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Fixed point solutions of generalized mixed equilibrium problems and variational inclusion problems for nonexpansive semigroups

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Abstract

In this paper, we introduce a composite iterative method for solving a common element of the set of solutions of fixed points for nonexpansive semigroups, the set of solutions of generalized mixed equilibrium problems and the set of solutions of the variational inclusion for a β -inverse strongly monotone mapping in a real Hilbert space. We prove that the sequence converges strongly to a common element of the above sets under some mild conditions. Our results improve and extend the corresponding results of Kumam and Wattanawitoon (Math. Comput. Model. 53:998-1006, 2011), Shehu (Math. Comput. Model. 55:1301-1314, 2012), Plubtieng and Punpaeng (Math. Comput. Model. 48:279-286, 2008), Li *et al.* (Nonlinear Anal. 70:3065-3071, 2009), Plubtieng and Wangkeeree (Bull. Korean Math. Soc. 45:717-728, 2008) and some authors.

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1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H . Recall that a mapping $T : C \rightarrow C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T . Let P_C be the projection of H onto the convex subset C . Moreover, we also denote by \mathbb{R} the set of all real numbers.

A one-parameter family $\mathfrak{S} = \{T(t) : 0 \leq t < \infty\}$ is said to be a nonexpansive semigroup on C if it satisfies the following conditions:

- (1) $T(0)x = x$ for all $x \in C$;
- (2) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$;
- (3) $\|T(t)x - T(t)y\| \leq \|x - y\|$ for all $x, y \in C$ and $t \geq 0$;
- (4) for all $x \in C$, $t \mapsto T(t)x$ is continuous.

We denote by $F(\mathfrak{S})$ the set of all common fixed points of \mathfrak{S} , that is, $F(\mathfrak{S}) = \{x \in C : T(t)x = x, 0 \leq t < \infty\}$. It is well known that $F(\mathfrak{S})$ is closed and convex.

A mapping A of C into H is called monotone if $\langle Ax - Ay, x - y \rangle \geq 0$ for all $x, y \in C$. A mapping A is called α -inverse strongly monotone if there exists a positive real number α such that $\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2$ for all $x, y \in C$. A mapping A is called α -strongly monotone if there exists a positive real number α such that $\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2$ for all $x, y \in C$. It is obvious that any α -inverse strongly monotone mappings A is a monotone and $\frac{1}{\alpha}$ -Lipschitz continuous mapping. A linear bounded operator A is called strongly positive if there exists a constant $\bar{\gamma} > 0$ with the property $\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2$ for all $x \in H$.

Let $B : H \rightarrow H$ be a single-valued nonlinear mapping and $M : H \rightarrow 2^H$ be a set-valued mapping. The variational inclusion problem is to find $x \in H$ such that

$$0 \in B(x) + M(x), \tag{1.1}$$

where 0 is the zero vector in H .

The set of solutions of (1.1) is denoted by $I(B, M)$ (see [1–3] and the reference therein).

A set-valued mapping $M : H \rightarrow 2^H$ is called monotone if for all $x, y \in H, f \in M(x)$ and $g \in M(y)$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping M is maximal if its graph $G(M) = \{(f, x) \in H \times H : f \in M(x)\}$ of M is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping M is maximal if and only if for $(x, f) \in H \times H, \langle x - y, f - g \rangle \geq 0$ for all $(y, g) \in G(M)$ imply $f \in M(x)$.

Let $M : H \rightarrow 2^H$ be a set-valued maximal monotone mapping. Then the single-valued mapping $J_{M,\lambda} : H \rightarrow H$ defined by

$$J_{M,\lambda}(x) = (I + \lambda M)^{-1}(x), \quad x \in H,$$

is called the resolvent operator associated with M , where λ is any positive number and I is the identity mapping. It is well known that the resolvent operator $J_{M,\lambda}$ is nonexpansive, 1-inverse strongly monotone and that a solution of problem (1.1) is a fixed point of the operator $J_{M,\lambda}(I - \lambda B)$ for all $\lambda > 0$, where I denotes the identity operator on H (see [4]).

Peng and Yao [5] considered the following generalized mixed equilibrium problem of finding $x \in C$ such that

$$\theta(x, y) + \varphi(y) - \varphi(x) + \langle Fx, y - x \rangle \geq 0, \quad \forall y \in C, \tag{1.2}$$

where $F : C \rightarrow H$ is a nonlinear mapping, $\varphi : C \rightarrow \mathbb{R}$ is a function and $\theta : C \times C \rightarrow \mathbb{R}$ is a bifunction. The set of solutions of problem (1.2) is denoted by *GMEP*.

In the case of $F = 0$, problem (1.2) reduces to the following mixed equilibrium problem of finding $x \in C$ such that

$$\theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C,$$

which was considered by Ceng and Yao [6]. *GMEP* is denoted by *MEP*.

In the case of $\varphi = 0$, problem (1.2) reduces to the following generalized equilibrium problem of finding $x \in C$ such that

$$\theta(x, y) + \langle Fx, y - x \rangle \geq 0, \quad \forall y \in C,$$

which was studied by Takahashi and Takahashi [7].

In the case of $\varphi = 0$ and $F = 0$, problem (1.2) reduces to the equilibrium problem of finding $x \in C$ such that

$$\theta(x, y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solution of (1.3) is denoted by $EP(\theta)$.

In the case $\theta = 0$ and $\varphi = 0$, problem (1.2) reduces to the classical variational inequality of finding $x \in C$ such that

$$\langle Fx, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of problem (1.4) is denoted by $VI(F, C)$.

The problem (1.2) is very general in the sense that it includes, as special cases, optimization problem, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games and others (see [6, 8–10]).

Peng and Yao [5] considered iterative methods for finding a common element of the set of solutions of problem (1.2), the set of solutions of problem (1.4), and the set of fixed points of a nonexpansive mapping.

Let $G_1, G_2 : C \times C \rightarrow \mathbb{R}$ be two bifunctions, $\phi, \psi : C \rightarrow \mathbb{R}$ be two functions and $B_1, B_2 : C \rightarrow H$ be two nonlinear mappings. We consider the generalized mixed equilibria problem of finding $(\bar{x}, \bar{y}) \in C \times C$ such that

$$\begin{cases} G_1(\bar{x}, x) + \phi(x) - \phi(\bar{x}) + \langle B_1\bar{y}, x - \bar{x} \rangle + \frac{1}{\mu_1} \langle \bar{x} - \bar{y}, x - \bar{x} \rangle \geq 0, & \forall x \in C, \\ G_2(\bar{y}, y) + \psi(y) - \psi(\bar{y}) + \langle B_2\bar{x}, y - \bar{y} \rangle + \frac{1}{\mu_2} \langle \bar{y} - \bar{x}, y - \bar{y} \rangle \geq 0, & \forall y \in C, \end{cases} \tag{1.5}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants.

In the case $\phi = \psi = 0$, problem (1.5) reduces to the following problem of the general system of generalized equilibria of finding $(\bar{x}, \bar{y}) \in C \times C$ such that

$$\begin{cases} G_1(\bar{x}, x) + \langle B_1\bar{y}, x - \bar{x} \rangle + \frac{1}{\mu_1} \langle \bar{x} - \bar{y}, x - \bar{x} \rangle \geq 0, & \forall x \in C, \\ G_2(\bar{y}, y) + \langle B_2\bar{x}, y - \bar{y} \rangle + \frac{1}{\mu_2} \langle \bar{y} - \bar{x}, y - \bar{y} \rangle \geq 0, & \forall y \in C, \end{cases} \tag{1.6}$$

which was considered by Ceng and Yao [9].

In the case $G_1 = G_2 = 0$, problem (1.6) reduces to the following problem of the general system of variational inequalities of finding $(\bar{x}, \bar{y}) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1\bar{y} + \bar{x} - \bar{y}, x - \bar{x} \rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B_2\bar{x} + \bar{y} - \bar{x}, y - \bar{y} \rangle \geq 0, & \forall y \in C, \end{cases} \tag{1.7}$$

which was considered by Ceng, Wang and Yao [11].

In particular, if $B_1 = B_2 = D : C \rightarrow H$ is a nonlinear mapping, then problem (1.7) reduces to the following problem of the system of variational inequalities of finding $(\bar{x}, \bar{y}) \in C \times C$

such that

$$\begin{cases} \langle \mu_1 D\bar{y} + \bar{x} - \bar{y}, x - \bar{x} \rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 D\bar{x} + \bar{y} - \bar{x}, y - \bar{y} \rangle \geq 0, & \forall y \in C, \end{cases} \quad (1.8)$$

which was studied by Verma [12].

If $\bar{x} = \bar{y}$ in (1.8), then (1.8) reduces to the classical variational inequality (1.4).

For solving the mixed equilibrium problem, let us give the following assumptions for the bifunction θ , φ and the set C :

- (H1) $\theta(x, y) = 0$ for all $x \in C$;
- (H2) θ is monotone, i.e., $\theta(x, y) + \theta(y, x) \leq 0$ for all $x, y \in C$;
- (H3) for each $x, y, z \in C$

$$\limsup_{t \rightarrow 0} \theta(tz + (1-t)x, y) \leq \theta(x, y);$$

- (H4) for each $x \in C$, $y \mapsto \theta(x, y)$ is convex and lower semicontinuous;
- (A1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D \subset C$ and $y \in C$ such that for any $z \in C \setminus D$

$$\theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r} \langle y - z, z - x \rangle < 0;$$

- (A2) C is a bounded set.

Recently, Shehu [13] studied the problem of finding a common element of the set of common fixed points of a one-parameter nonexpansive semigroup, the set of solutions to a variational inclusion, and the set of solutions to a generalized equilibrium problem. More precisely, the author proved the following theorem.

Theorem 1.1 *Let C be a nonempty closed and convex subset of a real Hilbert space H . Let θ be a bifunction from $C \times C$ to \mathbb{R} satisfying (H1)-(H4), ψ a μ -inverse strongly monotone mapping of C into H , B an α -inverse strongly monotone mapping of C into H and $M : H \rightarrow 2^H$ a maximal monotone mapping. Let $\mathfrak{S} = \{T(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\mathcal{F} = F(\mathfrak{S}) \cap I(B, M) \cap EP(\theta) \neq \emptyset$ and suppose $f : H \rightarrow H$ is a contraction mapping with a constant $\gamma \in (0, 1)$. Let $\{t_n\} \subset (0, \infty)$ be a real sequence such that $\lim_{n \rightarrow \infty} t_n = \infty$. Suppose $\{x_n\}_{n=1}^\infty$ and $\{u_n\}_{n=1}^\infty$ are generated by $x_1 \in H$,*

$$\begin{aligned} \theta(u_n, y) + \langle \psi x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ x_{n+1} &= \beta_n x_n + (1 - \beta_n) \left(\frac{1}{t_n} \int_0^{t_n} T(u) [\alpha_n f(x_n) + (1 - \alpha_n) J_{M, \lambda}(u_n - \lambda A u_n)] du \right) \end{aligned}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ are sequences in $(0, 1)$ and $\{r_n\}_{n=1}^\infty \subset (0, \infty)$ satisfying:

- (i) $\lim_{n \rightarrow \infty} \beta_n = 0$, $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$,
- (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$,
- (iii) $\lambda \in (0, 2\alpha]$,

$$(iv) \quad 0 < a \leq r_n \leq b < 2\mu, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty,$$

$$(v) \quad \lim_{n \rightarrow \infty} \frac{|t_n - t_{n-1}|}{t_n} \frac{1}{\alpha_n(1-\beta_n)} = 0.$$

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to z , where $z = P_{\mathcal{F}}f(z)$.

In this paper, motivated by Shehu [13], Kumam and Wattanawitton [14], Li *et al.* [15], Plubtieng and Punpaeng [10], Plubtieng and Wangkeeree [16], we introduce the following general iterative scheme for finding a common element of the set of common fixed points of a one-parameter nonexpansive semigroup, the set of solutions of the generalized mixed equilibrium problem (1.2), the set of solutions to a variational inclusion (1.1), and the set of solutions of the generalized mixed equilibria problem (1.5), which solves the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{F},$$

where $\mathcal{F} = F(\mathfrak{S}) \cap GMEP \cap \Omega \cap I(B, M)$ and Ω is the set of solutions of the generalized equilibria problem (1.5).

The results obtained in this paper improve and extend the recent results announced by [10, 13–16] and many others.

2 Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H . For every point $x \in H$ there exists a unique nearest point of C , denoted by $P_C x$ such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection of H onto C . We know that P_C is a firmly nonexpansive mapping of H onto C , i.e.,

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

Further, for any $x \in H$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y \rangle \geq 0, \quad \forall y \in C.$$

It is also known that H satisfies Opial's condition [17], that is, if for each sequence $\{x_n\}_{n=1}^{\infty}$ in H which converges weakly to a point $x \in H$, we have

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|, \quad \forall y \in H, y \neq x.$$

In order to prove our main results in the next section, we need the following lemmas.

Lemma 2.1 ([6]) *Let C be a nonempty closed convex subset of H . Let $\theta : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying conditions (H1)-(H4) and let $\varphi : C \rightarrow \mathbb{R}$ be a lower semicontinuous and convex function. For $r > 0$ and $x \in H$ define a mapping*

$$T_r^{(\theta, \varphi)}(x) = \left\{ z \in C : \theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}$$

for all $x \in H$. Assume that either (A1) or (A2) holds.

Then the following results hold:

- (i) $T_r^{(\theta, \varphi)}(x) \neq \emptyset$ for each $x \in H$ and $T_r^{(\theta, \varphi)}$ is single-valued;
- (ii) $T_r^{(\theta, \varphi)}$ is firmly nonexpansive, i.e., for any $x, y \in H$

$$\|T_r^{(\theta, \varphi)}x - T_r^{(\theta, \varphi)}y\|^2 \leq \langle T_r^{(\theta, \varphi)}x - T_r^{(\theta, \varphi)}y, x - y \rangle;$$

- (iii) $F(T_r^{(\theta, \varphi)}) = MEP(\theta, \varphi)$;
- (iv) $MEP(\theta, \varphi)$ is closed and convex.

By similar argument as in the proof of Lemma 2.2 in [6], we have the following result.

Lemma 2.2 *Let C be a nonempty closed convex subset of H . Let $G_1, G_2 : C \times C \rightarrow \mathbb{R}$ be two bifunctions satisfying conditions (H1)-(H4). Let $\phi, \psi : C \rightarrow \mathbb{R}$ be two lower semicontinuous and convex functions with restriction (A1) or (A2) and let the mappings $B_1, B_2 : C \rightarrow H$ be β_1 -inverse strongly monotone and β_2 -inverse strongly monotone, respectively. Let $\mu_1 \in (0, 2\beta_1)$ and $\mu_2 \in (0, 2\beta_2)$. Then for given $\bar{x}, \bar{y} \in C$, (\bar{x}, \bar{y}) is a solution of problem (1.5) if and only if \bar{x} is a fixed point of the mapping $\Gamma : C \rightarrow C$ defined by*

$$\Gamma(x) = T_{\mu_1}^{(G_1, \phi)} \left[T_{\mu_2}^{(G_2, \psi)}(x - \mu_2 B_2 x) - \mu_1 B_1 T_{\mu_2}^{(G_2, \psi)}(x - \mu_2 B_2 x) \right], \quad \forall x \in C,$$

where $\bar{y} = T_{\mu_2}^{(G_2, \psi)}(\bar{x} - \mu_2 B_2 \bar{x})$.

Remark 2.1 Under the conditions of Lemma 2.2, the set of fixed points of the mapping Γ is denoted by Ω .

Proposition 2.1 ([7]) *Let C, H, θ, φ and $T_r^{(\theta, \varphi)}$ be as in Lemma 2.1. Then the following holds:*

$$\|T_s^{(\theta, \varphi)}x - T_t^{(\theta, \varphi)}x\|^2 \leq \frac{s-t}{s} \langle T_s^{(\theta, \varphi)}x - T_t^{(\theta, \varphi)}x, T_s^{(\theta, \varphi)}x - x \rangle$$

for all $s, t > 0$ and $x \in H$.

Lemma 2.3 ([18]) *Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset of C of a real Hilbert space H . If T has a fixed point, then $I - T$ is demiclosed, that is, when $\{x_n\}_{n=1}^\infty$ is a sequence in C converging weakly to some $x \in C$ and the sequence $\{(I - T)x_n\}_{n=1}^\infty$ converges strongly to some y , it follows that $(I - T)x = y$.*

Lemma 2.4 ([4]) *Let $M : H \rightarrow 2^H$ be a maximal monotone mapping and let $B : H \rightarrow H$ be a monotone and Lipschitz continuous mapping. Then the mapping $M + B : H \rightarrow 2^H$ is a maximal monotone mapping.*

Lemma 2.5 ([19]) *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $\mathfrak{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C . Then for any $h \geq 0$,*

$$\limsup_{t \rightarrow \infty} \sup_{x \in C} \left\| \frac{1}{t} \int_0^t T(s)x ds - T(h) \left(\frac{1}{t} \int_0^t T(s)x ds \right) \right\| = 0.$$

Lemma 2.6 ([12]) *Let $\{a_n\}_{n=1}^\infty$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n + \sigma_n, \quad n \in N,$$

where $\{\delta_n\}_{n=1}^\infty \subset (0, 1)$ and $\{b_n\}_{n=1}^\infty, \{\sigma_n\}_{n=1}^\infty$ are sequences in \mathbb{R} such that

- (i) $\lim_{n \rightarrow \infty} \delta_n = 0, \sum_{n=1}^\infty \delta_n = \infty$;
- (ii) $\limsup_{n \rightarrow \infty} \frac{b_n}{\delta_n} \leq 0$;
- (iii) $\sigma_n \geq 0, \sum_{n=1}^\infty \sigma_n < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.7 ([19]) *Assume that A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho \bar{\gamma}$.*

3 Main results

Now we state and prove our main results.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\theta, G_1, G_2 : C \times C \rightarrow \mathbb{R}$ be three bifunctions which satisfy assumptions (H1)-(H4) and $\varphi, \phi, \psi : C \rightarrow \mathbb{R}$ be three lower semicontinuous and convex functions with restriction (A1) or (A2). Let $F, B, B_1, B_2 : C \rightarrow H$ be ζ -inverse strongly monotone, β -inverse strongly monotone, β_1 -inverse strongly monotone and β_2 -inverse strongly monotone, respectively and $M : H \rightarrow 2^H$ be a maximal monotone mapping. Let $\mathfrak{S} = \{T(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\mathcal{F} = F(\mathfrak{S}) \cap GMEP \cap \Omega \cap I(B, M) \neq \emptyset$. Let $\{t_n\} \subset (0, \infty)$ be a real sequence such that $\lim_{n \rightarrow \infty} t_n = \infty$. Let f be a contraction from C into itself with a constant α ($0 < \alpha < 1$) and let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ such that $\|A\| \leq 1$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha(1-\alpha_n\bar{\gamma})}$. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} z_n = T_{\delta_n}^{(\theta, \varphi)}(x_n - \delta_n Fx_n), \\ y_n = T_{\mu_1}^{(G_1, \phi_1)}[T_{\mu_2}^{(G_2, \psi)}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T_{\mu_2}^{(G_2, \psi)}(z_n - \mu_2 B_2 z_n)], \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n \\ \quad + ((1 - \beta_n)I - \alpha_n A)[\frac{1}{t_n} \int_0^{t_n} T(u)(J_{M, \lambda}(y_n - \lambda B y_n)) du], \quad n \geq 1, \end{cases} \quad (3.1)$$

where $\alpha_n \in (0, 1), \mu_1 \in (0, 2\beta_1], \mu_2 \in (0, 2\beta_2], \lambda \in (0, 2\beta], \delta_n \in (0, 2\zeta]$ satisfy the following conditions:

- (C1) $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty$ and $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
- (C2) $\sum_{n=1}^\infty \beta_n < \infty$ and $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$;
- (C3) $\liminf_{n \rightarrow \infty} \delta_n > 0$ and $\sum_{n=1}^\infty |\delta_{n+1} - \delta_n| < \infty$;
- (C4) $\lim_{n \rightarrow \infty} \frac{t_n - t_{n-1}}{t_n} \frac{1}{\alpha_n} = 0$.

Then $\{x_n\}$ converges strongly to $x^* = P_{\mathcal{F}}(\gamma f + (I - A))(x^*)$, which solves the following variational inequality:

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{F},$$

and (x^*, y^*) is a solution of problem (1.5), where $y^* = T_{\mu_2}^{(G_2, \psi)}(x^* - \mu_2 B_2 x^*)$.

Proof Since F is a ζ -inverse strongly monotone mapping, we have

$$\begin{aligned} \|(I - \delta_n F)x - (I - \delta_n F)y\|^2 &= \|x - y\|^2 - 2\delta_n \langle x - y, Fx - Fy \rangle + \delta_n^2 \|Fx - Fy\|^2 \\ &\leq \|x - y\|^2 + \delta_n(\delta_n - 2\zeta) \|Fx - Fy\|^2. \end{aligned} \quad (3.2)$$

In similar way, we can obtain

$$\begin{aligned} \|(I - \lambda B)x - (I - \lambda B)y\| &\leq \|x - y\|^2 + \lambda(\lambda - 2\beta) \|Bx - By\|^2, \\ \|(I - \mu_1 B_1)x - (I - \mu_1 B_1)y\| &\leq \|x - y\|^2 + \mu_1(\mu_1 - 2\beta_1) \|B_1x - B_1y\|^2, \\ \|(I - \mu_2 B_2)x - (I - \mu_2 B_2)y\| &\leq \|x - y\|^2 + \mu_2(\mu_2 - 2\beta_2) \|B_2x - B_2y\|^2. \end{aligned} \quad (3.3)$$

Noticing that $\lim_{n \rightarrow \infty} \alpha_n = 0 = \lim_{n \rightarrow \infty} \beta_n$, we may assume, with no loss of generality, that $\frac{\alpha_n}{1 - \beta_n} < \|A\|^{-1}$ for all $n \geq 1$. From Lemma 2.7 we know that if $0 \leq \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho \bar{\gamma}$. Since A is a strongly positive bounded linear operator on H , we have

$$\|A\| = \sup\{|\langle Ax, x \rangle| : x \in H, \|x\| = 1\}.$$

Observe that

$$\langle ((1 - \beta_n)I - \alpha_n A)x, x \rangle = 1 - \beta_n - \alpha_n \langle Ax, x \rangle \geq 1 - \beta_n - \alpha_n \|A\| \geq 0.$$

This shows that $(1 - \beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned} \|(1 - \beta_n)I - \alpha_n A\| &= \sup\{\langle ((1 - \beta_n)I - \alpha_n A)x, x \rangle : x \in H, \|x\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n \langle Ax, x \rangle : x \in H, \|x\| = 1\} \\ &\leq 1 - \beta_n - \alpha_n \bar{\gamma}. \end{aligned}$$

We divide the proof into several steps.

Step 1. $\{x_n\}$ is bounded.

Indeed, take $p \in \mathcal{F}$ arbitrarily. Since $p = T_{\delta_n}^{(\theta, \varphi)}(p - \delta_n Fp)$, F is ζ -inverse strongly monotone and $0 < \delta_n \leq 2\zeta$, we obtain for any $n \geq 1$

$$\begin{aligned} \|z_n - p\|^2 &= \|T_{\delta_n}^{(\theta, \varphi)}(x_n - \delta_n Fx_n) - T_{\delta_n}^{(\theta, \varphi)}(p - \delta_n Fp)\|^2 \\ &\leq \|(x_n - p) - \delta_n(Fx_n - Fp)\|^2 \\ &= \|x_n - p\|^2 - 2\delta_n \langle x_n - p, Fx_n - Fp \rangle + \delta_n^2 \|Fx_n - Fp\|^2 \\ &\leq \|x_n - p\|^2 + \delta_n(\delta_n - 2\zeta) \|Fx_n - Fp\|^2 \\ &\leq \|x_n - p\|^2. \end{aligned} \quad (3.4)$$

Putting $u_n = T_{\mu_2}^{(G_2, \psi)}(z_n - \mu_2 B_2 z_n)$, $w_n = J_{M, \lambda}(y_n - \lambda B y_n)$, and $u = T_{\mu_2}^{(G_2, \psi)}(p - \mu_2 B_2 p)$, we have

$$\begin{aligned} \|u_n - u\|^2 &= \|T_{\mu_2}^{(G_2, \psi)}(z_n - \mu_2 B_2 z_n) - T_{\mu_2}^{(G_2, \psi)}(p - \mu_2 B_2 p)\|^2 \\ &\leq \|(z_n - p) - \mu_2(B_2 z_n - B_2 p)\|^2 \end{aligned}$$

$$\begin{aligned}
 &= \|z_n - p\|^2 - 2\mu_2 \langle z_n - p, B_2 z_n - B_2 p \rangle + \mu_2^2 \|B_2 z_n - B_2 p\|^2 \\
 &\leq \|z_n - p\|^2 + \mu_2(\mu_2 - 2\beta_2) \|B_2 z_n - B_2 p\|^2 \\
 &\leq \|z_n - p\|^2
 \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
 \|w_n - p\|^2 &= \|J_{M,\lambda}(y_n - \lambda B y_n) - J_{M,\lambda}(p - \lambda B p)\|^2 \\
 &\leq \|(y_n - p) - \lambda(B y_n - B p)\|^2 \\
 &= \|y_n - p\|^2 - 2\lambda \langle y_n - p, B y_n - B p \rangle + \lambda^2 \|B y_n - B p\|^2 \\
 &\leq \|y_n - p\|^2 + \lambda(\lambda - 2\beta) \|B y_n - B p\|^2 \\
 &\leq \|y_n - p\|^2.
 \end{aligned} \tag{3.6}$$

And since

$$\begin{aligned}
 p &= T_{\mu_1}^{(G_1, \phi)} [T_{\mu_2}^{(G_2, \psi)}(p - \mu_2 B_2 p) - \mu_1 B_1 T_{\mu_2}^{(G_2, \psi)}(p - \mu_2 B_2 p)] \\
 &= T_{\mu_1}^{(G_1, \phi)}(u - \mu_1 B_1 u),
 \end{aligned}$$

we know that for any $n \geq 1$

$$\begin{aligned}
 \|y_n - p\|^2 &= \|T_{\mu_1}^{(G_1, \phi)}(u_n - \mu_1 B_1 u_n) - T_{\mu_1}^{(G_1, \phi)}(u - \mu_1 B_1 u)\|^2 \\
 &\leq \|(u_n - u) - \mu_1(B_1 u_n - B_1 u)\|^2 \\
 &= \|u_n - u\|^2 - 2\mu_1 \langle u_n - u, B_1 u_n - B_1 u \rangle + \mu_1^2 \|B_1 u_n - B_1 u\|^2 \\
 &\leq \|u_n - u\|^2 + \mu_1(\mu_1 - 2\beta_1) \|B_1 u_n - B_1 u\|^2 \\
 &\leq \|u_n - u\|^2 \\
 &\leq \|z_n - p\|^2.
 \end{aligned} \tag{3.7}$$

Since A is a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$, we have

$$\begin{aligned}
 &\|x_{n+1} - p\| \\
 &= \left\| \alpha_n (\gamma f(x_n) - Ap) + \beta_n (x_n - p) + ((1 - \beta_n)I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} [T(u)w_n - T(u)p] du \right\| \\
 &\leq \alpha_n \|\gamma f(x_n) - Ap\| + \beta_n \|x_n - p\| + (1 - \beta_n - \alpha_n \bar{\gamma}) \|w_n - p\| \\
 &\leq \alpha_n \gamma \|f(x_n) - f(p)\| + \alpha_n \|\gamma f(p) - Ap\| + \beta_n \|x_n - p\| + (1 - \beta_n - \alpha_n \bar{\gamma}) \|x_n - p\| \\
 &\leq [1 - \alpha_n(\bar{\gamma} - \gamma\alpha)] \|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\|.
 \end{aligned}$$

By induction, we obtain for all $n \geq 1$

$$\|x_n - p\| \leq \max \left\{ \|x_1 - p\|, \frac{1}{\bar{\gamma} - \gamma\alpha} \|\gamma f(p) - Ap\| \right\}.$$

Hence $\{x_n\}_{n=1}^\infty$ is bounded. So are $\{y_n\}_{n=1}^\infty$, $\{z_n\}_{n=1}^\infty$, $\{w_n\}_{n=1}^\infty$.

Step 2. We show that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$.

We estimate $\|y_{n+1} - y_n\|$. From (3.1), we have

$$\begin{aligned}
 \|y_{n+1} - y_n\|^2 &= \|T_{\mu_1}^{(G_1, \phi)}(u_{n+1} - \mu_1 B_1 u_{n+1}) - T_{\mu_1}^{(G_1, \phi)}(u_n - \mu_1 B_1 u_n)\|^2 \\
 &\leq \|(u_{n+1} - u_n) - \mu_1(B_1 u_{n+1} - B_1 u_n)\|^2 \\
 &\leq \|u_{n+1} - u_n\|^2 + \mu_1(\mu_1 - 2\beta_1)\|B_1 u_{n+1} - B_1 u_n\|^2 \\
 &\leq \|u_{n+1} - u_n\|^2 \\
 &= \|T_{\mu_2}^{(G_2, \psi)}(z_{n+1} - \mu_2 B_2 z_{n+1}) - T_{\mu_2}^{(G_2, \psi)}(z_n - \mu_2 B_2 z_n)\|^2 \\
 &\leq \|(z_{n+1} - z_n) - \mu_2(B_2 z_{n+1} - B_2 z_n)\|^2 \\
 &\leq \|z_{n+1} - z_n\|^2 + \mu_2(\mu_2 - 2\beta_2)\|B_2 z_{n+1} - B_2 z_n\|^2 \\
 &\leq \|z_{n+1} - z_n\|^2,
 \end{aligned} \tag{3.8}$$

$$\begin{aligned}
 &\|(x_{n+1} - \delta_{n+1} Fx_{n+1}) - (x_n - \delta_n Fx_n)\| \\
 &= \|x_{n+1} - x_n - \delta_{n+1}(Fx_{n+1} - Fx_n) + (\delta_n - \delta_{n+1})Fx_n\| \\
 &\leq \|x_{n+1} - x_n - \delta_{n+1}(Fx_{n+1} - Fx_n)\| + |\delta_n - \delta_{n+1}|\|Fx_n\| \\
 &\leq \|x_{n+1} - x_n\| + |\delta_n - \delta_{n+1}|\|Fx_n\|
 \end{aligned}$$

and

$$\begin{aligned}
 \|z_{n+1} - z_n\| &= \|T_{\delta_{n+1}}^{(\theta, \varphi)}(x_{n+1} - \delta_{n+1} Fx_{n+1}) - T_{\delta_n}^{(\theta, \varphi)}(x_n - \delta_n Fx_n)\| \\
 &\leq \|T_{\delta_{n+1}}^{(\theta, \varphi)}(x_{n+1} - \delta_{n+1} Fx_{n+1}) - T_{\delta_{n+1}}^{(\theta, \varphi)}(x_n - \delta_n Fx_n)\| \\
 &\quad + \|T_{\delta_{n+1}}^{(\theta, \varphi)}(x_n - \delta_n Fx_n) - T_{\delta_n}^{(\theta, \varphi)}(x_n - \delta_n Fx_n)\| \\
 &\leq \|(x_{n+1} - \delta_{n+1} Fx_{n+1}) - (x_n - \delta_n Fx_n)\| \\
 &\quad + \|T_{\delta_{n+1}}^{(\theta, \varphi)}(x_n - \delta_n Fx_n) - T_{\delta_n}^{(\theta, \varphi)}(x_n - \delta_n Fx_n)\| \\
 &\leq \|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\|Fx_n\| \\
 &\quad + \|T_{\delta_{n+1}}^{(\theta, \varphi)}(x_n - \delta_n Fx_n) - T_{\delta_n}^{(\theta, \varphi)}(x_n - \delta_n Fx_n)\|.
 \end{aligned} \tag{3.9}$$

Without loss of generality, let us assume that there exists a real number a such that $\delta_n > a > 0$ for all n . Utilizing Proposition 2.1 we have

$$\begin{aligned}
 &\|T_{\delta_{n+1}}^{(\theta, \varphi)}(x_n - \delta_n Fx_n) - T_{\delta_n}^{(\theta, \varphi)}(x_n - \delta_n Fx_n)\| \\
 &\leq \frac{|\delta_{n+1} - \delta_n|}{\delta_{n+1}} \|T_{\delta_{n+1}}^{(\theta, \varphi)}(I - \delta_n F)x_n - x_n\| \\
 &\leq \frac{|\delta_{n+1} - \delta_n|}{a} \|T_{\delta_{n+1}}^{(\theta, \varphi)}(I - \delta_n F)x_n - x_n\|.
 \end{aligned} \tag{3.10}$$

It follows from (3.8)-(3.10) that

$$\begin{aligned}
 \|w_{n+1} - w_n\| &= \|J_{M, \lambda}(y_{n+1} - \lambda B u_{n+1}) - J_{M, \lambda}(y_n - \lambda B y_n)\| \\
 &\leq \|(y_{n+1} - \lambda B y_{n+1}) - (y_n - \lambda B y_n)\| \\
 &\leq \|y_{n+1} - y_n\|
 \end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
 \|y_{n+1} - y_n\| &\leq \|z_{n+1} - z_n\| \\
 &\leq \|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n| \|Fx_n\| \\
 &\quad + \frac{|\delta_{n+1} - \delta_n|}{a} \|T_{\delta_{n+1}}^{(\theta, \varphi)}(I - \delta_n F)x_n - x_n\|. \tag{3.12}
 \end{aligned}$$

Put $h_n = \frac{1}{t_n} \int_0^{t_n} T(u)w_n du$ for all $n \geq 1$. We note that

$$\begin{aligned}
 \|h_n - h_{n-1}\| &= \left\| \frac{1}{t_n} \int_0^{t_n} T(u)w_n du - \frac{1}{t_{n-1}} \int_0^{t_{n-1}} T(u)w_{n-1} du \right\| \\
 &\leq \left\| \frac{1}{t_n} \int_0^{t_n} [T(u)w_n - T(u)w_{n-1}] du \right\| \\
 &\quad + \left| \frac{1}{t_n} - \frac{1}{t_{n-1}} \right| \left\| \int_0^{t_{n-1}} [T(u)w_{n-1} - T(u)p] du \right\| \\
 &\quad + \frac{1}{t_n} \left\| \int_{t_{n-1}}^{t_n} [T(u)w_{n-1} - T(u)p] du \right\| \\
 &\leq \|y_n - y_{n-1}\| + \frac{2|t_n - t_{n-1}|}{t_n} \|y_{n-1} - p\|. \tag{3.13}
 \end{aligned}$$

Using (3.12) and (3.13) we get

$$\begin{aligned}
 &\|x_{n+1} - x_n\| \\
 &= \left\| [\alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)h_n] \right. \\
 &\quad \left. - [\alpha_{n-1} \gamma f(x_{n-1}) + \beta_{n-1} x_{n-1} + ((1 - \beta_{n-1})I - \alpha_{n-1} A)h_{n-1}] \right\| \\
 &\leq \alpha_n \gamma \alpha \|x_n - x_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + \beta_n \|x_n - x_{n-1}\| \\
 &\quad + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + (1 - \alpha_n \bar{\gamma}) \|h_n - h_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|Ah_{n-1}\| \\
 &\quad + |\beta_n - \beta_{n-1}| \|h_n - h_{n-1}\| \\
 &\leq \alpha_n \gamma \alpha \|x_n - x_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + \beta_n \|x_n - x_{n-1}\| \\
 &\quad + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + (1 - \alpha_n \bar{\gamma}) \left[\|y_n - y_{n-1}\| + \frac{2|t_n - t_{n-1}|}{t_n} \|y_{n-1} - p\| \right] \\
 &\quad + |\alpha_n - \alpha_{n-1}| \|Ah_{n-1}\| + |\beta_n - \beta_{n-1}| \left[\|y_n - y_{n-1}\| + \frac{2|t_n - t_{n-1}|}{t_n} \|y_{n-1} - p\| \right] \\
 &\leq \alpha_n \gamma \alpha \|x_n - x_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + \beta_n \|x_n - x_{n-1}\| \\
 &\quad + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + (1 - \alpha_n \bar{\gamma}) \left[\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}| \|Fx_{n-1}\| \right. \\
 &\quad \left. + \frac{|\delta_n - \delta_{n-1}|}{a} \|T_{\delta_n}^{(\theta, \varphi)}(I - \delta_{n-1} F)x_{n-1} - x_{n-1}\| \right] \\
 &\quad + (1 - \alpha_n \bar{\gamma}) \frac{2|t_n - t_{n-1}|}{t_n} \|y_{n-1} - p\| + |\alpha_n - \alpha_{n-1}| \|Ah_{n-1}\| \\
 &\quad + |\beta_n - \beta_{n-1}| \left[\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}| \|Fx_{n-1}\| \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{|\delta_n - \delta_{n-1}|}{a} \left\| T_{\delta_n}^{(\theta, \varphi)}(I - \delta_{n-1}F)x_{n-1} - x_{n-1} \right\| \Big] \\
 & + |\beta_n - \beta_{n-1}| \frac{2|t_n - t_{n-1}|}{t_n} \|y_{n-1} - p\| \\
 \leq & \left[1 - (\bar{\gamma} - \gamma\alpha)\alpha_n \right] \|x_n - x_{n-1}\| + \frac{4|t_n - t_{n-1}|}{t_n} \|y_{n-1} - p\| \\
 & + |\alpha_n - \alpha_{n-1}| (\gamma \|f(x_{n-1})\| + \|Ah_{n-1}\|) + \beta_n [\|x_n\| + \|x_{n-1}\|] \\
 & + |\beta_n - \beta_{n-1}| [\|x_n\| + 2\|x_{n-1}\|] \\
 & + 2|\delta_n - \delta_{n-1}| \left[\|Fx_{n-1}\| + \frac{1}{a} \left\| T_{\delta_n}^{(\theta, \varphi)}(I - \delta_{n-1}F)x_{n-1} - x_{n-1} \right\| \right] \\
 = & \left[1 - (\bar{\gamma} - \gamma\alpha)\alpha_n \right] \|x_n - x_{n-1}\| + D \left[\frac{4|t_n - t_{n-1}|}{t_n} + |\alpha_n - \alpha_{n-1}| + \beta_n \right. \\
 & \left. + |\beta_n - \beta_{n-1}| + 2|\delta_n - \delta_{n-1}| \right],
 \end{aligned}$$

where $D = \max\{3 \sup_{n \geq 1} \|x_n\|, \sup_{n \geq 1} (\gamma \|f(x_n)\| + \|Ah_n\|), \sup_{n \geq 1} (\|Fx_n\| + \frac{1}{a} \|T_{\delta_n}^{(\theta, \varphi)}(I - \delta_n F) \times x_n - x_n\|), \sup_{n \geq 1} \|y_n - p\|\}$. From Lemma 2.6, taking $\delta_n = (\bar{\gamma} - \gamma\alpha)\alpha_n$, $b_n = \frac{4|t_n - t_{n-1}|}{t_n} D$, $\sigma_n = D(|\alpha_n - \alpha_{n-1}| + \beta_n + |\beta_n - \beta_{n-1}| + 2|\delta_n - \delta_{n-1}|)$, it follows that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$.

Step 3. $\lim_{n \rightarrow \infty} \|Fx_n - Fp\| = 0$, $\lim_{n \rightarrow \infty} \|B_1u_n - B_1u\| = 0$, $\lim_{n \rightarrow \infty} \|B_2z_n - B_2p\| = 0$ and $\lim_{n \rightarrow \infty} \|By_n - Bp\| = 0$.

Indeed, from (3.1), (3.4), (3.5), and (3.7) we get

$$\begin{aligned}
 & \|x_{n+1} - p\|^2 \\
 = & \left\| \alpha_n (\gamma f(x_n) - Ap) + \beta_n (x_n - p) + (1 - \beta_n)I \right. \\
 & \left. - \alpha_n A \frac{1}{t_n} \int_0^{t_n} [T(u)(J_{M,\lambda}(y_n - \lambda By_n)) - T(u)(J_{M,\lambda}(p - \lambda Bp))] du \right\|^2 \\
 \leq & \alpha_n \|\gamma f(x_n) - Ap\|^2 + \beta_n \|x_n - p\|^2 + (1 - \beta_n - \alpha_n \bar{\gamma}) \|y_n - p\|^2 \\
 \leq & \alpha_n \|\gamma f(x_n) - Ap\|^2 + \beta_n \|x_n - p\|^2 \\
 & + (1 - \beta_n - \alpha_n \bar{\gamma}) [\|u_n - u\|^2 + \mu_1(\mu_1 - 2\beta_1) \|B_1u_n - B_1u\|^2] \\
 \leq & \alpha_n \|\gamma f(x_n) - Ap\|^2 + \beta_n \|x_n - p\|^2 + (1 - \beta_n - \alpha_n \bar{\gamma}) [\|z_n - p\|^2 \\
 & + \mu_2(\mu_2 - 2\beta_2) \|B_2z_n - B_2p\|^2 + \mu_1(\mu_1 - 2\beta_1) \|B_1u_n - B_1u\|^2] \\
 \leq & \alpha_n \|\gamma f(x_n) - Ap\|^2 + \beta_n \|x_n - p\|^2 + (1 - \beta_n - \alpha_n \bar{\gamma}) [\|x_n - p\|^2 \\
 & + \delta_n(\delta_n - 2\zeta) \|Fx_n - Fp\|^2 + \mu_2(\mu_2 - 2\beta_2) \|B_2z_n - B_2p\|^2 \\
 & + \mu_1(\mu_1 - 2\beta_1) \|B_1u_n - B_1u\|^2].
 \end{aligned}$$

Therefore

$$\begin{aligned}
 & \delta_n(2\zeta - \delta_n) \|Fx_n - Fp\|^2 + \mu_2(2\beta_2 - \mu_2) \|B_2z_n - B_2p\|^2 \\
 & + \mu_1(2\beta_1 - \mu_1) \|B_1u_n - B_1u\|^2
 \end{aligned}$$

$$\begin{aligned} &\leq \alpha_n \|\gamma f(x_n) - Ap\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\leq \alpha_n \|\gamma f(x_n) - Ap\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\|. \end{aligned}$$

Since $\alpha_n \rightarrow 0$ and $\|x_n - x_{n+1}\| \rightarrow 0$ as $n \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} \|Fx_n - Fp\| = 0$, $\lim_{n \rightarrow \infty} \|B_2z_n - B_2p\| = 0$ and $\lim_{n \rightarrow \infty} \|B_1u_n - B_1u\| = 0$. Similarly, from (3.4), (3.6), and (3.7) we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n \|\gamma f(x_n) - Ap\|^2 + \beta_n \|x_n - p\|^2 + (1 - \beta_n - \alpha_n \bar{\gamma}) \|w_n - p\|^2 \\ &\leq \alpha_n \|\gamma f(x_n) - Ap\|^2 + (1 - \alpha_n \bar{\gamma}) \|x_n - p\|^2 + \lambda(\lambda - 2\beta) \|By_n - Bp\|^2, \end{aligned}$$

which implies that

$$\lambda(2\beta - \lambda) \|By_n - Bp\|^2 \leq \alpha_n \|\gamma f(x_n) - Ap\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\|.$$

We also have $\lim_{n \rightarrow \infty} \|By_n - Bp\| = 0$.

Step 4. We claim that $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$, $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$ and $\lim_{n \rightarrow \infty} \|w_n - y_n\| = 0$.

Indeed, from Lemma 2.1, (3.4), (3.5), and (3.7) we have

$$\begin{aligned} \|u_n - u\|^2 &= \|T_{\mu_2}^{(G_2, \psi)}(z_n - \mu_2 B_2 z_n) - T_{\mu_2}^{(G_2, \psi)}(p - \mu_2 B_2 p)\|^2 \\ &\leq \langle (z_n - \mu_2 B_2 z_n) - (p - \mu_2 B_2 p), u_n - u \rangle \\ &= \frac{1}{2} [\|(z_n - \mu_2 B_2 z_n) - (p - \mu_2 B_2 p)\|^2 + \|u_n - u\|^2 \\ &\quad - \|(z_n - \mu_2 B_2 z_n) - (p - \mu_2 B_2 p) - (u_n - u)\|^2] \\ &\leq \frac{1}{2} [\|x_n - p\|^2 + \|u_n - u\|^2 - \|(z_n - u_n) - (p - u)\|^2 \\ &\quad + 2\mu_2 \langle (z_n - u_n) - (p - u), B_2 z_n - B_2 p \rangle - \mu_2^2 \|B_2 z_n - B_2 p\|^2] \end{aligned}$$

and

$$\begin{aligned} \|y_n - p\|^2 &= \|T_{\mu_1}^{(G_1, \phi)}(u_n - \mu_1 B_1 u_n) - T_{\mu_1}^{(G_1, \phi)}(u - \mu_1 B_1 u)\|^2 \\ &\leq \langle (u_n - \mu_1 B_1 u_n) - (u - \mu_1 B_1 u), y_n - p \rangle \\ &= \frac{1}{2} [\|(u_n - \mu_1 B_1 u_n) - (u - \mu_1 B_1 u)\|^2 + \|y_n - p\|^2 \\ &\quad - \|(u_n - \mu_1 B_1 u_n) - (u - \mu_1 B_1 u) - (y_n - p)\|^2] \\ &\leq \frac{1}{2} [\|u_n - u\|^2 + \|y_n - p\|^2 - \|(u_n - y_n) + (p - u)\|^2 \\ &\quad + 2\mu_1 \langle B_1 u_n - B_1 u, (u_n - y_n) + (p - u) \rangle - \mu_1^2 \|B_1 u_n - B_1 u\|^2] \\ &\leq \frac{1}{2} [\|x_n - p\|^2 + \|y_n - p\|^2 - \|(u_n - y_n) + (p - u)\|^2 \\ &\quad + 2\mu_1 \langle B_1 u_n - B_1 u, (u_n - y_n) + (p - u) \rangle], \end{aligned}$$

which imply that

$$\begin{aligned} \|u_n - u\|^2 &\leq \|x_n - p\|^2 - \|(z_n - u_n) - (p - u)\|^2 \\ &\quad + 2\mu_2 \langle (z_n - u_n) - (p - u), B_2 z_n - B_2 p \rangle - \mu_1^2 \|B_2 z_n - B_2 p\|^2 \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \|y_n - p\|^2 &\leq \|x_n - p\|^2 - \|(u_n - y_n) + (p - u)\|^2 \\ &\quad + 2\mu_1 \|B_1 u_n - B_1 u\| \|(u_n - y_n) + (p - u)\|. \end{aligned} \tag{3.15}$$

It follows from (3.15) that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n \|\gamma f(x_n) - Ap\|^2 + \beta_n \|x_n - p\|^2 + (1 - \beta_n - \alpha_n \bar{\gamma}) \|y_n - p\|^2 \\ &\leq \alpha_n \|\gamma f(x_n) - Ap\|^2 + (1 - \alpha_n \bar{\gamma}) \|x_n - p\|^2 \\ &\quad - (1 - \beta_n - \alpha_n \bar{\gamma}) \|(u_n - y_n) + (p - u)\|^2 \\ &\quad + 2\mu_1 (1 - \beta_n - \alpha_n \bar{\gamma}) \|B_1 u_n - B_1 u\| \|(u_n - y_n) + (p - u)\|, \end{aligned}$$

which gives

$$\begin{aligned} &(1 - \beta_n - \alpha_n \bar{\gamma}) \|(u_n - y_n) + (p - u)\|^2 \\ &\leq \alpha_n \|\gamma f(x_n) - Ap\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\| \\ &\quad + 2\mu_1 (1 - \beta_n - \alpha_n \bar{\gamma}) \|B_1 u_n - B_1 u\| \|(u_n - y_n) + (p - u)\|. \end{aligned}$$

Since $\alpha_n \rightarrow 0$, $\|x_{n+1} - x_n\| \rightarrow 0$ and $\|B_1 u_n - B_1 u\| \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} \|(u_n - y_n) + (p - u)\| = 0. \tag{3.16}$$

Also, from (3.14) we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n \|\gamma f(x_n) - Ap\|^2 + \beta_n \|x_n - p\|^2 + (1 - \beta_n - \alpha_n \bar{\gamma}) \|y_n - p\|^2 \\ &\leq \alpha_n \|\gamma f(x_n) - Ap\|^2 + \beta_n \|x_n - p\|^2 + (1 - \beta_n - \alpha_n \bar{\gamma}) \|u_n - u\|^2 \\ &\leq \alpha_n \|\gamma f(x_n) - Ap\|^2 + \beta_n \|x_n - p\|^2 + (1 - \beta_n - \alpha_n \bar{\gamma}) [\|x_n - p\|^2 \\ &\quad - \|(z_n - u_n) - (p - u)\|^2 + 2\mu_2 \langle (z_n - u_n) - (p - u), B_2 z_n - B_2 p \rangle]. \end{aligned}$$

So, we have

$$\begin{aligned} &(1 - \beta_n - \alpha_n \bar{\gamma}) \|(z_n - u_n) - (p - u)\|^2 \\ &\leq \alpha_n \|\gamma f(x_n) - Ap\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\| \\ &\quad + 2\mu_2 \|(z_n - u_n) - (p - u)\| \|B_2 z_n - B_2 p\|. \end{aligned}$$

Note that $\|B_2z_n - B_2p\| \rightarrow 0$ as $n \rightarrow \infty$. Then we have

$$\lim_{n \rightarrow \infty} \|(z_n - u_n) - (p - u)\| = 0. \tag{3.17}$$

In addition, from the firm nonexpansivity of $T_{\delta_n}^{(\theta, \varphi)}$, we obtain

$$\begin{aligned} \|z_n - p\|^2 &= \|T_{\delta_n}^{(\theta, \varphi)}(x_n - \delta_n Fx_n) - T_{\delta_n}^{(\theta, \varphi)}(p - \delta_n Fp)\|^2 \\ &\leq \langle (x_n - \delta_n Fx_n) - (p - \delta_n Fp), z_n - p \rangle \\ &= \frac{1}{2} [\|(x_n - \delta_n Fx_n) - (p - \delta_n Fp)\|^2 + \|z_n - p\|^2 \\ &\quad - \|(x_n - \delta_n Fx_n) - (p - \delta_n Fp) - (z_n - p)\|^2] \\ &\leq \frac{1}{2} [\|x_n - p\|^2 + \|z_n - p\|^2 - \|x_n - z_n - \delta_n(Fx_n - Fp)\|^2] \\ &= \frac{1}{2} [\|x_n - p\|^2 + \|z_n - p\|^2 - \|x_n - z_n\|^2 \\ &\quad + 2\delta_n \langle Fx_n - Fp, x_n - z_n \rangle - \delta_n^2 \|Fx_n - Fp\|^2], \end{aligned}$$

which implies that

$$\begin{aligned} \|z_n - p\|^2 &\leq \|x_n - p\|^2 - \|x_n - z_n\|^2 + 2\delta_n \langle Fx_n - Fp, x_n - z_n \rangle \\ &\leq \|x_n - p\|^2 - \|x_n - z_n\|^2 + 2\delta_n \|Fx_n - Fp\| \|x_n - z_n\|. \end{aligned} \tag{3.18}$$

From (3.7) and (3.18), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n \|\gamma f(x_n) - Ap\|^2 + \beta_n \|x_n - p\|^2 + (1 - \beta_n - \alpha_n \bar{\gamma}) \|y_n - p\|^2 \\ &\leq \alpha_n \|\gamma f(x_n) - Ap\|^2 + \beta_n \|x_n - p\|^2 + (1 - \beta_n - \alpha_n \bar{\gamma}) \|z_n - p\|^2 \\ &\leq \alpha_n \|\gamma f(x_n) - Ap\|^2 + \beta_n \|x_n - p\|^2 + (1 - \beta_n - \alpha_n \bar{\gamma}) [\|x_n - p\|^2 \\ &\quad - \|x_n - z_n\|^2 + 2\delta_n \|Fx_n - Fp\| \|x_n - z_n\|]. \end{aligned}$$

It follows that

$$\begin{aligned} (1 - \beta_n - \alpha_n \bar{\gamma}) \|x_n - z_n\|^2 &\leq \alpha_n \|\gamma f(x_n) - Ap\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\| \\ &\quad + 2(1 - \beta_n - \alpha_n \bar{\gamma}) \delta_n \|Fx_n - Fp\| \|x_n - z_n\|. \end{aligned}$$

Since $\|Fx_n - Fp\| \rightarrow 0$ as $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0. \tag{3.19}$$

Thus, from (3.16), (3.17), and (3.19) we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \|z_n - y_n\| &= \lim_{n \rightarrow \infty} \|(z_n - u_n) - (p - u) + (u_n - y_n) + (p - u)\| \\ &\leq \lim_{n \rightarrow \infty} \|(z_n - u_n) - (p - u)\| + \lim_{n \rightarrow \infty} \|(u_n - y_n) + (p - u)\| \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - y_n\| &\leq \lim_{n \rightarrow \infty} \|x_n - z_n\| + \lim_{n \rightarrow \infty} \|z_n - y_n\| \\ &= 0. \end{aligned}$$

Since $J_{M,\lambda}$ is 1-inverse strongly monotone, we have

$$\begin{aligned} \|w_n - p\|^2 &= \|J_{M,\lambda}(y_n - \lambda By_n) - J_{M,\lambda}(p - \lambda Bp)\|^2 \\ &\leq \langle (y_n - \lambda By_n) - (p - \lambda Bp), w_n - p \rangle \\ &= \frac{1}{2} [\|(y_n - \lambda By_n) - (p - \lambda Bp)\|^2 + \|w_n - p\|^2 \\ &\quad - \|(y_n - \lambda By_n) - (p - \lambda Bp) - (w_n - p)\|^2] \\ &\leq \frac{1}{2} (\|y_n - p\|^2 + \|w_n - p\|^2 - \|(y_n - w_n) - \lambda(By_n - Bp)\|^2) \\ &\leq \frac{1}{2} (\|y_n - p\|^2 + \|w_n - p\|^2 - \|y_n - w_n\|^2 + 2\lambda \langle y_n - w_n, By_n - Bp \rangle \\ &\quad - \lambda^2 \|By_n - Bp\|^2), \end{aligned}$$

which implies that

$$\|w_n - p\|^2 \leq \|y_n - p\|^2 - \|y_n - w_n\|^2 + 2\lambda \|y_n - w_n\| \|By_n - Bp\|. \tag{3.20}$$

Substituting (3.15) into (3.20), we have

$$\begin{aligned} \|w_n - p\|^2 &\leq \|x_n - p\|^2 - \|(u_n - y_n) + (p - u)\|^2 \\ &\quad + 2\mu_1 \|B_1 u_n - B_1 u\| \|(u_n - y_n) + (p - u)\| \\ &\quad - \|y_n - w_n\|^2 + 2\lambda \|y_n - w_n\| \|By_n - Bp\|. \end{aligned} \tag{3.21}$$

It follows from (3.21) that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n \|\gamma f(x_n) - Ap\|^2 + \beta_n \|x_n - p\|^2 + (1 - \beta_n - \alpha_n \bar{\gamma}) \|w_n - p\|^2 \\ &\leq \alpha_n \|\gamma f(x_n) - Ap\|^2 + (1 - \alpha_n \bar{\gamma}) \|x_n - p\|^2 \\ &\quad - (1 - \beta_n - \alpha_n \bar{\gamma}) \|(u_n - y_n) + (p - u)\|^2 \\ &\quad + 2\mu_1 (1 - \beta_n - \alpha_n \bar{\gamma}) \|B_1 u_n - B_1 u\| \|(u_n - y_n) + (p - u)\| \\ &\quad - (1 - \beta_n - \alpha_n \bar{\gamma}) \|y_n - w_n\|^2 + 2\lambda (1 - \beta_n - \alpha_n \bar{\gamma}) \|y_n - w_n\| \|By_n - Bp\|, \end{aligned}$$

which gives

$$\begin{aligned} (1 - \beta_n - \alpha_n \bar{\gamma}) \|y_n - w_n\|^2 &\leq \alpha_n \|\gamma f(x_n) - Ap\|^2 + \|x_n - x_{n+1}\| (\|x_n - p\| + \|x_{n+1} - p\|) \end{aligned}$$

$$\begin{aligned}
 & - (1 - \beta_n - \alpha_n \bar{\gamma}) \|(u_n - y_n) + (p - u)\|^2 \\
 & + 2\mu_1(1 - \beta_n - \alpha_n \bar{\gamma}) \|B_1 u_n - B_1 u\| \|(u_n - y_n) + (p - u)\| \\
 & + 2\lambda(1 - \beta_n - \alpha_n \bar{\gamma}) \|y_n - w_n\| \|B y_n - B p\|.
 \end{aligned}$$

Since $\alpha_n \rightarrow 0$, $\|x_{n+1} - x_n\| \rightarrow 0$, $\|(u_n - y_n) + (p - u)\| \rightarrow 0$, $\|B y_n - B p\| \rightarrow 0$ as $n \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} \|y_n - w_n\| = 0$.

Step 5. We show $\lim_{n \rightarrow \infty} \|T(u)y_n - y_n\| = 0$.

Denote $h_n = \frac{1}{t_n} \int_0^{t_n} T(u)w_n \, du$. From (3.1), $\lim_{n \rightarrow \infty} \alpha_n = 0$, and $\lim_{n \rightarrow \infty} \beta_n = 0$ we have

$$\begin{aligned}
 \|x_{n+1} - h_n\| &= \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)h_n - h_n\| \\
 &\leq \alpha_n \|\gamma f(x_n) - A h_n\| + \beta_n \|x_n - h_n\| \\
 &\rightarrow 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned} \tag{3.22}$$

Let $K = \{w \in C : \|w - p\| \leq \frac{1}{\bar{\gamma} - \gamma \alpha} \|\gamma f(p) - A p\|\}$. Then K is a nonempty bounded closed convex subset of C which is $T(u)$ -invariant for each $u \in [0, \infty)$ and contains $\{x_n\}$. It follows from Lemma 2.5 that

$$\lim_{n \rightarrow \infty} \|h_n - T(u)h_n\| = 0, \quad u \geq 0 \tag{3.23}$$

and from (3.22) and (3.23), we have

$$\begin{aligned}
 \|x_{n+1} - T(u)x_{n+1}\| &\leq \|x_{n+1} - h_n\| + \|h_n - T(u)h_n\| + \|T(u)h_n - T(u)x_{n+1}\| \\
 &\leq 2\|x_{n+1} - h_n\| + \|h_n - T(u)h_n\| \\
 &\rightarrow 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} \|x_n - T(u)x_n\| = 0. \tag{3.24}$$

Furthermore, from Step 4 we have for every $u \in [0, \infty)$ that

$$\|T(u)y_n - T(u)x_n\| \leq \|y_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.25}$$

So, we obtain from (3.24)

$$\begin{aligned}
 \|T(u)y_n - x_n\| &\leq \|T(u)y_n - T(u)x_n\| + \|T(u)x_n - x_n\| \\
 &\rightarrow 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned} \tag{3.26}$$

Hence, we have for every $u \in [0, \infty)$ that

$$\begin{aligned}
 \|T(u)y_n - y_n\| &\leq \|T(u)y_n - x_n\| + \|x_n - y_n\| \\
 &\rightarrow 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

Step 6. We show that $\limsup_{n \rightarrow \infty} \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle \leq 0$ and $\limsup_{n \rightarrow \infty} \langle \gamma f(x^*) - Ax^*, \frac{1}{t_n} \int_0^{t_n} T(u)w_n du - x^* \rangle \leq 0$, where $x^* = P_{\mathcal{F}}(\gamma f(x^*) + (I - A)(x^*))$.

Indeed, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle = \lim_{i \rightarrow \infty} \langle \gamma f(x^*) - Ax^*, x_{n_i} - x^* \rangle. \tag{3.27}$$

Since $\{y_{n_i}\}$ is bounded, we can assume that $y_{n_i} \rightharpoonup w$. First, we prove that $w \in F(\mathfrak{S})$.

Assume the contrary that $w \neq T(u)w$ for some $u \in [0, \infty)$. Then by Opial's condition, we obtain from Step 5 that

$$\begin{aligned} \liminf_{j \rightarrow \infty} \|y_{n_j} - w\| &< \liminf_{j \rightarrow \infty} \|y_{n_j} - T(u)w\| \\ &\leq \liminf_{j \rightarrow \infty} (\|y_{n_j} - T(u)y_{n_j}\| + \|T(u)y_{n_j} - T(u)w\|) \\ &\leq \liminf_{j \rightarrow \infty} \|y_{n_j} - w\|. \end{aligned}$$

This is a contraction. Hence, $w \in F(\mathfrak{S})$.

Next, let us show that $w \in GMEP$.

From $z_n = T_{\delta_n}^{(\theta, \varphi)}(x_n - \delta_n Fx_n)$, we obtain

$$\theta(z_n, y) + \varphi(y) - \varphi(z_n) + \frac{1}{\delta_n} \langle y - z_n, z_n - (x_n - \delta_n Fx_n) \rangle \geq 0, \quad \forall y \in C.$$

It follows from (H2) that

$$\varphi(y) - \varphi(z_n) + \frac{1}{\delta_n} \langle y - z_n, z_n - (x_n - \delta_n Fx_n) \rangle \geq \theta(y, z_n), \quad \forall y \in C.$$

Replacing n by n_i , we have

$$\varphi(y) - \varphi(z_{n_i}) + \langle y - z_{n_i}, Fx_{n_i} \rangle + \left\langle y - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\delta_{n_i}} \right\rangle \geq \theta(y, z_{n_i}), \quad \forall y \in C. \tag{3.28}$$

Let $z_t = ty + (1 - t)w$ for all $t \in [0, 1]$ and $y \in C$. Then we have $z_t \in C$. It follows from (3.28) that

$$\begin{aligned} \langle z_t - z_{n_i}, Fz_t \rangle &\geq \langle z_t - z_{n_i}, Fz_t \rangle - \varphi(z_t) + \varphi(z_{n_i}) - \langle z_t - z_{n_i}, Fx_{n_i} \rangle \\ &\quad - \left\langle z_t - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\delta_{n_i}} \right\rangle + \theta(z_t, z_{n_i}) \\ &\leq \langle z_t - z_{n_i}, Fz_t - Fz_{n_i} \rangle + \langle z_t - z_{n_i}, Fz_{n_i} - Fx_{n_i} \rangle \\ &\quad - \varphi(z_t) + \varphi(z_{n_i}) - \left\langle z_t - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\delta_{n_i}} \right\rangle + \theta(z_t, z_{n_i}). \end{aligned}$$

Since $\|z_{n_i} - x_{n_i}\| \rightarrow 0$, we have $\|Fz_{n_i} - Fx_{n_i}\| \rightarrow 0$ as $n_i \rightarrow \infty$. From the monotonicity of F , we have

$$\langle Fz_t - Fz_{n_i}, z_t - z_{n_i} \rangle \geq 0.$$

From (H4), $\frac{z_{n_i} - x_{n_i}}{\delta_{n_i}} \rightarrow 0$ and $z_{n_i} \rightarrow w$, we have

$$\langle z_t - w, Fz_t \rangle \geq -\varphi(z_t) + \varphi(w) + \theta(z_t, w) \tag{3.29}$$

as $n_i \rightarrow \infty$. By (H1), (H4), and (3.29), we obtain

$$\begin{aligned} 0 &= \theta(z_t, z_t) + \varphi(z_t) - \varphi(z_t) \\ &\leq t\theta(z_t, y) + (1-t)\theta(z_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(z_t) \\ &\leq t[\theta(z_t, y) + \varphi(y) - \varphi(z_t)] + (1-t)\langle z_t - w, Fz_t \rangle \\ &= t[\theta(z_t, y) + \varphi(y) - \varphi(z_t)] + (1-t)t\langle y - w, Fz_t \rangle. \end{aligned}$$

Hence we obtain

$$0 \leq \theta(z_t, y) + \varphi(y) - \varphi(z_t) + (1-t)\langle y - w, Fz_t \rangle.$$

Putting $t \rightarrow 0$, we have

$$0 \leq \theta(w, y) + \varphi(y) - \varphi(w) + \langle y - w, Fw \rangle, \quad \forall y \in C.$$

This implies that $w \in GMEP$.

Next, we prove that $w \in \Omega$.

Utilizing Lemma 2.1, we have for all $x, y \in C$

$$\begin{aligned} &\|\Gamma(x) - \Gamma(y)\|^2 \\ &= \|T_{\mu_1}^{(G_1, \phi)} [T_{\mu_2}^{(G_2, \psi)}(x - \mu_2 B_2 x) - \mu_1 B_1 T_{\mu_2}^{(G_2, \psi)}(x - \mu_2 B_2 x)] \\ &\quad - T_{\mu_1}^{(G_1, \phi)} [T_{\mu_2}^{(G_2, \psi)}(y - \mu_2 B_2 y) - \mu_1 B_1 T_{\mu_2}^{(G_2, \psi)}(y - \mu_2 B_2 y)]\|^2 \\ &\leq \|T_{\mu_2}^{(G_2, \psi)}(x - \mu_2 B_2 x) - T_{\mu_2}^{(G_2, \psi)}(y - \mu_2 B_2 y) \\ &\quad - \mu_1 (B_1 T_{\mu_2}^{(G_2, \psi)}(x - \mu_2 B_2 x) - B_1 T_{\mu_2}^{(G_2, \psi)}(y - \mu_2 B_2 y))\|^2 \\ &\leq \|T_{\mu_2}^{(G_2, \psi)}(x - \mu_2 B_2 x) - T_{\mu_2}^{(G_2, \psi)}(y - \mu_2 B_2 y)\|^2 \\ &\quad + \mu_1(\mu_1 - 2\beta_1) \|B_1 T_{\mu_2}^{(G_2, \psi)}(x - \mu_2 B_2 x) - B_1 T_{\mu_2}^{(G_2, \psi)}(y - \mu_2 B_2 y)\|^2 \\ &\leq \|T_{\mu_2}^{(G_2, \psi)}(x - \mu_2 B_2 x) - T_{\mu_2}^{(G_2, \psi)}(y - \mu_2 B_2 y)\|^2 \\ &\leq \|x - y - \mu_2(B_2 x - B_2 y)\|^2 \\ &\leq \|x - y\|^2 + \mu_2(\mu_2 - 2\beta_2) \|B_2 x - B_2 y\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

This shows that $\Gamma : C \rightarrow C$ is nonexpansive. Note that

$$\begin{aligned} \|y_n - \Gamma(y_n)\| &= \|\Gamma(z_n) - \Gamma(y_n)\| \\ &\leq \|z_n - y_n\| \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

According to Lemma 2.2 and Lemma 2.3, we obtain $w \in \Omega$.

Lastly, we show that $w \in I(B, M)$.

In fact, since B is a β -inverse strongly monotone, B is monotone and Lipschitz continuous mapping. It follows from Lemma 2.4 that $M + B$ is a maximal monotone mapping. Let $(v, g) \in G(M + B)$. Then $g - B(v) \in M(v)$. Since $w_{n_i} = J_{M, \lambda}(y_{n_i} - \lambda B y_{n_i})$, we have $y_{n_i} - \lambda B y_{n_i} \in (I + \lambda M)(w_{n_i})$, i.e., $\frac{1}{\lambda}(y_{n_i} - w_{n_i} - \lambda A y_{n_i}) \in M(w_{n_i})$. By virtue of the maximal monotonicity of $M + B$, we have

$$\left\langle v - w_{n_i}, g - B(v) - \frac{1}{\lambda}(y_{n_i} - w_{n_i} - \lambda B y_{n_i}) \right\rangle \geq 0$$

and hence

$$\begin{aligned} \langle v - w_{n_i}, g \rangle &\geq \left\langle v - w_{n_i}, B(v) + \frac{1}{\lambda}(y_{n_i} - w_{n_i} - \lambda B y_{n_i}) \right\rangle \\ &= \langle v - w_{n_i}, Bv - Bw_{n_i} \rangle + \langle v - w_{n_i}, Bw_{n_i} - B y_{n_i} \rangle \\ &\quad + \left\langle v - w_{n_i}, \frac{1}{\lambda}(y_{n_i} - w_{n_i}) \right\rangle. \end{aligned}$$

It follows from $\lim_{n \rightarrow \infty} \|w_n - y_n\| = 0$ that we have

$$\lim_{n \rightarrow \infty} \|Bw_n - B y_n\| = 0$$

and

$$\begin{aligned} w_{n_i} &\rightharpoonup w, \\ \limsup_{n \rightarrow \infty} \langle v - w_{n_i}, g \rangle &= \langle v - w, g \rangle \geq 0. \end{aligned}$$

It follows from the maximal monotonicity of $M + B$ that $\theta \in (M + B)(w)$, that is, $w \in I(B, M)$. Therefore $\omega \in \mathcal{F} = F(\mathfrak{N}) \cap GMEP \cap \Omega \cap I(B, M)$.

By $x^* = P_{\mathcal{F}}(\gamma f + (I - A))(x^*)$, we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle (\gamma f - A)x^*, x_n - x^* \rangle &= \lim_{i \rightarrow \infty} \langle (\gamma f - A)x^*, x_{n_i} - x^* \rangle \\ &= \langle (\gamma f - A)x^*, w - x^* \rangle \\ &\leq 0 \end{aligned}$$

and

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left\langle (\gamma f - A)x^*, \frac{1}{t_n} \int_0^{t_n} T(u)w_n \, du - x^* \right\rangle \\ &= \lim_{i \rightarrow \infty} \left\langle (\gamma f - A)x^*, \frac{1}{t_{n_i}} \int_0^{t_{n_i}} T(u)w_{n_i} \, du - x^* \right\rangle \\ &= \langle (\gamma f - A)x^*, w - x^* \rangle \\ &\leq 0, \end{aligned}$$

as required.

Step 7. We prove $x_n \rightarrow x^*$ as $n \rightarrow \infty$.

By using (3.1), we have

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ &= \left\| \alpha_n(\gamma f(x_n) - Ax^*) + \beta_n(x_n - x^*) \right. \\ &\quad \left. + ((1 - \beta_n)I - \alpha_n A) \left(\frac{1}{t_n} \int_0^{t_n} T(u)w_n \, du - x^* \right) \right\|^2 \\ &\leq \alpha_n^2 \|\gamma f(x_n) - Ax^*\|^2 \\ &\quad + \left\| \beta_n(x_n - x^*) + ((1 - \beta_n)I - \alpha_n A) \left(\frac{1}{t_n} \int_0^{t_n} T(u)w_n \, du - x^* \right) \right\|^2 \\ &\quad + 2\alpha_n \left\langle \gamma f(x_n) - Ax^*, \beta_n(x_n - x^*) + ((1 - \beta_n)I - \alpha_n A) \left(\frac{1}{t_n} \int_0^{t_n} T(u)w_n \, du - x^* \right) \right\rangle \\ &\leq \alpha_n^2 \|\gamma f(x_n) - Ax^*\|^2 + \beta_n^2 \|x_n - x^*\|^2 + (1 - \beta_n - \alpha_n \bar{\gamma})^2 \|w_n - x^*\|^2 \\ &\quad + 2\beta_n \left\langle x_n - x^*, ((1 - \beta_n)I - \alpha_n A) \left(\frac{1}{t_n} \int_0^{t_n} T(u)w_n \, du - x^* \right) \right\rangle \\ &\quad + 2\alpha_n \beta_n \langle \gamma f(x_n) - \gamma f(x^*), x_n - x^* \rangle + 2\alpha_n \beta_n \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle \\ &\quad + 2\alpha_n (1 - \beta_n - \alpha_n \bar{\gamma}) \left\langle \gamma f(x_n) - \gamma f(x^*), \frac{1}{t_n} \int_0^{t_n} T(u)w_n \, du - x^* \right\rangle \\ &\quad + 2\alpha_n (1 - \beta_n - \alpha_n \bar{\gamma}) \left\langle \gamma f(x^*) - Ax^*, \frac{1}{t_n} \int_0^{t_n} T(u)w_n \, du - x^* \right\rangle. \end{aligned}$$

It follows from (3.4), (3.6), and (3.7) that

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ &\leq \alpha_n^2 \|\gamma f(x_n) - Ax^*\|^2 + \beta_n^2 \|x_n - x^*\|^2 + (1 - \beta_n - \alpha_n \bar{\gamma})^2 \|x_n - x^*\|^2 \\ &\quad + 2\beta_n (1 - \beta_n - \alpha_n \bar{\gamma}) \|x_n - x^*\|^2 \\ &\quad + 2\alpha_n \beta_n \gamma \alpha \|x_n - x^*\|^2 + 2\alpha_n \beta_n \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle \\ &\quad + 2\alpha_n (1 - \beta_n - \alpha_n \bar{\gamma}) \gamma \alpha \|x_n - x^*\|^2 \\ &\quad + 2\alpha_n (1 - \beta_n - \alpha_n \bar{\gamma}) \left\langle \gamma f(x^*) - Ax^*, \frac{1}{t_n} \int_0^{t_n} T(u)w_n \, du - x^* \right\rangle \\ &\leq [1 - 2\alpha_n (\bar{\gamma} - \gamma \alpha + \alpha_n \bar{\gamma} \gamma \alpha)] \|x_n - x^*\|^2 + \alpha_n^2 \|\gamma f(x_n) - Ax^*\|^2 \\ &\quad + 2\alpha_n \beta_n \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle \\ &\quad + 2\alpha_n (1 - \beta_n - \alpha_n \bar{\gamma}) \left\langle \gamma f(x^*) - Ax^*, \frac{1}{t_n} \int_0^{t_n} T(u)w_n \, du - x^* \right\rangle \\ &\quad + \alpha_n^2 \bar{\gamma}^2 \|x_n - x^*\|^2 \\ &= (1 - \bar{\alpha}_n) \|x_n - x^*\|^2 + \bar{\beta}_n, \end{aligned}$$

where

$$\begin{aligned} \bar{\alpha}_n &= 2\alpha_n(\bar{\gamma} - \gamma\alpha + \alpha_n\bar{\gamma}\gamma\alpha), \\ \bar{\beta}_n &= \alpha_n^2 \|\gamma f(x_n) - Ax^*\|^2 + 2\alpha_n\beta_n \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle \\ &\quad + 2\alpha_n(1 - \beta_n - \alpha_n\bar{\gamma}) \left\langle \gamma f(x^*) - Ax^*, \frac{1}{t_n} \int_0^{t_n} T(u)w_n du - x^* \right\rangle \\ &\quad + \alpha_n^2\bar{\gamma}^2 \|x_n - x^*\|^2. \end{aligned}$$

It is easily seen that $\bar{\alpha}_n \rightarrow 0$ as $n \rightarrow \infty$, $\sum_{n=1}^{\infty} \bar{\alpha}_n = +\infty$ and $\limsup_{n \rightarrow \infty} \frac{\bar{\beta}_n}{\bar{\alpha}_n} \leq 0$. Hence, applying Lemma 2.6 we immediately obtain $x_n \rightarrow x^*$ as $n \rightarrow \infty$. This completes the proof. \square

Remark 3.1 Let us consider the following sequences:

$$\alpha_n = \frac{1}{n^{\frac{1}{4}}}, \quad \beta_n = \frac{1}{n^2}, \quad \delta_n = \frac{1}{n^{\frac{1}{4}}} + \frac{1}{2} \quad \text{and} \quad t_n = n, \quad \forall n \geq 1.$$

It is easy to see that all hypotheses (C1)-(C4) of Theorem 3.1 are satisfied.

By Theorem 3.1, we can obtain the following results immediately.

Corollary 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\theta, G_1, G_2 : C \times C \rightarrow \mathbb{R}$ be three bifunctions which satisfy assumptions (H1)-(H4) and $\varphi, \phi, \psi : C \rightarrow \mathbb{R}$ be three lower semicontinuous and convex functions with restriction (A1) or (A2). Let $B, B_1, B_2 : C \rightarrow H$ be β -inverse strongly monotone, β_1 -inverse strongly monotone and β_2 -inverse strongly monotone, respectively and $M : H \rightarrow 2^H$ be a maximal monotone mapping. Let $\mathfrak{S} = \{T(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\mathcal{F}_1 = F(\mathfrak{S}) \cap \text{MEP} \cap \Omega \cap I(B, M) \neq \emptyset$. Let $\{t_n\} \subset (0, \infty)$ be a real sequence such that $\lim_{n \rightarrow \infty} t_n = \infty$. Let f be a contraction from C into itself with a constant α ($0 < \alpha < 1$) and let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ such that $\|A\| \leq 1$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha(1-\alpha_n\bar{\gamma})}$. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} \theta(z_n, y) + \varphi(y) - \varphi(z_n) + \frac{1}{\delta_n} \langle y - z_n, z_n - x_n \rangle \geq 0, & \forall y \in C, \\ y_n = T_{\mu_1}^{(G_1, \phi)} [T_{\mu_2}^{(G_2, \psi)}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T_{\mu_2}^{(G_2, \psi)}(z_n - \mu_2 B_2 z_n)], \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n \\ \quad + ((1 - \beta_n)I - \alpha_n A) \left[\frac{1}{t_n} \int_0^{t_n} T(u) (J_{M, \lambda}(y_n - \lambda B y_n)) du \right], & \forall n \geq 1, \end{cases}$$

where $\alpha_n \in (0, 1)$, $\mu_1 \in (0, 2\beta_1]$, $\mu_2 \in (0, 2\beta_2]$, $\lambda \in (0, 2\beta]$, $\delta_n \in (0, 2\zeta]$ satisfy the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
- (ii) $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
- (iii) $\liminf_{n \rightarrow \infty} \delta_n > 0$ and $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$;
- (iv) $\lim_{n \rightarrow \infty} \frac{|t_n - t_{n-1}|}{t_n} \frac{1}{\alpha_n} = 0$.

Then $\{x_n\}$ converges strongly to $x^* = P_{\mathcal{F}_1}(\gamma f + (I - A))(x^*)$, which solves the following variational inequality:

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{F}_1$$

and (x^*, y^*) is a solution of problem (1.5), where $y^* = T_{\mu_2}^{(G_2, \psi)}(x^* - \mu_2 B_2 x^*)$.

Proof In Theorem 3.1, for all $n \geq 0$, $z_n = T_{\delta_n}^{(\theta, \varphi)}(x_n - \delta_n Fx_n)$ is equivalent to

$$\theta(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Fx_n, y - z_n \rangle + \frac{1}{\delta_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C. \quad (3.30)$$

Putting $F = 0$, we obtain

$$\theta(z_n, y) + \varphi(y) - \varphi(z_n) + \frac{1}{\delta_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C.$$

By Theorem 3.1, we can easily get the desired conclusion. □

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $G_1, G_2 : C \times C \rightarrow \mathbb{R}$ be two bifunctions which satisfy assumptions (H1)-(H4) and $\phi, \psi : C \rightarrow \mathbb{R}$ be two lower semicontinuous and convex functions with restriction (A1) or (A2). Let $F, B, B_1, B_2 : C \rightarrow H$ be ζ -inverse strongly monotone, β -inverse strongly monotone, β_1 -inverse strongly monotone and β_2 -inverse strongly monotone, respectively, and let $M : H \rightarrow 2^H$ be a maximal monotone mapping. Let $\mathfrak{S} = \{T(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\mathcal{F}_2 = F(\mathfrak{S}) \cap VI(F, C) \cap \Omega \cap I(B, M) \neq \emptyset$. Let $\{t_n\} \subset (0, \infty)$ be a real sequence such that $\lim_{n \rightarrow \infty} t_n = \infty$. Let f be a contraction from C into itself with a constant α ($0 < \alpha < 1$) and let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ such that $\|A\| \leq 1$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha(1-\alpha\bar{\gamma})}$. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} z_n = P_C(x_n - \delta_n Fx_n), \\ y_n = T_{\mu_1}^{(G_1, \phi)} [T_{\mu_2}^{(G_2, \psi)}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T_{\mu_2}^{(G_2, \psi)}(z_n - \mu_2 B_2 z_n)], \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n \\ \quad + ((1 - \beta_n)I - \alpha_n A) \left[\frac{1}{t_n} \int_0^{t_n} T(u) (J_{M, \lambda}(y_n - \lambda B y_n)) du \right], \quad \forall n \geq 1, \end{cases}$$

where $\alpha_n \in [0, 1]$, $\mu_1 \in (0, 2\beta_1]$, $\mu_2 \in (0, 2\beta_2]$, $\lambda \in (0, 2\beta]$, $\delta_n \in (0, 2\zeta]$ satisfy the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
- (ii) $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
- (iii) $\liminf_{n \rightarrow \infty} \delta_n > 0$ and $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$;
- (iv) $\lim_{n \rightarrow \infty} \frac{t_n - t_{n-1}}{t_n} \frac{1}{\alpha_n} = 0$.

Then $\{x_n\}$ converges strongly to $x^* = P_{\mathcal{F}_2}(\gamma f + (I - A))(x^*)$, which solves the following variational inequality:

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{F}_2$$

and (x^*, y^*) is a solution of problem (1.5), where $y^* = T_{\mu_2}^{(G_2, \psi)}(x^* - \mu_2 B_2 x^*)$.

Proof Put $\theta = 0$ and $\varphi = 0$ in Theorem 3.1. Then we have from (3.30)

$$\langle Fx_n, y - z_n \rangle + \frac{1}{\delta_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C, n \geq 1.$$

That is,

$$\langle y - z_n, x_n - \delta_n Fx_n - z_n \rangle \leq 0, \quad \forall y \in C.$$

It follows that $P_C(x_n - \delta_n Fx_n) = z_n$ for all $n \geq 1$. We easily obtain the desired conclusion. \square

Corollary 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\theta, G_1, G_2 : C \times C \rightarrow \mathbb{R}$ be three bifunctions which satisfy assumptions (H1)-(H4) and $\varphi, \phi, \psi : C \rightarrow \mathbb{R}$ be three lower semicontinuous and convex functions with restriction (A1) or (A2). Let $F, B, B_1, B_2 : C \rightarrow H$ be ζ -inverse strongly monotone, β -inverse strongly monotone, β_1 -inverse strongly monotone, and β_2 -inverse strongly monotone, respectively. Let $\mathfrak{S} = \{T(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\mathcal{F}_3 = F(\mathfrak{S}) \cap GMEP \cap \Omega \cap VI(F, C) \neq \emptyset$. Let $\{t_n\} \subset (0, \infty)$ be a real sequence such that $\lim_{n \rightarrow \infty} t_n = \infty$. Let f be a contraction from C into itself with a constant α ($0 < \alpha < 1$) and let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ such that $\|A\| \leq 1$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha(1-\alpha_n\bar{\gamma})}$. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} z_n = T_{\delta_n}^{(\theta, \varphi)}(x_n - \delta_n Fx_n), \\ y_n = T_{\mu_1}^{(G_1, \phi)} [T_{\mu_2}^{(G_2, \psi)}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T_{\mu_2}^{(G_2, \psi)}(z_n - \mu_2 B_2 z_n)], \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n \\ \quad + ((1 - \beta_n)I - \alpha_n A) \left[\frac{1}{t_n} \int_0^{t_n} T(u)(P_C(y_n - \lambda B y_n)) du \right], \quad n \geq 1, \end{cases}$$

where $\alpha_n \in (0, 1)$, $\mu_1 \in (0, 2\beta_1]$, $\mu_2 \in (0, 2\beta_2]$, $\lambda \in (0, 2\beta]$, $\delta_n \in (0, 2\zeta]$ satisfy the conditions (C1)-(C4). Then $\{x_n\}$ converges strongly to $x^* = P_{\mathcal{F}_3}(\gamma f + (I - A))(x^*)$, which solves the following variational inequality:

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{F}_3,$$

and (x^*, y^*) is a solution of problem (1.5), where $y^* = T_{\mu_2}^{(G_2, \psi)}(x^* - \mu_2 B_2 x^*)$.

Proof Taking $J_{M, \lambda} = P_C$ in Theorem 3.1, we can obtain desired conclusion immediately. \square

Remark 3.2 Theorem 3.1 generalizes and improves Theorem 3.1 of Kumam and Wattanawitton [14], Theorem 3.3 of Plubtieng and Punpaeng [10] and Theorem 3.1 of Shehu [13] in the following aspects:

- (1) Algorithm of Theorem 3.1 is different from algorithms in [10, 13, 14].
- (2) Theorem 3.1 includes Theorem 3.3 of Plubtieng and Punpaeng [10] as a special case.
- (3) Theorem 3.1 improves Theorem 3.1 of Kumam and Wattanawitton [14] since the generalized equilibrium problem that is within [14] is extended to the generalized mixed equilibrium problem.

Competing interests

The author declares that he has no competing interests.

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