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Uniqueness and value distribution for difference operators of meromorphic function

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Abstract

We investigate the value distribution of difference operator for meromorphic functions. In addition, we study the sharing value problems related to a meromorphic function $f(z)$ and its shift $f(z+c)$.

1 Introduction and main results

A meromorphic function means meromorphic in the whole complex plane. We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna Theory [1]. As usual, the abbreviation CM stands for "counting multiplicities", while IM means "ignoring multiplicities", and we denote the order of meromorphic function f by $\sigma(f)$. For a non-constant meromorphic function f and a set S of complex numbers, we define the set $E(S, f) = \cup_{a \in S}\{z | f(z) - a = 0\}$, where a zero of $f - a$ with multiplicity m counts m times in $E(S, f)$.

We define difference operator as $\Delta_c f = f(z+c) - f(z)$, where c is a non-zero constant. In particular, we denote by $S(f)$ the family of all meromorphic functions $a(z)$ that satisfy $T(r, a) = S(r, f) = o(T(r, f))$, where $r \to \infty$ outside a possible exceptional set of finite logarithmic measure. For convenience, we set $\hat{S}(f) := S(f) \cup \{\infty\}$.

The difference Nevanlinna theory and its applications to the uniqueness theory have become a subject of great interest [2-4], recently. With these fundamental results, Heittokangas et al. considered a meromorphic function $f(z)$ sharing values with its shift $f(z+c)$, we recall a key result from [5].

Theorem A [[5], **Theorem 2**]. *Let f be a non-constant meromorphic function of finite order, let $c \in \mathbb{C}$, and let $a, b, c \in \hat{S}(f)$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a, b CM and c IM, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

Recently, Yang and Liu and one of the present authors [6] considered the case $F = f^n$, where f is a meromorphic function, assuming value sharing with F and $F(z+c)$:

Theorem B [[6], **Theorem 1.4**]. *Let f be a non-constant meromorphic function of finite order, $n \geq 7$ be an integer, let $c \in \mathbb{C}$, and let $F = f^n$. If $F(z)$ and $F(z+c)$ share $a \in S(f) \backslash \{0\}$ and ∞ CM, then $f(z) = \omega f(z+c)$, for a constant ω that satisfies $\omega^n = 1$.*

Next, we consider the problem that related to the Theorem B, and have the following result, where a is a periodic function with period c. However, our proof is different to the one in [6].

Theorem 1.1. *Let f be a non-constant meromorphic function of finite order, let $c \in \mathbb{C}$, and let $a \in S(f) \setminus \{0\}$ be a periodic function with period c. If $f(z)^n$ and $f(z+c)^n$ share a*

and ∞ *CM, and* $n \geq 4$ *is an integer, then* $f(z) = \omega f(z + c)$, *for a constant* ω *that satisfies* $\omega^n = 1$.

Remarks.

(1) Theorem 1.1 is not true, if $a = 0$. This can be seen by considering $f(z) = e^{z^2}$. Then $f(z)^n$ and $f(z + c)^n$ share 0 and ∞ CM, however, $f(z) \neq \omega f(z + c)$, where n is a positive integer.

(2) Theorem 1.1 does not remain valid when $n = 1$. For example, $f(z) = e^z + 1$ and $f(z + c) = e^{z+c} + 1$, where $c \neq 2\pi i$. Clearly, $f(z)$ and $f(z + c)$ share 1 and ∞ CM, however, $f(z) \neq \omega f(z + c)$ for $\omega^n = 1$. Unfortunately, we have not succeeded in reducing the condition $n \geq 4$ to $n \geq 2$ in Theorem 1.1, and we also cannot give a counterexample when $n = 2, 3$ at present.

(3) We give an example to show that the restriction of finite order in Theorem 1.1 cannot be deleted. This can be seen by taking $f(z) = e^{e^z}, ne^c = -1$. Then $f(z)^n$ and $f(z + c)^n$ share 1 and ∞ CM, however, $f(z) \neq \omega f(z + c)$, where n is a positive integer.

In 1976, Gross asked the following question [[7], Question 6]:

Question. Can one find (even one set) finite sets S_j $(j = 1, 2)$ such that any two entire functions f and g satisfying $E(S_j, f) = E(S_j, g)$ $(j = 1, 2)$ must be identical?

Since then, many results have been obtained for this and related topics (see [8-11]). We recall the following result given by Yi [9].

Theorem C [[9], **Theorem 1**]. *Let* $S_1 = \{\omega \mid \omega^n + a\omega^{n-1} + b = 0\}$, *where* $n \geq 7$ *is an integer, a and b are two non-zero constants such that the algebraic equation* $\omega^n + a\omega^{n-1} + b = 0$ *has no multiple roots. If f and g are two entire functions satisfying* $E(S_1, f) = E(S_1, g)$, *then* $f = g$.

Afterwards, Fang and Lahiri [12] got the result for meromorphic functions.

Theorem D [[12], **Theorem 1**]. *Let* S_1 *be defined as Theorem C and* $S_2 = \{\infty\}$. *Assume that f and g are two meromorphic functions satisfying* $E(S_j, f) = E(S_j, g)$ *for* $j = 1,2$. *If f has no simple poles and* $n \geq 7$, *then* $f = g$.

Next, we give a difference analog of Theorem D that replacing g with $f(z + c)$, and obtain the following result.

Theorem 1.2. *Let* S_1 *be defined as Theorem C and* $S_2 = \{\infty\}$. *Assume that f is a meromorphic function of finite order satisfying* $E(S_j, f) = E(S_j, f(z + c))$ *for* $j = 1,2$. *If* $n \geq 6$ *and* $\overline{N}(r,f) < \frac{n-3}{n-1}T(r,f) + S(r,f)$, *then* $f(z) = f(z + c)$ *for all* $z \in \mathbb{C}$.

We investigate the value distribution of difference polynomials of meromorphic (entire) functions. Let f be a transcendental meromorphic function, and let n be a positive integer. Concerning to the value distribution of $f^n f'$, Hayman [[13], Corollary to Theorem 9] proved that $f^n f'$ takes every non-zero complex value infinitely often if $n \geq 3$. Mues [[14], Satz 3] proved that $f^2 f' - 1$ has infinitely many zeros. Later on, Bergweiler and Eremenko [[15], Theorem 2] showed that $ff' - 1$ has infinitely many zeros also. As an analog result in difference, Laine and Yang [16] investigated the value distribution of difference products of entire functions, and obtained the following:

Theorem E [[16], **Theorem 2**]. *Let f be a transcendental entire function of finite order, and let c be a non-zero complex constant. Then for* $n \geq 2$, $f(z)^n f(z + c)$ *assumes every non-zero value* $a \in \mathbb{C}$ *infinitely often.*

In a recent article, one of the present authors considered the value distribution of $f(z)^n (f(z) - 1) f(z + c)$, the result may be stated as follows:

Theorem F [[17], **Theorem 1**]. *Let f be a transcendental meromorphic function of finite order* $\sigma(f)$, *let* $a \neq 0$ *be a small function with respect to f, and let c be a non-zero complex constant. If the exponent of convergence of the poles of f satisfies* $\lambda(\frac{1}{f}) < \sigma(f)$ *and* $n \geq 2$, *then* $f(z)^n (f - 1) f(z + c) - a$ *has infinitely many zeros.*

In this article, we replace $f(z + c)$ with $\Delta_c f$, and consider the value distribution of $f(z)^n(f(z) - 1)\Delta_c f$. We get the following results:

Theorem 1.3. *Let f be a transcendental meromorphic function of finite order* $\sigma(f)$ *and* $\Delta_c f \neq 0$, *let* $a \neq 0$ *be a small function with respect to f, and let c be a non-zero complex constant. If the exponent of convergence of the poles off satisfies* $\lambda(\frac{1}{f}) < \sigma(f)$ *and* $n \geq 3$, *then* $f(z)^n (f - 1)\Delta_c f - a$ *has infinitely many zeros.*

Corollary 1.4. *Let f be a transcendental entire function of finite order and* $\Delta_c f \neq 0$, *let* $a \neq 0$ *be a small function with respect to f, and let c be a non-zero complex constant. Then for* $n \geq 3$, $f(z)^n(f - 1)\Delta_c f - a$ *has infinitely many zeros.*

In particular, if *a* is a non-zero polynomial in Corollary 1.4, then Corollary 1.4 can be improved.

Theorem 1.5. *Let f be a transcendental entire function of finite order and* $\Delta_c f \neq 0$, *let a be a non-zero polynomial, and let c be a non-zero complex constant. Then for* $n \geq 2$, $f(z)^n (f - 1)\Delta_c f - a$ *has infinitely many zeros.*

2 Preliminary lemmas

Lemma 2.1. [[4], *Theorem 2.1*] *Let f be a meromorphic function of finite order, and let* $c \in \mathbb{C}$ *and* $\delta \in (0, 1)$. *Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = o\left(\frac{T(r,f)}{r^{\delta}}\right) = S(r,f).$$

Chiang and Feng have obtained similar estimates for the logarithmic difference [[3], Corollary 2.5], and this study is independent from [4].

Lemma 2.2. [[4], *Lemma 2.3*] *Let f be a meromorphic function of finite order and* $c \in \mathbb{C}$. *Then for any small function* $a \in S(f)$ *with period c,*

$$m\left(r, \frac{\Delta_c f}{f - a}\right) = S(r,f).$$

Lemma 2.3. [[3], *Theorem 2.1*] *Let f be a meromorphic function of finite order* $\sigma(f)$, *and let c be a non-zero constant. Then, for each* $\varepsilon > 0$, *we have*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r).$$

Lemma 2.4. [[18], *Theorem 2.4.2*] *Let f be a transcendental meromorphic solution of*

$$f^n A(z,f) = B(z,f),$$

where $A(z, f)$, $B(z, f)$ are differential polynomials in f and its derivatives with small meromorphic coefficients a_λ, in the sense of $m(r, a_\lambda) = S(r, f)$ for all $\lambda \in I$. If the $\deg(B(z, f)) \leq n$, *then $m(r, A(z, f)) = S(r, f)$.*

Lemma 2.5. *Let f be a finite order entire function and $\Delta_c f \neq 0$, and let c be a non-zero constant. Then*

$$m(r, ff'\Delta_c f) \geq T(r, f) + S(r, f).$$

Proof. Since f is an entire function with finite order, we deduce from Lemma 2.2 and the Lemma of logarithmic derivative that

$$\begin{aligned} 3T(r, f) &= T(r, f^3) = m(r, f^3) + S(r, f) \\ &\leq m\left(r, \frac{f^3}{ff'\Delta_c f}\right) + m(r, ff'\Delta_c f) + S(r, f) \\ &= m\left(r, \frac{f^2}{f'\Delta_c f}\right) + m(r, ff'\Delta_c f) + S(r, f) \\ &\leq T\left(r, \frac{f'}{f}\right) + T\left(r, \frac{\Delta_c f}{f}\right) + m(r, ff'\Delta_c f) + S(r, f) \\ &\leq 2N\left(r, \frac{1}{f}\right) + m(r, ff'\Delta_c f) + S(r, f) \\ &\leq 2T(r, f) + m(r, ff'\Delta_c f) + S(r, f). \end{aligned}$$

Hence, we get

$$m(r, ff'\Delta_c f) \geq T(r, f) + S(r, f). \tag{1}$$

3 Proof of Theorem 1.1

Since $f(z)^n$ and $f(z + c)^n$ share a and ∞ CM, we obtain that

$$\frac{f(z+c)^n - a(z+c)}{f(z)^n - a(z)} = e^{Q(z)}, \tag{2}$$

where $Q(z)$ is a polynomial. From Lemma 2.1, we know that $T(r, e^{Q(z)}) = m(r, e^{Q(z)}) = S(r, f)$. Rewrite (2) as

$$f(z+c)^n = e^{Q(z)}(f(z)^n - a(z) + a(z)e^{-Q(z)}). \tag{3}$$

Set

$$G(z) = \frac{f(z)^n}{a(z)(1 - e^{-Q(z)})}.$$

If $e^{Q(z)} \not\equiv 1$, then we apply the Valiron-Mohon'ko theorem and the second main theorem *to* $G(z)$, and get

$$\begin{aligned} nT(r, f) + S(r, f) = T(r, G) &\leq \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r, G) + \overline{N}\left(r, \frac{1}{G-1}\right) + S(r, G) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f(z)^n - a(z) + a(z)e^{-Q(z)}}\right) + S(r, f) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f(z+c)}\right) + S(r, f) \\ &\leq 2T(r, f) + T(r, f(z+c)) + S(r, f). \end{aligned} \tag{4}$$

Combining (4) with Lemma 2.3, we get

$$nT(r,f) \leq 3T(r,f) + O(r^{\sigma(f)-1+\varepsilon}) + S(r,f),$$

which contradicts that $n \geq 4$. Therefore, $e^{Q(z)} \equiv 1$, that is, $f(z)^n = f(z+c)^n$, so we have $f(z) = \omega f(z+c)$, for a constant ω with $\omega^n = 1$.

4 Proof of Theorem 1.2

From the assumption of Theorem 1.2, we get that

$$\frac{f(z+c)^n + af(z+c)^{n-1} + b}{f(z)^n + af(z)^{n-1} + b} = e^{Q(z)}, \tag{5}$$

where $Q(z)$ is a polynomial. Applying Lemma 2.1, we obtain that $T(r, e^{Q(z)}) = m(r, e^{Q(z)}) = S(r, f)$. Rewrite (5) as

$$f(z+c)^n + af(z+c)^{n-1} = e^{Q(z)}\left(f(z)^n + af(z)^{n-1} + b - \frac{b}{e^{Q(z)}}\right). \tag{6}$$

If $e^{Q(z)} \not\equiv 1$, applying the second main theorem for three small functions, we get

$$\begin{aligned} nT(r,f) + S(r,f) &= T(r, f(z)^n + af(z)^{n-1}) \\ &\leq \overline{N}\left(r, \frac{1}{f(z)^n + af(z)^{n-1}}\right) + \overline{N}(r, f(z)^n + af(z)^{n-1}) \\ &+ \overline{N}\left(r, \frac{1}{f(z)^n + af(z)^{n-1} + b - \frac{b}{e^{Q(z)}}}\right) + S(r,f) \\ &\leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f(z+c)^{n-1}(f(z+c)+a)}\right) \\ &+ \overline{N}\left(r, \frac{1}{f(z)^{n-1}(f(z)+a)}\right) + S(r,f) \\ &\leq 3T(r,f) + 2T(r, f(z+c)) + S(r,f). \end{aligned} \tag{7}$$

Combining (4.3) with Lemma 2.3, we get

$$nT(r,f) \leq 5T(r,f) + O(r^{\sigma(f)-1+\varepsilon}) + S(r,f),$$

which contradicts $n \geq 6$. Hence, $e^{Q(z)} \equiv 1$, we conclude by (5) that

$$f(z+c)^n + af(z+c)^{n-1} = f(z)^n + af(z)^{n-1}. \tag{8}$$

Set $G(z) = \frac{f(z)}{f(z+c)}$. If $G(z)$ is non-constant, then we have from (8)

$$f(z) = -\frac{aG(G^{n-1} - 1)}{G^n - 1} = -a\frac{G^{n-1} + \cdots + G}{G^{n-1} + \cdots + 1}. \tag{9}$$

Making use of the standard Valiron-Mohon'ko lemma, we get from (9) that

$$T(r,f) = (n-1)T(r,G) + S(r,f). \tag{10}$$

Noting that $n \geq 6$, we deduce that 1 is not a Picard value of G^n. Suppose that $a_j \in \{\mathbb{C} \setminus 1\}$ $(j = 1, 2, \ldots, n-1)$ are the distinct roots of equation $h^n - 1 = 0$. Applying the second main theorem to G, we conclude by (9) that

$$(n-3)T(r,G) \leq \sum_{j=1}^{n-1} \overline{N}\left(r, \frac{1}{G-a_j}\right) + S(r,G) = \overline{N}(r,f). \tag{11}$$

From (10) and (11), we get $\overline{N}(r,f) \geq \frac{n-3}{n-1}T(r,f) + S(r,f)$, which contradicts the assumption.

So $G(z)$ is a constant, and we get $f(z) = tf(z+c)$, where t is a non-zero constant. From (8), we know $t = 1$, therefore, $f = g$.

5 Proof of Theorem 1.3

The main idea of this proof is from [[17], Theorem 1], while the details are somewhat different. For the convenience of the reader, we give a complete proof.

Set $F(z) = f^n(z)(f(z) - 1)\Delta_c f$. Since f is a transcendental meromorphic function with finite order $\sigma(f)$, we conclude by Lemma 2.3 that

$$\begin{aligned} T(r,F) &\leq T(r, f^n(z)(f(z)-1)) + T(r, \Delta_c f) + S(r,f) \\ &\leq (n+2)T(r,f) + T(r, f(z+c)) + S(r,f) \\ &\leq (n+3)T(r,f) + O(r^{\sigma(f)-1+\varepsilon}) + S(r,f). \end{aligned}$$

Thus, we get $S(r,F) = o(T(r,f)) = S(r,f)$. Moreover, we get

$$\begin{aligned} T(r, \Delta_c f) &\leq m(r, \Delta_c f) + O\left(r^{\lambda(\frac{1}{f})+\varepsilon}\right) + S(r,f) \\ &\leq m\left(r, \frac{\Delta_c f}{f}\right) + m(r,f) + O\left(r^{\lambda(\frac{1}{f})+\varepsilon}\right) + S(r,f) \\ &\leq T(r,f) + O\left(r^{\lambda(\frac{1}{f})+\varepsilon}\right) + S(r,f). \end{aligned} \tag{12}$$

On the other hand, we deduce by Lemma 2.2 that

$$\begin{aligned} (n+2)T(r,f) &= T(r, f^{n+1}(f-1)) + S(r,f) \\ &= m(r, f^{n+1}(f-1)) + O\left(r^{\lambda(\frac{1}{f})+\varepsilon}\right) + S(r,f) \\ &\leq m\left(r, \frac{f^{n+1}(f-1)}{F}\right) + m(r,F) + O\left(r^{\lambda(\frac{1}{f})+\varepsilon}\right) + S(r,f) \\ &\leq T\left(r, \frac{\Delta_c f}{f}\right) + m(r,F) + O\left(r^{\lambda(\frac{1}{f})+\varepsilon}\right) + S(r,f) \\ &\leq m\left(r, \frac{\Delta_c f}{f}\right) + N\left(r, \frac{1}{f}\right) + m(r,F) + O\left(r^{\lambda(\frac{1}{f})+\varepsilon}\right) + S(r,f) \\ &\leq T(r,f) + T(r,F) + O\left(r^{\lambda(\frac{1}{f})+\varepsilon}\right) + S(r,f). \end{aligned}$$

Hence

$$(n+1)T(r,f) \leq T(r,F) + O\left(r^{\lambda(\frac{1}{f})+\varepsilon}\right) + S(r,f). \tag{13}$$

The second main theorem yields

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,\frac{1}{F-a}) + S(r,F) \\ &\leq \overline{N}(r,\frac{1}{F-a}) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-1}\right) \\ &\quad + \overline{N}(r,\frac{1}{\Delta_c f}) + O\left(r^{\lambda(\frac{1}{f})+\varepsilon}\right) + S(r,f) \\ &\leq \overline{N}\left(r,\frac{1}{F-a}\right) + 2T(r,f) + T(r,\Delta_c f) + O\left(r^{\lambda(\frac{1}{f})+\varepsilon}\right) + S(r,f). \end{aligned}$$

From (12) and above inequality, we get that

$$T(r,F) \leq \overline{N}\left(r,\frac{1}{F-a}\right) + 3T(r,f) + O\left(r^{\lambda(\frac{1}{f})+\varepsilon}\right) + S(r,f). \tag{14}$$

Combining (13) and (14), we have

$$(n-2)T(r,f) \leq \overline{N}\left(r,\frac{1}{F-a}\right) + O\left(r^{\lambda(\frac{1}{f})+\varepsilon}\right) + S(r,f),$$

which is a contradiction to the fact that f is of order σ (f) if F - a has finitely many zeros. The conclusion follows.

6 Proof of Theorem 1.5

Suppose that $f^n(f - 1)\Delta_c f - a$ admits finitely many zeros only. Then, there are two non-zero polynomials $P(z)$, $Q(z)$ such that

$$f^n(f-1)\Delta_c f - a = P(z)e^{Q(z)}. \tag{15}$$

Differentiating (15) and eliminating $e^{Q(z)}$, we obtain

$$(f^n - f^{n-1})F(z,f) = a'P(z) - aP^*(z) - P(z)f(z)^{n-1}f'(z)\Delta_c f, \tag{16}$$

where

$$F(z,f) = (n+1)P(z)f'(z)\Delta_c f + P(z)f(z)(\Delta_c f)' - P^*(z)f(z)\Delta_c f$$

and $P^*(z) = P'(z) + P(z)Q'(z)$.

First, we conclude that $a'P(z) - aP^*(z) \not\equiv 0$. Otherwise, if $a'P(z) - aP^*(z) = 0$, by integrating, then we have

$$\frac{a}{P(z)} = Ae^{Q(z)},$$

where A is a non-zero constant. Hence, we get $e^{Q(z)}$ is a constant and

$$f^n(z)(f(z)-1)\Delta_c f = BP(z) + a, \tag{17}$$

where B is a non-zero constant. Then, from Lemma 2.3 and (17), we obtain that

$$(n+1)T(r,f) \leq 2T(r,f) + O\left(r^{\sigma(f)-1+\varepsilon}\right) + S(r,f),$$

which is a contradiction when $n \geq 2$.

If $F(z, f)$ vanish identically, then

$$aP^*(z) + P(z)f(z)^{n-1}f'(z)\Delta_c f - a'P(z) \equiv 0. \tag{18}$$

Rewrite (18), we get

$$f^{n-2}ff'(z)\Delta_c f = \frac{a'P(z) - aP^*(z)}{P(z)},$$

hence

$$f^{n-2}f^2f'(z)\frac{\Delta_c f}{f} = \frac{a'P(z) - aP^*(z)}{P(z)}. \tag{19}$$

Then, combining Lemmas 2.2, 2.4 and Equation (19), we conclude that

$$m(r, ff'(z)\Delta_c f) = S(r, f),$$

which contradicts (1).

It remains to consider the case that $F(z, f) \not\equiv 0$. We rewrite (16) in the form that

$$(f(z)^{n+2} - f(z)^{n+1})\frac{F(z,f)}{f(z)^2} = a'P(z) - aP^*(z) - P(z)f(z)^{n-1}f'(z)\Delta_c f \tag{20}$$

and

$$f(z)^{n+1}\left((f(z)-1)\frac{F(z,f)}{f(z)^2}\right) = a'P(z) - aP^*(z) - P(z)f(z)^{n-1}\frac{f'(z)\Delta_c f}{f(z)^2}.$$

By Lemmas 2.2 and 2.4, we know that

$$m\left(r, \frac{F(z,f)}{f(z)^2}\right) = S(r,f)$$

and

$$m\left(r, (f(z)-1)\frac{F(z,f)}{f(z)^2}\right) = S(r,f).$$

As $f(z)$ is entire, we get that the poles of $\frac{F(z,f)}{f(z)^2}$ may be located only at the zeros of $f(z)$. If $\frac{F(z,f)}{f(z)^2}$ has infinitely many poles, then from that a zero of $f(z)$ with multiplicity t should be a pole of $t + 1$ of $\frac{F(z,f)}{f(z)^2}$. Since $n \geq 2$, we know that the left side of (20) must have infinitely many zeros, which is a contradiction that $a'P(z) - aP^*(z)$ is a non-zero polynomial. We get

$$N\left(r, \frac{F(z,f)}{f(z)^2}\right) = O(\log r) \quad \text{and} \quad N\left(r, (f(z)-1)\frac{F(z,f)}{f(z)^2}\right) = O(\log r).$$

Hence

$$T\left(r, \frac{F(z,f)}{f(z)^2}\right) = S(r,f)$$

and

$$T\left(r, (f(z)-1)\frac{F(z,f)}{f(z)^2}\right) = S(r,f)$$

as well. Combining these two estimates, we obtain

$$T(r,f) = S(r,f)$$

contradiction. This completes the proof of Theorem 1.5.

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Authors' contributions
XQ completed the main part of this article, JD and LY corrected the main theorems. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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References

1. Yang, CC, Yi, HX: Uniqueness Theory of Meromorphic Functions. Kluwer Academic Publishers, Dordrecht (2003)
2. Bergweiler, W, Langley, JK: Zeros of difference of meromorphic functions. Math Proc Cambridge Philos Soc. **142**, 133–147 (2007). doi:10.1017/S0305004106009777
3. Chiang, YM, Feng, SJ: On the Nevanlinna characteristic of $f(z + \eta)$ and difference equations in the complex plane. Ramanujan J. **16**, 105–129 (2008). doi:10.1007/s11139-007-9101-1
4. Halburd, RG, Korhonen, RJ: Nevanlinna theory for the difference operator. Ann Acad Sci Fenn. **31**, 463–478 (2006)
5. Heittokangas, J, Korhonen, R, Laine, I, Rieppo, J, Zhang, JL: Value sharing results for shifts of meromorphic function, and sufficient conditions for periodicity. J Math Anal Appl. **355**, 352–363 (2009). doi:10.1016/j.jmaa.2009.01.053
6. Qi, XG, Yang, LZ, Liu, K: Uniqueness and periodicity of meromorphic functions concerning difference operator. Comput Math Appl. **60**, 1739–1746 (2010). doi:10.1016/j.camwa.2010.07.004
7. Gross, F: Factorization of meromorphic functions and some open problems, Complex analysis (Proc. Conf., Univ. Kentucky, Lexington, Ky., 1976). In Lecture Notes in Math, vol. 599, pp. 51–67.Springer, Berlin (1977). doi:10.1007/BFb0096825
8. Frank, G, Reinders, M: A unique range set for meromorphic functions with 11 elements. Complex var Theory Appl. **37**, 185–193 (1998). doi:10.1080/17476939808815132
9. Yi, HX: A question of Gross and the uniqueness of entire functions. Nagoya Math J. **138**, 169–177 (1995)
10. Yi, HX: Unicity theorems for meromorphic or entire functions II. Bull Austral Math Soc. **52**, 215–224 (1995). doi:10.1017/S0004972700014635
11. Yi, HX: Unicity theorems for meromorphic or entire functions III. Bull Austral Math Soc. **53**, 71–82 (1996). doi:10.1017/S0004972700016737
12. Fang, ML, Lahiri, I: Unique range set for certain meromorphic functions. Indian J Math. **45**, 141–150 (2003)
13. Hayman, WK: Picard values of meromorphic functions and their derivatives. Ann Math. **70**(2):9–42 (1959)
14. Mues, E: *Über* ein Problem von Hayman. Math Z. **164**, 239–259 (1979). doi:10.1007/BF01182271
15. Bergweiler, W, Eremenko, A: On the singularities of the inverse to a meromorphic function of finite order. Rev Mat Iberoamericana. **11**, 355–373 (1995)
16. Laine, I, Yang, CC: Value Distribution of Difference Polynomials. Proc Japan Acad Ser A Math Sci. **83**, 148–151 (2007). doi:10.3792/pjaa.83.148
17. Qi, XG: Value distribution and uniqueness of difference polynomials and entire solutions of difference equations. Ann Polon Math. **102**(2):129–142 (2011). doi:10.4064/ap102-2-3
18. Laine, I: Nevanlinna Theory and Complex Differential Equations. Walter de Gruyter. Berlin-New York (1993)

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