcal{F}$-continuous. Then the mapping $s \mapsto C(sx)$ from $\mathbb{R}$ to Y is $\mathcal{F}$-continuous; in this case, $C(rx) = r^3C(x)$ for all $r \in \mathbb{R}$.

Proof. Suppose that a function $\phi : X \times X \to Z$ satisfies $\phi(x, y) = (\|x\|^p+\|y\|^q)z_o$. Existence and uniqueness of the cubic mapping C satisfying (9) are deduced from Theorem 2.1. Note that for each $x \in X$, $t \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$\mathcal{F}_{C(x)-\frac{f(2^n x)}{8^n},z}(t) = \mathcal{F}_{C(2^n x)-f(2^n x),z}(8^n t) \geq \mathcal{F}'_{2^{np}\|x\|^p z_\circ,z}(8^n(8-2^p)t) = \mathcal{F}'_{\|x\|^p z_\circ,z}\left(\frac{8^n(8-2^p)t}{2^{np}}\right). \tag{10}$$

Fix $x \in X$ and $s_o \in \mathbb{R}$. Given $\epsilon > 0$ and $0 < \alpha < 1$. From (10) follows that

$$\mathcal{F}_{C(sx)-\frac{f(2^n sx)}{8^n},z}(t) \geq \mathcal{F}'_{\|x\|^p z_\circ,z}\left(\frac{8^n(8-2^p)t}{|s|^p 2^{np}}\right) \geq \mathcal{F}'_{\|x\|^p z_\circ,z}\left(\frac{8^n(8-2^p)t}{(1+|s_\circ|)^p 2^{np}}\right),$$

for all $|s - s_o| < 1$ and $s \in \mathbb{R}$. Since $\lim_{n\to\infty} \frac{8^n(8-2^p)t}{(1+|s_\circ|)^p 2^{np}} = \infty$, there exists $n_o \in \mathbb{N}$ such that

$$\mathcal{F}_{C(sx)-\frac{f(2^{n_\circ} sx)}{8^{n_\circ}},z}\left(\frac{\varepsilon}{3}\right) \geq \alpha,$$

for all $|s - s_o| < 1$ and $s \in \mathbb{R}$. By the $\mathcal{F}$-continuity of the mapping $t \to f(2^{n_\circ} tx)$, there exists $\delta < 1$ such that for each s with $0 < |s - s_o| < \delta$, we have

$$\mathcal{F}_{\frac{f(2^{n_\circ} sx)}{8^{n_\circ}}-\frac{f(2^{n_\circ} s_\circ x)}{8^{n_\circ}},z}\left(\frac{\varepsilon}{3}\right) \geq \alpha.$$

It follows that

$$\begin{aligned}&\mathcal{F}_{C(sx)-C(s_\circ x),z}(\varepsilon)\\ &\quad\geq \mathcal{F}_{C(sx)-\frac{f(2^{n_\circ} sx)}{8^{n_\circ}},z}\left(\frac{\varepsilon}{3}\right) * \mathcal{F}_{\frac{f(2^{n_\circ} sx)}{8^{n_\circ}}-\frac{f(2^{n_\circ} s_\circ x)}{8^{n_\circ}},z}\left(\frac{\varepsilon}{3}\right) * \mathcal{F}_{C(s_\circ x)-\frac{f(2^{n_\circ} s_\circ x)}{8^{n_\circ}},z}\left(\frac{\varepsilon}{3}\right) \geq \alpha,\end{aligned}$$

for each s with $0 < |s - s_o| < \delta$. Hence, the mapping $s \mapsto C(sx)$ is $\mathcal{F}$-continuous.

Now, we use the $\mathcal{F}$-continuity of $s \mapsto C(sx)$ to establish that $C(r_\circ x) = r_\circ^3 C(x)$ for all $r_o \in \mathbb{R}$. For each r, $\mathbb{Q}$ is a dense subset of $\mathbb{R}$, we have $C(rx) = r^3C(x)$. Fix $r_o \in \mathbb{R}$ and $t > 0$. Then, for $0 < \alpha < 1$ there exists $\delta > 0$ such that

$$\mathcal{F}_{C(rx)-C(r_\circ x),z}(t/3) \geq \alpha,$$

for each $r \in \mathbb{R}$ and $0 < |r - r_o| < \delta$. Choose a rational number r with $0 < |r - r_o| < \delta$ and $|r^3 - r_\circ^3| < 1-\alpha$. Then

$$\begin{aligned}\mathcal{F}_{C(r_\circ x)-r_\circ^3 C(x),z}(t) &\geq \mathcal{F}_{C(r_\circ x)-C(rx),z}(t/3) * \mathcal{F}_{C(rx)-r^3 C(x),z}(t/3) * \mathcal{F}_{r^3 C(x)-r_\circ^3 C(x),z}(t/3)\\ &\geq \alpha * 1 * \mathcal{F}_{C(x),z}(t/3(1-\alpha)).\end{aligned}$$

Thus $\mathcal{F}_{C(r_\circ x)-r_\circ^3 C(x),z}(t) = 1$. Hence, we conclude that $C(r_\circ x) = r_\circ^3 C(x)$.

Remark 3.2. We can also prove Theorem 3.1 for the case when p, $q > 3$. In this case, there exists a unique cubic mapping $C : X \to Y$ such that $\mathcal{F}_{C(x)-f(x),z}(t) \geq \mathcal{F}'_{\|x\|^p z_\circ,z}((2^p-8)t)$ for all $x \in X$, $t > 0$ and nonzero $z \in X$.

4 Approximately and conditional cubic mapping in random 2-normed spaces

In this section, we obtain completeness in RTN-space through the existence of some solution of a stability problem for cubic functional equation.

A mapping $f : \mathbb{N}\cup\{0\} \to X$ is said to be *approximately cubic* if for each $\alpha \in (0, 1)$ there exists some $n_\alpha \in \mathbb{N}$ such that $\mathcal{F}_{E(n,m),z}(1) \geq \alpha$, for all $n \geq 2m \geq n_\alpha$ and nonzero $z \in X$.

By a *conditional cubic mapping*, we mean a mapping $f : \mathbb{N} \cup \{0\} \to X$ such that (1) holds whenever $x \geq 2y$.

It can be easily verified that for each conditional cubic mapping $f : \mathbb{N} \cup \{0\} \to X$, we have $f(2^n) = 2^{3n}f(1)$.

Theorem 4.1. Let $(X, \mathcal{F}, *)$ be a RTN-space such that for each approximately cubic mapping $f : \mathbb{N} \cup \{0\} \to X$, there exists a conditional cubic mapping $C : \mathbb{N} \cup \{0\} \to X$, such that

$$\lim_{n\to\infty} \mathcal{F}_{C(n)-f(n),z}(1) = 1,$$

for nonzero $z \in X$. Then $(X, \mathcal{F}, *)$ is a random 2-Banach space.

Proof. Let (x_n) be a Cauchy sequence in a RTN-space. By induction on k, we can find a strictly increasing sequence (n_k) of natural numbers such that

$$\mathcal{F}_{x_n-x_m,z}\left(\frac{1}{(10k)^3}\right) \geq 1 - \frac{1}{k},$$

for each $n, m \geq n_k$ and nonzero $z \in X$. Let $y_k = x_{n_k}$ and define $f : \mathbb{N} \cup \{0\} \to X$ by $f(k) = k^3y_k$. Let $\alpha \in (0, 1)$. and find some $n_\circ \in \mathbb{N}$ such that $1 - \frac{1}{n_\circ} > \alpha$. One can easily verified that

$$\begin{aligned}\mathcal{F}_{E_{k,j},z}(1) \geq\ & \mathcal{F}_{y_{2k+j}-y_{k+j},z}\left(\frac{1}{20k^3}\right) * \mathcal{F}_{y_{2k+j}-y_{k-j},z}\left(\frac{1}{20k^3}\right) * \mathcal{F}_{y_{2k+j}-y_k,z}\left(\frac{1}{40k^3}\right)\\ & * \mathcal{F}_{y_{2k-j}-yk,z}\left(\frac{1}{80k^3}\right) * \mathcal{F}_{y_{2k+j}-y_{2k-j},z}\left(\frac{1}{120k^2j}\right) * \mathcal{F}_{y_{k-j}-y_{k+j},z}\left(\frac{1}{60k^2j}\right)\\ & * \mathcal{F}_{y_{2k+j}-y_{k+j},z}\left(\frac{1}{60kj^2}\right) * \mathcal{F}_{y_{2k-j}-y_{k-j},z}\left(\frac{1}{60kj^2}\right)\\ & * \mathcal{F}_{y_{2k+j}-y_{2k-j},z}\left(\frac{1}{10j^3}\right) * \mathcal{F}_{y_{k-j}-y_{k+j},z}\left(\frac{1}{20j^3}\right),\end{aligned}$$

for each $k \geq 2j$, and nonzero $z \in X$. Then

$$\mathcal{F}_{y_{2k+j}-y_{k+j},z}\left(\frac{1}{20k^3}\right) \geq \mathcal{F}_{y_{2k+j}-y_{k+j},z}\left(\frac{1}{20(k+j)^3}\right) \geq \mathcal{F}_{y_{2k+j}-y_{k+j},z}\left(\frac{1}{10^3(k+j)^3}\right) \geq \alpha,$$

for $j > n_o$ and nonzero $z \in X$. Since $k - j \geq \frac{k}{2}$ and $k - j \geq j$, we have

$$\mathcal{F}_{y_{2k+j}-y_{k-j},z}\left(\frac{1}{20k^3}\right) \geq \mathcal{F}_{y_{2k+j}-y_{k-j},z}\left(\frac{1}{10^3(k-j)^3}\right) \geq \alpha,$$
$$\mathcal{F}_{y_{k-j}-y_{k+j},z}\left(\frac{1}{60k^2j}\right) \geq \mathcal{F}_{y_{k-j}-y_{k+j},z}\left(\frac{1}{10^3(k-j)^3}\right) \geq \alpha,$$
$$\mathcal{F}_{y_{2k-j}-y_{k-j},z}\left(\frac{1}{60kj^2}\right) \geq \mathcal{F}_{y_{2k-j}-y_{k-j},z}\left(\frac{1}{10^3(k-j)^3}\right) \geq \alpha,$$
$$\mathcal{F}_{y_{k-j}-y_{k+j},z}\left(\frac{1}{20j^3}\right) \geq \mathcal{F}_{y_{2k-j}-y_{k-j},z}\left(\frac{1}{10^3(k-j)^3}\right) \geq \alpha.$$

Clearly

$$\mathcal{F}_{y_{2k+j}-y_k,z}\left(\frac{1}{40k^3}\right) \geq \mathcal{F}_{y_{2k+j}-y_k,z}\left(\frac{1}{10^3k^3}\right) \geq \alpha, \mathcal{F}_{y_{2k-j}-y_k,z}\left(\frac{1}{80k^3}\right) \geq \mathcal{F}_{y_{2k-j}-y_k,z}\left(\frac{1}{10^3k^3}\right) \geq \alpha,$$
$$\text{and } \mathcal{F}_{y_{2k+j}-y_{k+j},z}\left(\frac{1}{60kj^2}\right) \geq \mathcal{F}_{y_{2k+j}-y_{k+j},z}\left(\frac{1}{10^3(k+j)^3}\right) \geq \alpha.$$

The inequalities $2k - j \geq j$ and $2k - j > k$ imply

$$\mathcal{F}_{y_{2k+j}-y_{2k-j},z}\left(\frac{1}{120k^2j}\right) \geq \mathcal{F}_{y_{2k+j}-y_{2k-j},z}\left(\frac{1}{120(2k-j)^3}\right) \geq \mathcal{F}_{y_{2k+j}-y_{2k-j},z}\left(\frac{1}{10^3(2k-j)^3}\right) \geq \alpha$$
$$\mathcal{F}_{y_{2k+j}-y_{2k-j},z}\left(\frac{1}{10j^3}\right) \geq \mathcal{F}_{y_{2k+j}-y_{2k-j},z}\left(\frac{1}{10(2k-j)^3}\right) \geq \mathcal{F}_{y_{2k+j}-y_{2k-j},z}\left(\frac{1}{10^3(2k-j)^3}\right) \geq \alpha.$$

Therefore $\mathcal{F}_{E_{k,j},z}(1) \geq \alpha$. This shows that f is approximately cubic type mapping. By our assumption, there exists a conditional cubic mapping $C : \mathbb{N} \cup \{0\} \to X$, such that $\lim_{k\to\infty} \mathcal{F}_{C(k)-f(k),z}(1) = 1$. In particular, $\lim_{k\to\infty} \mathcal{F}_{C(2^k)-f(2^k),z}(1) = 1$. This means that

$$\lim_{k\to\infty} \mathcal{F}_{C(1)-y_{2^k},z}\left(\frac{1}{2^{3k}}\right) = 1$$

Hence the subsequence (y_{2^k}) converges to $y = C(1)$. Therefore, the Cauchy sequence (x_n) also converges to y.

Acknowledgements
The authors would like to thank the anonymous reviewers for their valuable comments.

Authors' contributions
Both the authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 16 December 2011 Accepted: 29 March 2012 Published: 29 March 2012

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doi:10.1186/1687-1847-2012-39
Cite this article as: Alotaibi and Mohiuddine: **On the stability of a cubic functional equation in random 2-normed spaces.** *Advances in Difference Equations* 2012 **2012**:39.