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Suzuki type theorems in triangular and non-Archimedean fuzzy metric spaces with application

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Abstract

In this paper, first we introduce certain new classes of Suzuki type contractions in triangular and non-Archimedean fuzzy metric spaces. Further we establish fixed point theorems for such kind of mappings in non-Archimedean and triangular fuzzy metric spaces. We also prove Suzuki type fixed point results in non-Archimedean and triangular ordered fuzzy metric spaces. The results presented here improve and generalize certain recent results from the literature. Two illustrative examples and an application to integral equations are given to support the usability of our results.

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1 Introduction and preliminaries

The concept of fuzzy metric space was introduced in different ways by some authors (see, *i.e.*, [1, 2]) and further to this, the fixed point theory in this kind of spaces has been intensively studied (see [3–11]). Here, we consider the notion of fuzzy metric space introduced by Kramosil and Michálek [2] and modified by George and Veeramani [12, 13] who obtained a Hausdorff topology for the class of fuzzy metric spaces. Recently, Miheţ [14] enlarged the class of fuzzy contractive mappings of Gregori and Sapena [7] and proved a fuzzy Banach contraction result for complete non-Archimedean fuzzy metric spaces [7] (see also Vetro [15]).

The applications of fixed point theorems are remarkable in different disciplines of mathematics, engineering and economics in dealing with problems arising in approximation theory, game theory and many others (see [16] and the references therein). Consequently, many researchers, following the Banach contraction principle, investigated the existence of weaker contractive conditions or extended previous results under relatively weak hypotheses on the metric space. On the other hand, Samet *et al.* [17] introduced the concepts of α-ψ-contractive and α-admissible mappings and established various fixed point theorems for such mappings defined on complete metric spaces. Afterwards Salimi *et al.* [18] and Hussain *et al.* [19, 20] modified the notions of α-ψ-contractive and α-admissible mappings and established certain fixed point theorems (see also [21–25]). In this paper, we introduce certain new classes of contraction mappings and establish fixed point theorems for such kind of mappings in non-Archimedean fuzzy metric spaces. The results presented in this paper generalize and extend some recent results in non-Archimedean fuzzy metric spaces. Some examples are given to support the usability of our results. For the sake of completeness, we now briefly recall some basic concepts.

Definition 1.1 A binary operation $\star : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if it satisfies the following assertions:

(TN1) $\star$ is commutative and associative;

(TN2) $\star$ is continuous;

(TN3) $a \star 1 = a$ for all $a \in [0,1]$;

(TN4) $a \star b \leq c \star d$ when $a \leq c$ and $b \leq d$ and $a,b,c,d \in [0,1]$.

Definition 1.2 (George and Veeramani [12]) A fuzzy metric space is an ordered triple $(X,M,\star)$ such that X is a nonempty set, $\star$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0,+\infty)$ satisfying the following conditions, for all $x,y,z \in X$ and $t,s > 0$:

(FM1) $M(x,y,t) > 0$ for all $t > 0$;

(FM2) $M(x,y,t) = 1$ if and only if $x = y$;

(FM3) $M(x,y,t) = M(y,x,t)$;

(FM4) $M(x,y,t) \star M(y,z,s) \leq M(x,z,t+s)$;

(FM5) $M(x,y,\cdot) : (0,+\infty) \to (0,1]$ is left continuous.

Then the triple $(X,M,\star)$ is called a fuzzy metric space. If we replace (FM4) by

(FM6) $M(x,y,t) \star M(y,z,s) \leq M(x,z,\max\{t,s\})$,

then the triple $(X,M,\star)$ is called a non-Archimedean fuzzy metric space. Since (FM6) implies (FM4), then each non-Archimedean fuzzy metric space is a fuzzy metric space.

Definition 1.3 Let $(X,M,\star)$ be a fuzzy metric space (or non-Archimedean fuzzy metric space). Then

(i) a sequence $\{x_n\}$ converges to $x \in X$ if and only if $\lim_{n\to+\infty} M(x_n,x,t) = 1$ for all $t > 0$;

(ii) a sequence $\{x_n\}$ in X is a Cauchy sequence if and only if for all $\epsilon \in (0,1)$ and $t > 0$, there exists n_0 such that $M(x_n,x_m,t) > 1-\epsilon$ for all $m,n \geq n_0$;

(iii) the fuzzy metric space (or the non-Archimedean fuzzy metric space) is called complete if every Cauchy sequence converges to some $x \in X$.

If $(X,M,\star)$ is a fuzzy metric space and $(X,\preceq)$ is partially ordered, then $(X,M,\star)$ is called a partially ordered fuzzy metric space. Then $x,y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. Let $f : X \to X$ be a mapping, f is said to be non-decreasing if $fx \preceq fy$ whenever $x,y \in X$ and $x \preceq y$.

Definition 1.4 [3] Let $(X,M,*)$ be a triangular fuzzy metric space. The fuzzy metric M is called triangular whenever

$$\frac{1}{M(x,y,t)} - 1 \leq \frac{1}{M(x,z,t)} - 1 + \frac{1}{M(z,y,t)} - 1.$$

Definition 1.5 [17] Let $f : X \to X$ and $\alpha : X \times X \to [0,+\infty)$. f is an α-admissible mapping if

$$\alpha(x,y) \geq 1 \quad \text{implies} \quad \alpha(fx,fy) \geq 1, \quad x,y \in X.$$

Definition 1.6 [26] Let $(X,M,*)$ be a fuzzy metric space, $T:X\to X$ and $\alpha:X\times X\times(0,\infty)\to[0,\infty)$. We say that T is an α-admissible mapping if

$$x,y\in X,\quad \alpha(x,y,t)\geq t \quad\Longrightarrow\quad \alpha(Tx,Ty,t)\geq t$$

for all $t>0$.

2 Fixed point results in triangular fuzzy metric spaces

Let $(X,M,*)$ be a fuzzy metric space, $f:X\to X$ be a self-mapping on X. We define $P^f(x,y,t)$, $Q^f(x,y,t)$ and $R^f(x,y,t)$ as follows:

$$P^f(x,y,t)=\max\left\{\frac{1}{M(x,y,t)},\frac{1}{M(x,fx,t)},\frac{1}{M(y,fy,t)},\frac{1}{2}\left[\frac{1}{M(x,fy,t)}+\frac{1}{M(y,fx,t)}-1\right]\right\},$$
$$Q^f(x,y,t)=\max\left\{M(x,fx,t),M(y,fy,t),M(x,fy,t),M(y,fx,t)\right\}$$

and

$$R^f(x,y,t)=\min\left\{1-\frac{1}{M(x,fx,t)},1-\frac{1}{M(y,fy,t)},1-\frac{1}{M(x,fy,t)},1-\frac{1}{M(y,fx,t)}\right\}.$$

We now state and prove our first result of this section.

Theorem 2.1 *Let $(X,M,*)$ be a complete triangular fuzzy metric space and f be a self-mapping on X. Also suppose that $\alpha:X\times X\times[0,\infty)\to[0,\infty)$ is a mapping. Assume that the following assertions hold:*

(i) *there exists $x_0\in X$ such that $\alpha(x_0,fx_0,t)\geq t$ for all $t>0$;*

(ii) *f is an α-admissible mapping;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1},t)\geq t$ for all $n\in\mathbb{N}$ and all $t>0$ with $x_n\to x$ as $n\to\infty$, then $\alpha(x_n,x,t)\geq t$ for all $n\in\mathbb{N}\cup\{0\}$;*

(iv) *for all $x,y\in X$ and all $t>0$ with $\frac{1}{1+\lambda}(\frac{1}{M(x,fx,t)}-1)\leq\frac{1}{M(x,y,t)}-1$, we have*

$$\frac{\alpha(x,y,t)}{tM(fx,fy,t)}\leq\lambda P^f(x,y,t)+\left|Q^f(x,y,t)-\lambda\right|+LR^f(x,y,t),\tag{2.1}$$

where $\lambda\in(0,1)$ and $L\geq 0$.

Then f has a fixed point.

Proof Let $x_0\in X$ be such that $\alpha(x_0,fx_0,t)\geq t$ for all $t>0$. Define a sequence $\{x_n\}$ in X by $x_n=f^nx_0=fx_{n-1}$ for all $n\in\mathbb{N}$. If $x_{n+1}=x_n$ for some $n\in\mathbb{N}$, then $x=x_n$ is a fixed point for f and the result is proved. Hence, we suppose that $x_{n+1}\neq x_n$ for all $n\in\mathbb{N}$. Since f is an α-admissible mapping and $\alpha(x_0,fx_0,t)\geq t$, we deduce that $\alpha(x_1,x_2,t)=\alpha(fx_0,f^2x_0,t)\geq t$. Continuing this process, we get

$$\alpha(x_n,x_{n+1},t)\geq t\tag{2.2}$$

for all $n\in\mathbb{N}\cup\{0\}$ and all $t>0$. Now since

$$\frac{1}{1+\lambda}\left(\frac{1}{M(x_{n-1},fx_{n-1},t)}-1\right)\leq\frac{1}{M(x_{n-1},fx_{n-1},t)}-1=\frac{1}{M(x_{n-1},x_n,t)}-1,$$

then from (iv) with $x = x_{n-1}$ and $y = x_n$ we get

$$\begin{aligned}\frac{1}{M(x_n,x_{n+1},t)} &\leq \frac{\alpha(x_n,x_{n+1},t)}{tM(x_n,x_{n+1},t)} = \frac{\alpha(x_n,x_{n+1},t)}{tM(fx_{n-1},fx_n,t)} \\ &\leq \lambda P^f(x_{n-1},x_n,t) + \left|Q^f(x_{n-1},x_n,t) - \lambda\right| + LR^f(x_{n-1},x_n,t).\end{aligned}$$

That is,

$$\frac{1}{M(x_n,x_{n+1},t)} \leq \lambda P^f(x_{n-1},x_n,t) + \left|Q^f(x_{n-1},x_n,t) - \lambda\right| + LR^f(x_{n-1},x_n,t) \tag{2.3}$$

for all $t > 0$ and all $n \in \mathbb{N}$, where

$$\begin{aligned}P^f(x_{n-1},x_n,t) &= \max\left\{\frac{1}{M(x_{n-1},x_n,t)}, \frac{1}{M(x_{n-1},fx_{n-1},t)},\right. \\ &\quad \left.\frac{1}{M(x_n,fx_n,t)}, \frac{1}{2}\left[\frac{1}{M(x_{n-1},fx_n,t)} + \frac{1}{M(x_n,fx_{n-1},t)} - 1\right]\right\} \\ &= \max\left\{\frac{1}{M(x_{n-1},x_n,t)}, \frac{1}{M(x_n,x_{n+1},t)}, \frac{1}{2M(x_{n-1},x_{n+1},t)}\right\} \\ &\leq \max\left\{\frac{1}{M(x_{n-1},x_n,t)}, \frac{1}{M(x_n,x_{n+1},t)},\right. \\ &\quad \left.\frac{1}{2}\left[\frac{1}{M(x_{n-1},x_n,t)} - 1 + \frac{1}{M(x_n,x_{n+1},t)} - 1\right] + \frac{1}{2}\right\} \\ &\leq \max\left\{\frac{1}{M(x_{n-1},x_n,t)}, \frac{1}{M(x_n,x_{n+1},t)},\right. \\ &\quad \left.\frac{1}{2}\left[\frac{1}{M(x_{n-1},x_n,t)} + \frac{1}{M(x_n,x_{n+1},t)}\right] - \frac{1}{2}\right\} \\ &= \max\left\{\frac{1}{M(x_{n-1},x_n,t)}, \frac{1}{M(x_n,x_{n+1},t)}\right\} \leq P^f(x_{n-1},x_n,t),\end{aligned}$$

which implies

$$P^f(x_{n-1},x_n,t) = \max\left\{\frac{1}{M(x_{n-1},x_n,t)}, \frac{1}{M(x_n,x_{n+1},t)}\right\} \tag{2.4}$$

and

$$\begin{aligned}Q^f(x_{n-1},x_n,t) &= \max\left\{M(x_{n-1},fx_{n-1},t), M(x_n,fx_n,t), M(x_{n-1},fx_n,t), M(x_n,fx_{n-1},t)\right\} \\ &= \max\left\{M(x_{n-1},x_n,t), M(x_n,x_{n+1},t), M(x_{n-1},x_{n+1},t), M(x_n,x_n,t)\right\} \\ &= 1. \end{aligned} \tag{2.5}$$

Also,

$$\begin{aligned}R^f(x_{n-1},x_n,t) &= \min\left\{1 - \frac{1}{M(x_{n-1},fx_{n-1},t)}, 1 - \frac{1}{M(x_n,fx_n,t)},\right. \\ &\quad \left.1 - \frac{1}{M(x_{n-1},fx_n,t)}, 1 - \frac{1}{M(x_n,fx_{n-1},t)}\right\}\end{aligned}$$

$$= \min\left\{1-\frac{1}{M(x_{n-1},x_n,t)},1-\frac{1}{M(x_n,x_{n+1},t)},\right.$$
$$\left.1-\frac{1}{M(x_{n-1},x_{n+1},t)},1-\frac{1}{M(x_n,x_n,t)}\right\} = 0. \tag{2.6}$$

From (2.3), (2.4), (2.5) and (2.6) we obtain

$$\begin{aligned}\frac{1}{M(x_n,x_{n+1},t)} &\le \lambda P^f(x_{n-1},x_n,t) + |1-\lambda| \\ &= \lambda P^f(x_{n-1},x_n,t) + 1 - \lambda \\ &= \lambda\big(P^f(x_{n-1},x_n,t) - 1\big) + 1,\end{aligned}$$

which implies

$$\frac{1}{M(x_n,x_{n+1},t)} - 1 \le \lambda\big(P^f(x_{n-1},x_n,t) - 1\big).$$

Now if $P^f(x_{n-1},x_n,t) = \frac{1}{M(x_n,x_{n+1},t)}$, then we get

$$\begin{aligned}\frac{1}{M(x_n,x_{n+1},t)} - 1 &\le \lambda\left[\frac{1}{M(x_n,x_{n+1},t)} - 1\right] \\ &< \frac{1}{M(x_n,x_{n+1},t)} - 1,\end{aligned}$$

which is a contradiction. Hence,

$$\frac{1}{M(x_n,x_{n+1},t)} - 1 \le \lambda\left[\frac{1}{M(x_{n-1},x_n,t)} - 1\right] \tag{2.7}$$

for all $n \in \mathbb{N}$ and all $t > 0$. This implies

$$\frac{1}{M(x_n,x_{n+1},t)} - 1 \le \lambda^n\left[\frac{1}{M(x_0,x_1,t)} - 1\right]. \tag{2.8}$$

Thus, for all $n > m$, we have

$$\begin{aligned}\frac{1}{M(x_n,x_m,t)} - 1 &\le \frac{1}{M(x_n,x_{n-1},t)} - 1 + \cdots + \frac{1}{M(x_{m+1},x_m,t)} - 1 \\ &\le \big(\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda^m\big)\left(\frac{1}{M(x_0,x_1,t)} - 1\right) \\ &\le \frac{\lambda^m}{1-\lambda}\left(\frac{1}{M(x_0,x_1,t)} - 1\right).\end{aligned} \tag{2.9}$$

That is, $\{x_n\}$ is a Cauchy sequence. Since, X is a complete fuzzy metric space, then there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. By (iii), $\alpha(x_n,x^*,t) \ge t$ holds for all $n \in \mathbb{N}$ and all $t > 0$. Suppose that there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{1+\lambda}\left(\frac{1}{M(x_{n_0},fx_{n_0},t)} - 1\right) > \frac{1}{M(x_{n_0},x^*,t)} - 1$$

and

$$\frac{1}{1+\lambda}\left(\frac{1}{M(x_{n_0-1},fx_{n_0-1},t)}-1\right) > \frac{1}{M(x_{n_0-1},x^*,t)}-1.$$

By (2.7) we deduce

$$\begin{aligned}
\frac{1}{M(x_{n_0-1},fx_{n_0-1},t)}-1 &= \frac{1}{M(x_{n_0-1},x_{n_0},t)}-1 \\
&\le \frac{1}{M(x_{n_0-1},x,t)}-1+\frac{1}{M(x_{n_0},x,t)}-1 \\
&< \frac{1}{1+\lambda}\left(\frac{1}{M(x_{n_0-1},fx_{n_0-1},t)}-1\right) \\
&\quad + \frac{1}{1+\lambda}\left(\frac{1}{M(x_{n_0},fx_{n_0},t)}-1\right) \\
&\le \frac{1}{1+\lambda}\left(\frac{1}{M(x_{n_0-1},fx_{n_0-1},t)}-1\right) \\
&\quad + \frac{\lambda}{1+\lambda}\left(\frac{1}{M(x_{n_0-1},fx_{n_0-1},t)}-1\right) \\
&= \frac{1}{M(x_{n_0-1},fx_{n_0-1},t)}-1,
\end{aligned}$$

which is a contradiction. Hence, either

$$\frac{1}{1+r}\left(\frac{1}{M(x_n,fx_n,t)}-1\right) \le \frac{1}{M(x_n,x^*,t)}-1$$

or

$$\frac{1}{1+r}\left(\frac{1}{M(x_{n-1},fx_{n-1},t)}-1\right) \le \frac{1}{M(x_{n-1},x^*,t)}-1$$

holds for all $n \in \mathbb{N}$. Let

$$\frac{1}{1+r}\left(\frac{1}{M(x_n,fx_n,t)}-1\right) \le \frac{1}{M(x_n,x^*,t)}-1.$$

Then from (iv) we have

$$\begin{aligned}
\frac{1}{M(x_n,fx^*,t)} &\le \frac{\alpha(x_n,x^*,t)}{tM(x_{n+1},fx^*,t)} = \frac{\alpha(x_n,x^*,t)}{tM(fx_n,fx^*,t)} \\
&\le \lambda P^f\left(x_n,x^*,t\right) + \left|Q^f\left(x_n,x^*,t\right)-\lambda\right|
\end{aligned} \tag{2.10}$$

for all $t > 0$, where

$$\begin{aligned}
P^f\left(x_n,x^*,t\right) = \max\biggl\{&\frac{1}{M(x_n,x^*,t)},\frac{1}{M(x_n,fx_n,t)},\frac{1}{M(x^*,fx^*,t)}, \\
&\frac{1}{2}\left[\frac{1}{M(x_n,fx^*,t)}+\frac{1}{M(x^*,fx_n,t)}-1\right]\biggr\}
\end{aligned}$$

$$= \max\left\{\frac{1}{M(x_n,x^*,t)},\frac{1}{M(x_n,x_{n+1},t)},\frac{1}{M(x^*,fx^*,t)},\right.$$
$$\left.\frac{1}{2}\left[\frac{1}{M(x_n,fx^*,t)}+\frac{1}{M(x^*,x_{n+1},t)}-1\right]\right\},$$

and so

$$\lim_{n\to\infty} P^f\left(x_n,x^*,t\right) = \frac{1}{M(x^*,fx^*,t)}.$$

Also,

$$Q^f\left(x_n,x^*,t\right) = \max\left\{M(x_n,x_{n+1},t),M\left(x^*,fx^*,t\right),M\left(x_n,fx^*,t\right),M\left(x^*,x_{n+1},t\right)\right\},$$

and so

$$\lim_{n\to\infty} Q^f\left(x_n,x^*,t\right) = 1.$$

Similarly,

$$\lim_{n\to\infty} R^f\left(x_n,x^*,t\right) = 0.$$

By taking limit as $n\to\infty$ in (2.10) we get

$$\frac{1}{M(x^*,fx^*,t)}-1 \le \lambda\left(\frac{1}{M(x^*,fx^*,t)}-1\right)$$

for all $t>0$. Now, assume that there exists t_0 such that $M(x^*,fx^*,t_0)<1$. Then by the above inequality we have $1\le\lambda$, which is a contradiction. Hence, $M(x^*,fx^*,t)=1$ for all $t>0$; *i.e.*, $x^*=fx^*$. Similarly we can deduce that x^* is a fixed point of f when

$$\frac{1}{1+r}\left(\frac{1}{M(x_{n-1},fx_{n-1},t)}-1\right) \le \frac{1}{M(x_{n-1},x^*,t)}-1. \qquad \square$$

Example 2.1 Let $X=\mathbb{R}^2$. We define $\alpha: X\times X\times[0,\infty)\to[0,\infty)$ by

$$\alpha(x,y,t) = \begin{cases} t, & x,y\in U=\{(0,0),(4,0),(0,4),(4,5),(5,4)\},\\ 0, & \text{otherwise.}\end{cases}$$

Define M on $X\times X\times(0,\infty)$ by $M((x_1,x_2),(y_1,y_2),t)=\frac{1}{1+|x_1-y_1|+|x_2-y_2|}$ and $a\star b=\min\{a,b\}$. Clearly, $(M,X,\star)$ is a complete triangular fuzzy metric space. Also, define $f: X\to X$ and $\psi:[0,\infty)\to[0,\infty)$ by

$$f(x_1,x_2) = \begin{cases} (x_1,0) & \text{if } x_1\le x_2 \text{ and } x_1,x_2\in U,\\ (0,x_2) & \text{if } x_1> x_2 \text{ and } x_1,x_2\in U,\\ (2x_1^2,3x_2^3) & \text{if } x_1,x_2\in\mathbb{R}^2\backslash U\end{cases} \quad\text{and}\quad \psi(t)=0.99t.$$

First we assume

$$\frac{1}{1+\frac{9}{10}}\left(\frac{1}{M(x,fx,t)}-1\right) = \frac{10}{19}\left(\frac{1}{M(x,fx,t)}-1\right) \le \frac{1}{M(x,y,t)}-1$$

and $\alpha(x,y,t) \geq t$ (or $x,y \in U$). Then

$$\begin{aligned}(x,y) \in \big\{&\big((0,0),(4,0)\big),\big((0,0),(0,4)\big),\big((0,0),(4,5)\big),\big((0,0),(5,4)\big),\\&\big((4,0),(0,0)\big),\big((4,0),(0,4)\big),\big((4,0),(5,4)\big),\big((4,0),(4,5)\big),\\&\big((0,4),(0,0)\big),\big((0,4),(4,0)\big),\big((0,4),(5,4)\big),\big((0,4),(4,5)\big),\\&\big((4,5),(0,0)\big),\big((4,5),(4,0)\big),\big((4,5),(0,4)\big),\big((4,5),(5,4)\big)\\&\big((5,4),(0,0)\big),\big((5,4),(4,0)\big),\big((5,4),(0,4)\big),\big((5,4),(4,5)\big)\big\}.\end{aligned}$$

Since $M(fx,fy,t) = M(fy,fx,t)$, $P^f(x,y,t) = P^f(y,x,t)$ and $Q^f(x,y,t) = Q^f(y,x,t)$, hence without any loss of generality we can reduce the above set to the following:

$$\begin{aligned}(x,y) \in \big\{&\big((0,0),(4,0)\big),\big((0,0),(0,4)\big),\big((0,0),(4,5)\big),\big((0,0),(5,4)\big),\\&\big((4,0),(0,4)\big),\big((4,0),(5,4)\big),\big((4,0),(4,5)\big),\big((0,4),(5,4)\big),\big((0,4),(4,5)\big)\big\}.\end{aligned}$$

Now, we consider the following cases:

• Let $(x,y) = ((0,0),(4,0))$, then

$$\begin{aligned}P^f\big((0,0),(4,0),t\big) &= \max\left\{\frac{1}{M((0,0),(4,0),t)},\frac{1}{M((0,0),f(0,0),t)},\frac{1}{M((4,0),f(4,0),t)},\right.\\&\quad\left.\frac{1}{2}\left[\frac{1}{M((0,0),f(4,0),t)}+\frac{1}{M((4,0),f(0,0),t)}-1\right]\right\}\\&= \max\left\{\frac{1}{M((0,0),(4,0),t)},\frac{1}{M((0,0),(0,0),t)},\frac{1}{M((4,0),(0,0),t)},\right.\\&\quad\left.\frac{1}{2}\left[\frac{1}{M((0,0),(0,0),t)}+\frac{1}{M((4,0),(0,0),t)}-1\right]\right\}\\&= \max\left\{5,1,5,\frac{1}{2}[1+5-1]\right\} = 5\end{aligned}$$

and

$$\begin{aligned}Q^f\big((0,0),(4,0),t\big) &= \max\big\{M\big((0,0),f(0,0),t\big),M\big((4,0),f(4,0),t\big),\\&\quad M\big((0,0),f(4,0),t\big),M\big((4,0),f(0,0),t\big)\big\}\\&= \max\big\{M\big((0,0),(0,0),t\big),M\big((4,0),(0,0),t\big),\\&\quad M\big((0,0),(0,0),t\big),M\big((4,0),(0,0),t\big)\big\}\\&= \max\left\{1,\frac{1}{5},1,\frac{1}{5}\right\} = 1,\end{aligned}$$

and so

$$\begin{aligned}\frac{\alpha((0,0),(4,0),t)}{tM(f(0,0),f(4,0),t)} &= 1 \leq \frac{46}{10} = \frac{9}{10}\cdot 5 + \left|1-\frac{9}{10}\right|\\&= \frac{9}{10}P^f\big((0,0),(4,0),t\big) + \left|Q^f\big((0,0),(4,0),t\big)-\frac{9}{10}\right|.\end{aligned}$$

• Let $(x, y) = ((0,0), (0,4))$, then

$$\begin{aligned} P^f\big((0,0),(4,0),t\big) &= \max\left\{\frac{1}{M((0,0),(4,0),t)}, \frac{1}{M((0,0),f(0,0),t)}, \frac{1}{M((4,0),f(4,0),t)},\right. \\ &\quad \left.\frac{1}{2}\left[\frac{1}{M((0,0),f(4,0),t)} + \frac{1}{M((4,0),f(0,0),t)} - 1\right]\right\} \\ &= \max\left\{\frac{1}{M((0,0),(4,0),t)}, \frac{1}{M((0,0),(0,0),t)}, \frac{1}{M((4,0),(0,0),t)},\right. \\ &\quad \left.\frac{1}{2}\left[\frac{1}{M((0,0),(0,0),t)} + \frac{1}{M((4,0),(0,0),t)} - 1\right]\right\} \\ &= \max\left\{5, 1, 5, \frac{1}{2}[1 + 5 - 1]\right\} = 5 \end{aligned}$$

and

$$\begin{aligned} Q^f\big((0,0),(4,0),t\big) &= \max\big\{M\big((0,0),f(0,0),t\big), M\big((4,0),f(4,0),t\big), \\ &\quad M\big((0,0),f(4,0),t\big), M\big((4,0),f(0,0),t\big)\big\} \\ &= \max\big\{M\big((0,0),(0,0),t\big), M\big((4,0),(0,0),t\big), \\ &\quad M\big((0,0),(0,0),t\big), M\big((4,0),(0,0),t\big)\big\} \\ &= \max\left\{1, \frac{1}{5}, 1, \frac{1}{5}\right\} = 1, \end{aligned}$$

and so

$$\begin{aligned} \frac{\alpha((0,0),(4,0),t)}{tM(f(0,0),f(4,0),t)} &= 1 \le \frac{46}{10} = \frac{9}{10}\cdot 5 + \left|1 - \frac{9}{10}\right| \\ &= \frac{9}{10}P^f\big((0,0),(4,0),t\big) + \left|Q^f\big((0,0),(4,0),t\big) - \frac{9}{10}\right|. \end{aligned}$$

• Let $(x, y) = ((0,0), (4,5))$, then

$$\begin{aligned} P^f\big((0,0),(4,5),t\big) &= \max\left\{\frac{1}{M((0,0),(4,5),t)}, \frac{1}{M((0,0),f(0,0),t)}, \frac{1}{M((4,5),f(4,5),t)},\right. \\ &\quad \left.\frac{1}{2}\left[\frac{1}{M((0,0),f(4,5),t)} + \frac{1}{M((4,5),f(0,0),t)} - 1\right]\right\} \\ &= \max\left\{\frac{1}{M((0,0),(4,5),t)}, \frac{1}{M((0,0),(0,0),t)}, \frac{1}{M((4,5),(4,0),t)},\right. \\ &\quad \left.\frac{1}{2}\left[\frac{1}{M((0,0),(4,0),t)} + \frac{1}{M((4,5),(0,0),t)} - 1\right]\right\} \\ &= \max\left\{10, 1, 6, \frac{1}{2}[5 + 10 - 1]\right\} = 10 \end{aligned}$$

and

$$\begin{aligned} Q^f\big((0,0),(4,5),t\big) &= \max\big\{M\big((0,0),f(0,0),t\big), M\big((4,5),f(4,5),t\big) \\ &\quad M\big((0,0),f(4,5),t\big), M\big((4,5),f(0,0),t\big)\big\} \end{aligned}$$

$$= \max\{M((0,0),(0,0),t), M((4,5),(4,0),t), M((0,0),(4,0),t), M((4,5),(0,0),t)\}$$
$$= \max\left\{1, \frac{1}{6}, \frac{1}{5}, \frac{1}{10}\right\} = 1,$$

and so

$$\frac{\alpha((0,0),(4,5),t)}{tM(f(0,0),f(4,5),t)} = 5 \le \frac{91}{10} = \frac{9}{10} \cdot 10 + \left|1 - \frac{9}{10}\right| = \frac{9}{10} P^f((0,0),(4,5),t) + \left|Q^f((0,0),(4,5),t) - \frac{9}{10}\right|.$$

- Let $(x,y) = ((0,0),(5,4))$, then

$$P^f((0,0),(5,4),t) = \max\left\{\frac{1}{M((0,0),(5,4),t)}, \frac{1}{M((0,0),f(0,0),t)}, \frac{1}{M((5,4),f(5,4),t)}, \frac{1}{2}\left[\frac{1}{M((0,0),f(5,4),t)} + \frac{1}{M((5,4),f(0,0),t)} - 1\right]\right\}$$
$$= \max\left\{\frac{1}{M((0,0),(5,4),t)}, \frac{1}{M((0,0),(0,0),t)}, \frac{1}{M((5,4),(0,4),t)}, \frac{1}{2}\left[\frac{1}{M((0,0),(0,4),t)} + \frac{1}{M((5,4),(0,0),t)} - 1\right]\right\}$$
$$= \max\left\{10, 1, 6, \frac{1}{2}[5 + 10 - 1]\right\} = 10$$

and

$$Q^f((0,0),(5,4),t) = \max\{M((0,0),f(0,0),t), M((5,4),f(5,4),t), M((0,0),f(5,4),t), M((5,4),f(0,0),t)\}$$
$$= \max\{M((0,0),(0,0),t), M((5,4),(0,4),t), M((0,0),(0,4),t), M((5,4),(0,0),t)\}$$
$$= \max\left\{1, \frac{1}{6}, \frac{1}{5}, \frac{1}{10}\right\} = 1,$$

and so

$$\frac{\alpha((0,0),(5,4),t)}{tM(f(0,0),f(5,4),t)} = 5 \le \frac{91}{10} = \frac{9}{10} \cdot 10 + \left|1 - \frac{9}{10}\right| = \frac{9}{10} P^f((0,0),(5,4),t) + \left|Q^f((0,0),(5,4),t) - \frac{9}{10}\right|.$$

- Let $(x,y) = ((4,0),(0,4))$, then

$$P^f((4,0),(0,4),t) = \max\left\{\frac{1}{M((4,0),(0,4),t)}, \frac{1}{M((4,0),f(4,0),t)}, \frac{1}{M((0,4),f(0,4),t)}, \frac{1}{2}\left[\frac{1}{M((4,0),f(0,4),t)} + \frac{1}{M((0,4),f(4,0),t)} - 1\right]\right\}$$

$$\begin{aligned}
&= \max\left\{\frac{1}{M((4,0),(0,4),t)}, \frac{1}{M((4,0),(0,0),t)}, \frac{1}{M((0,4),(0,0),t)},\right.\\
&\quad \left.\frac{1}{2}\left[\frac{1}{M((4,0),(0,0),t)} + \frac{1}{M((0,4),(0,0),t)} - 1\right]\right\}\\
&= \max\left\{9,5,5,\frac{1}{2}[5+5-1]\right\} = 9
\end{aligned}$$

and

$$\begin{aligned}
Q^f\big((4,0),(0,4),t\big) &= \max\big\{M\big((4,0),f(4,0),t\big), M\big((0,4),f(0,4),t\big),\\
&\quad M\big((4,0),f(0,4),t\big), M\big((0,4),f(4,0),t\big)\big\}\\
&= \max\big\{M\big((4,0),(0,0),t\big), M\big((0,4),(0,0),t\big),\\
&\quad M\big((4,0),(0,0),t\big), M\big((0,4),(0,0),t\big)\big\}\\
&= \max\left\{\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}\right\} = \frac{1}{5},
\end{aligned}$$

and so

$$\begin{aligned}
\frac{\alpha((4,0),(0,4),t)}{tM(f(4,0),f(0,4),t)} = 1 \le \frac{88}{10} &= \frac{9}{10}\cdot 9 + \left|\frac{1}{5} - \frac{9}{10}\right|\\
&= \frac{9}{10}P^f\big((4,0),(0,4),t\big) + \left|Q^T\big((4,0),(0,4),t\big) - \frac{9}{10}\right|.
\end{aligned}$$

- Let $(x,y) = ((4,0),(5,4))$, then

$$\begin{aligned}
P^f\big((4,0),(5,4),t\big) &= \max\left\{\frac{1}{M((4,0),(5,4),t)}, \frac{1}{M((4,0),f(4,0),t)}, \frac{1}{M((5,4),f(5,4),t)},\right.\\
&\quad \left.\frac{1}{2}\left[\frac{1}{M((4,0),f(5,4),t)} + \frac{1}{M((5,4),f(4,0),t)} - 1\right]\right\}\\
&= \max\left\{\frac{1}{M((4,0),(5,4),t)}, \frac{1}{M((4,0),(0,0),t)}, \frac{1}{M((5,4),(0,4),t)},\right.\\
&\quad \left.\frac{1}{2}\left[\frac{1}{M((4,0),(0,4),t)} + \frac{1}{M((5,4),(0,0),t)} - 1\right]\right\}\\
&= \max\left\{6,5,6,\frac{1}{2}[9+10-1]\right\} = 9
\end{aligned}$$

and

$$\begin{aligned}
Q^f\big((4,0),(5,4),t\big) &= \max\big\{M\big((4,0),f(4,0),t\big), M\big((5,4),f(5,4),t\big),\\
&\quad M\big((4,0),f(5,4),t\big), M\big((5,4),f(4,0),t\big)\big\}\\
&= \max\big\{M\big((4,0),(0,0),t\big), M\big((5,4),(0,4),t\big),\\
&\quad M\big((4,0),(0,4),t\big), M\big((5,4),(0,0),t\big)\big\}\\
&= \max\left\{\frac{1}{5},\frac{1}{6},\frac{1}{9},\frac{1}{10}\right\} = \frac{1}{5},
\end{aligned}$$

and so

$$\begin{aligned}\frac{\alpha((4,0),(5,4),t)}{tM(f(4,0),f(5,4),t)} &= 5 \leq \frac{88}{10} = \frac{9}{10}\cdot 9 + \left|\frac{1}{5} - \frac{9}{10}\right| \\ &= \frac{9}{10}P^f\big((4,0),(5,4),t\big) + \left|Q^f\big((4,0),(5,4),t\big) - \frac{9}{10}\right|.\end{aligned}$$

- Let $(x,y) = ((4,0),(4,5))$, then

$$\begin{aligned}P^f\big((4,0),(4,5),t\big) &= \max\left\{\frac{1}{M((4,0),(4,5),t)}, \frac{1}{M((4,0),f(4,0),t)}, \frac{1}{M((4,5),f(4,5),t)},\right. \\ &\quad \left.\frac{1}{2}\left[\frac{1}{M((4,0),f(4,5),t)} + \frac{1}{M((4,5),f(4,0),t)} - 1\right]\right\} \\ &= \max\left\{\frac{1}{M((4,0),(4,5),t)}, \frac{1}{M((4,0),(0,0),t)}, \frac{1}{M((4,5),(4,0),t)},\right. \\ &\quad \left.\frac{1}{2}\left[\frac{1}{M((4,0),(4,0),t)} + \frac{1}{M((4,5),f(4,0),t)} - 1\right]\right\} \\ &= \max\left\{6,5,6,\frac{1}{2}[1+6-1]\right\} = 6\end{aligned}$$

and

$$\begin{aligned}Q^f\big((4,0),(4,5),t\big) &= \max\big\{M\big((4,0),f(4,0),t\big), M\big((4,5),f(4,5),t\big), \\ &\quad M\big((4,0),f(4,5),t\big), M\big((4,5),f(4,0),t\big)\big\} \\ &= \max\big\{M\big((4,0),(0,0),t\big), M\big((4,5),(4,0),t\big), \\ &\quad M\big((4,0),(4,0),t\big), M\big((4,5),(0,0),t\big)\big\} \\ &= \max\left\{\frac{1}{5},\frac{1}{6},1,\frac{1}{10}\right\} = 1,\end{aligned}$$

and so

$$\begin{aligned}\frac{\alpha((4,0),(5,4),t)}{tM(f(4,0),f(5,4),t)} &= 5 \leq \frac{88}{10} = \frac{9}{10}\cdot 9 + \left|\frac{1}{5} - \frac{9}{10}\right| \\ &= \frac{9}{10}P^f\big((4,0),(5,4),t\big) + \left|Q^f\big((4,0),(5,4),t\big) - \frac{9}{10}\right|.\end{aligned}$$

- Let $(x,y) = ((0,4),(5,4))$, then

$$\begin{aligned}P^f\big((0,4),(5,4),t\big) &= \max\left\{\frac{1}{M((0,4),(5,4),t)}, \frac{1}{M((0,4),f(0,4),t)}, \frac{1}{M((5,4),f(5,4),t)},\right. \\ &\quad \left.\frac{1}{2}\left[\frac{1}{M((0,4),f(5,4),t)} + \frac{1}{M((5,4),f(0,4),t)} - 1\right]\right\} \\ &= \max\left\{\frac{1}{M((0,4),(5,4),t)}, \frac{1}{M((0,4),(0,0),t)}, \frac{1}{M((5,4),(0,4),t)},\right. \\ &\quad \left.\frac{1}{2}\left[\frac{1}{M((0,4),(0,4),t)} + \frac{1}{M((5,4),(0,0),t)} - 1\right]\right\} \\ &= \max\left\{6,5,6,\frac{1}{2}[1+10-1]\right\} = 6\end{aligned}$$

and

$$\begin{aligned} Q^f\big((0,4),(5,4),t\big) &= \max\big\{M\big((0,4),f(0,4),t\big),M\big((5,4),f(5,4),t\big),\\ &\quad M\big((0,4),f(5,4),t\big),M\big((5,4),f(0,4),t\big)\big\}\\ &= \max\big\{M\big((0,4),(0,0),t\big),M\big((5,4),(0,4),t\big),\\ &\quad M\big((0,4),(0,4),t\big),M\big((5,4),(0,0),t\big)\big\}\\ &= \max\left\{\frac{1}{5},\frac{1}{6},1,\frac{1}{10}\right\} = 1, \end{aligned}$$

and so

$$\begin{aligned} \frac{\alpha((0,4),(5,4),t)}{tM(f(0,4),f(5,4),t)} = 5 \leq \frac{55}{10} &= \frac{9}{10}\cdot 6 + \left|1-\frac{9}{10}\right|\\ &= \frac{9}{10}P^f\big((0,4),(5,4),t\big) + \left|Q^f\big((0,4),(5,4),t\big)-\frac{9}{10}\right|. \end{aligned}$$

- Let $(x,y) = ((0,4),(4,5))$, then

$$\begin{aligned} P^f\big((0,4),(4,5),t\big) &= \max\left\{\frac{1}{M((0,4),(4,5),t)},\frac{1}{M((0,4),f(0,4),t)},\frac{1}{M((4,5),f(4,5),t)},\right.\\ &\quad \left.\frac{1}{2}\left[\frac{1}{M((0,4),f(4,5),t)}+\frac{1}{M((4,5),f(0,4),t)}-1\right]\right\}\\ &= \max\left\{\frac{1}{M((0,4),(4,5),t)},\frac{1}{M((0,4),(0,0),t)},\frac{1}{M((4,5),(4,0),t)},\right.\\ &\quad \left.\frac{1}{2}\left[\frac{1}{M((0,4),(4,0),t)}+\frac{1}{M((4,5),(0,0),t)}-1\right]\right\}\\ &= \max\left\{6,5,6,\frac{1}{2}[9+10-1]\right\} = 9 \end{aligned}$$

and

$$\begin{aligned} Q^f\big((0,4),(4,5),t\big) &= \max\big\{M\big((0,4),f(0,4),t\big),M\big((4,5),f(4,5),t\big),\\ &\quad M\big((0,4),f(4,5),t\big),M\big((4,5),f(0,4),t\big)\big\}\\ &= \max\big\{M\big((0,4),(0,0),t\big),M\big((4,5),(4,0),t\big),\\ &\quad M\big((0,4),(4,0),t\big),M\big((4,5),(0,0),t\big)\big\}\\ &= \max\left\{\frac{1}{5},\frac{1}{6},\frac{1}{9},\frac{1}{10}\right\} = \frac{1}{5}, \end{aligned}$$

and so

$$\begin{aligned} \frac{\alpha((0,4),(4,5),t)}{tM(f(0,4),f(4,5),t)} = 5 \leq \frac{88}{10} &= \frac{9}{10}\cdot 9 + \left|\frac{1}{5}-\frac{9}{10}\right|\\ &= \frac{9}{10}P^f\big((0,4),(4,5),t\big) + \left|Q^f\big((0,4),(4,5),t\big)-\frac{9}{10}\right|. \end{aligned}$$

Otherwise, $\alpha(x,y,t) = 0$, and so

$$\frac{\alpha(x,y,t)}{tM(fx,fy,t)} = 0 \leq \frac{9}{10}P^f(x,y,t) + \left|Q^f(x,y,t) - \frac{9}{10}\right|.$$

That is,

$$\frac{1}{1+\frac{9}{10}}\left(\frac{1}{M(x,fx,t)} - 1\right) \leq \frac{1}{M(x,y,t)} - 1$$

implies

$$\begin{aligned}\frac{\alpha(x,y,t)}{tM(fx,fy,t)} &\leq \frac{9}{10}P^f(x,y,t) + \left|Q^f(x,y,t) - \frac{9}{10}\right| \\ &\leq \frac{9}{10}P^f(x,y,t) + \left|Q^f(x,y,t) - \frac{9}{10}\right| + LR^f(x,y,t),\end{aligned}$$

where $L \geq 0$. Let $\alpha(x,y,t) \geq t$, then $x, y \in U$. On the other hand, $fw \in U$ for all $w \in U$. Then $\alpha(fx,fy,t) \geq t$. That is, f is an α-admissible mapping. If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq t$ with $x_n \to x$ as $n \to \infty$, then $x_n \in U$ for all $n \in \mathbb{N}$. Also, U is a closed set, so $x \in U$. That is, $\alpha(x_n, x, t) \geq t$ for all $n \in \mathbb{N} \cup \{0\}$. Clearly, $\alpha((0,0), f(0,0), t) \geq t$.

Therefore all conditions of Theorem 2.1 hold and f has a fixed point. Here, $x = (0,0)$ is a fixed point of f.

Corollary 2.1 *Let $(X, M, *)$ be a complete triangular fuzzy metric space and f be a self-mapping on X. Also suppose that $\alpha : X \times X \times [0,\infty) \to [0,\infty)$ is a mapping. Assume that the following assertions hold*:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq t$ for all $t > 0$*;
(ii) *f is an α-admissible mapping*;
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq t$ for all $n \in \mathbb{N}$ and all $t > 0$ with $x_n \to x$ as $n \to \infty$, then $\alpha(x, x_n, t) \geq t$ for all $n \in \mathbb{N} \cup \{0\}$*;
(iv) *for all $x, y \in X$ and all $t > 0$, we have*

$$\frac{\alpha(x,y,t)}{tM(fx,fy,t)} \leq \lambda P^f(x,y,t) + \left|Q^f(x,y,t) - \lambda\right| + LR^f(x,y,t)$$

holds where $\lambda \in (0,1)$ and $L \geq 0$.

Then f has a fixed point.

By taking $L = 0$ in Corollary 2.1, we obtain the following corollary.

Corollary 2.2 *Let $(X, M, *)$ be a complete triangular fuzzy metric space and f be a self-mapping on X. Also suppose that $\alpha : X \times X \times [0,\infty) \to [0,\infty)$ is a mapping. Assume that the following assertions hold*:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq t$ for all $t > 0$*;
(ii) *f is an α-admissible mapping*;
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq t$ for all $n \in \mathbb{N}$ and all $t > 0$ with $x_n \to x$ as $n \to \infty$, then $\alpha(x, x_n, t) \geq t$ for all $n \in \mathbb{N} \cup \{0\}$*;

(iv) *for all* $x, y \in X$ *and all* $t > 0$, *we have*

$$\frac{\alpha(x,y,t)}{tM(fx,fy,t)} \leq \lambda P^f(x,y,t) + \left|Q^f(x,y,t) - \lambda\right|$$

holds where $\lambda \in (0,1)$.

Then f *has a fixed point.*

By taking $\alpha(x,y,t) = t$ for all $x, y \in X$ and all $t > 0$ in Corollary 2.1, we obtain the following result.

Corollary 2.3 *Let* $(X, M, *)$ *be a complete triangular fuzzy metric space and* f *be a self-mapping on* X. *Assume that for all* $x, y \in X$ *and all* $t > 0$,

$$\frac{1}{M(Tx,Ty,t)} \leq \lambda P^f(x,y,t) + \left|Q^f(x,y,t) - \lambda\right| + LR^f(x,y,t)$$

holds where $\lambda \in (0,1)$ *and* $L \geq 0$. *Then* f *has a fixed point.*

By taking $L = 0$ in Corollary 2.3, we obtain the following corollary.

Corollary 2.4 *Let* $(X, M, *)$ *be a complete triangular fuzzy metric space and* f *be a self-mapping on* X. *Assume that for all* $x, y \in X$ *and all* $t > 0$,

$$\frac{1}{M(fx,fy,t)} \leq \lambda P^f(x,y,t) + \left|Q^f(x,y,t) - \lambda\right|$$

holds where $\lambda \in (0,1)$. *Then* f *has a fixed point.*

Theorem 2.2 *Let* $(X, M, *, \preceq)$ *be a complete triangular partially ordered fuzzy metric space and* f *be a self-mapping on* X. *Assume that the following assertions hold*:

(i) *there exists* $x_0 \in X$ *such that* $x_0 \preceq fx_0$;

(ii) f *is an increasing mapping*;

(iii) *if* $\{x_n\}$ *is an increasing sequence in* X *such that with* $x_n \to x$ *as* $n \to \infty$, *then* $x \preceq x_n$ *for all* $n \in \mathbb{N} \cup \{0\}$;

(iv) *for all* $x, y \in X$ *and all* $t > 0$ *with* $\frac{1}{1+\lambda}(\frac{1}{M(x,fx,t)} - 1) \leq \frac{1}{M(x,y,t)} - 1$ *and* $x \preceq y$, *we have*

$$\frac{1}{M(fx,fy,t)} \leq \lambda P^f(x,y,t) + \left|Q^f(x,y,t) - \lambda(t)\right| + LR^f(x,y,t),$$

where $\lambda \in (0,1)$ *and* $L \geq 0$.

Then f *has a fixed point.*

Proof Define $\alpha : X \times X \times (0,\infty) \to [0,+\infty)$ by $\alpha(x,y,t) = \begin{cases} t, & \text{if } x \preceq y, \\ 0, & \text{otherwise.}\end{cases}$ At first we prove that f is an α-admissible mapping. Let $\alpha(x,y,t) \geq t$, then $x \preceq y$. Now, since f is increasing, then we have $fx \preceq fy$. That is, $\alpha(fx,fy,t) \geq t$. Therefore f is an α-admissible mapping. From (i) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$. That is, $\alpha(x_0, fx_0, t) \geq t$ for all $t > 0$. If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq t$ for all $n \in \mathbb{N}$ and all $t > 0$ with $x_n \to x$ as $n \to \infty$, then $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ with $x_n \to x$ as $n \to \infty$. Hence from (iii) we get $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$. That is, $\alpha(x_n, x, t) \geq t$ for all $n \in \mathbb{N} \cup \{0\}$.

Let $\frac{1}{1+\lambda}(\frac{1}{M(x,fx,t)} - 1) \leq \frac{1}{M(x,y,t)} - 1$ and $x \preceq y$ (or $\alpha(x,y,t) = t$), then from (iv) we have

$$\frac{1}{M(fx,fy,t)} \leq \lambda P^f(x,y,t) + \left|Q^f(x,y,t) - \lambda(t)\right| + LR^f(x,y,t). \tag{2.11}$$

Also if $\alpha(x,y,t) = 0$, then

$$\frac{\alpha(x,y,t)}{tM(fx,fy,t)} = 0 \leq \lambda P^f(x,y,t) + \left|Q^f(x,y,t) - \lambda(t)\right| + LR^f(x,y,t). \tag{2.12}$$

That is, for all $x,y \in X$ and all $t > 0$ with $\frac{1}{1+\lambda}(\frac{1}{M(x,fx,t)} - 1) \leq \frac{1}{M(x,y,t)} - 1$, we have

$$\frac{\alpha(x,y,t)}{tM(fx,fy,t)} \leq \lambda P^f(x,y,t) + \left|Q^f(x,y,t) - \lambda(t)\right| + LR^f(x,y,t). \tag{2.13}$$

Hence, all conditions of Theorem 2.1 are satisfied and f has a fixed point. □

Corollary 2.5 *Let $(X,M,*,\preceq)$ be a complete triangular partially ordered fuzzy metric space and f be a self-mapping on X. Assume that the following assertions hold*:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq fx_0$*;
(ii) *f is an increasing mapping*;
(iii) *if $\{x_n\}$ is an increasing sequence in X such that with $x_n \to x$ as $n \to \infty$, then $x \preceq x_n$ for all $n \in \mathbb{N} \cup \{0\}$*;
(iv) *for all $x,y \in X$ and all $t > 0$ with $x \preceq y$, we have*

$$\frac{1}{M(fx,fy,t)} \leq \lambda P^f(x,y,t) + \left|Q^f(x,y,t) - \lambda\right| + LR^f(x,y,t),$$

where $\lambda \in (0,1)$ and $L \geq 0$.

Then f has a fixed point.

Corollary 2.6 *Let $(X,M,*,\preceq)$ be a complete triangular partially ordered fuzzy metric space and f be a self-mapping on X. Assume that the following assertions hold*:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq fx_0$*;
(ii) *f is an increasing mapping*;
(iii) *if $\{x_n\}$ is an increasing sequence in X such that with $x_n \to x$ as $n \to \infty$, then $x \preceq x_n$ for all $n \in \mathbb{N} \cup \{0\}$*;
(iv) *for all $x,y \in X$ and all $t > 0$ with $x \preceq y$, we have*

$$\frac{1}{M(fx,fy,t)} \leq \lambda P^f(x,y,t) + \left|Q^f(x,y,t) - \lambda\right|,$$

where $\lambda \in (0,1)$.

Then f has a fixed point.

3 Some results in non-Archimedean fuzzy metric spaces

In this section we state and prove certain fixed point results in the setting of non-Archimedean fuzzy metric space to generalize the work of Miheț [14].

Theorem 3.1 *Let $(X, M, \star, \preceq)$ be a partially ordered complete non-Archimedean fuzzy metric space and f be an α-admissible and non-increasing mapping. Also suppose that the following assertions hold*:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq t$ for all $t > 0$ and $x_0 \preceq fx_0$*;

(ii) *if $\{x_n\}$ is an increasing sequence such that $\alpha(x_n, x_{n+1}, t) \geq t$ for all $n \in \mathbb{N}$ and all $t > 0$ with $x_n \to x$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$ where $\alpha(x, fx, t) \geq t$*;

(iii) *for all comparable $x, y \in X$ and all $t > 0$, we have*

$$\begin{aligned}&\bigl[\alpha(x,fx,t)\alpha(y,fy,t)+1\bigr]^{\psi(M(fx,fy,t))}\left[t^2+\frac{\alpha(x,fx,t)+\alpha(y,fy,t)}{2t}\right]^{\phi(M(x,y,t))}\\&\quad\leq\bigl[t^2+1\bigr]^{\psi(M(x,y,t))},\end{aligned}$$

where $\psi, \phi : [0,1] \to [0,1]$ are two continuous functions such that ψ is decreasing, $\psi(t) = 0$ iff $t = 1$ and $\phi(t) > 0$ for all $t \in (0,1)$.

Then f has a fixed point.

Proof Let $x_0 \preceq fx_0$. If $x_0 = fx_0$, then the result is proved. Hence we suppose that $x_0 \prec fx_0$. Define a sequence $\{x_n\}$ by $x_n = f^n x_0 = fx_{n-1}$ for all $n \in \mathbb{N}$. Since f is non-decreasing and $x_0 \prec fx_0$, then

$$x_0 \prec x_1 \preceq x_2 \preceq \cdots, \tag{3.1}$$

and hence $\{x_n\}$ is a non-decreasing sequence. If $x_n = x_{n+1} = fx_n$ for some $n \in \mathbb{N}$, then the result is proved as x_n is a fixed point of f. In what follows we suppose that $0 < M(x_n, x_{n+1}, t) < 1$. Since f is an α-admissible mapping and

$$\alpha(x_0, fx_0, t) = \alpha(x_0, x_1, t) \geq t,$$

we deduce that

$$\alpha(x_1, x_2, t) = \alpha(fx_0, fx_1, t) \geq t.$$

Continuing this process, we get $\alpha(x_n, x_{n+1}, t) \geq t$ for all $n \in \mathbb{N} \cup \{0\}$ and all $t > 0$. From (iii) with $x = x_{n-1}$ and $y = x_n$, we obtain

$$\begin{aligned}&\bigl[t^2+1\bigr]^{\psi(M(x_n,x_{n+1},t))+\phi(M(x_{n-1},x_n,t))}\\&\quad=\bigl[t^2+1\bigr]^{\psi(M(x_n,x_{n+1},t))}\left[t^2+\frac{t+t}{2t}\right]^{\phi(M(x_{n-1},x_n,t))}\\&\quad\leq\bigl[\alpha(x_{n-1},x_n,t)\alpha(x_n,x_{n+1},t)+1\bigr]^{\psi(M(fx_{n-1},fx_n,t))}\\&\qquad\times\left[t^2+\frac{\alpha(x_{n-1},x_n,t)+\alpha(x_n,x_{n+1},t)}{2t}\right]^{\phi(M(x_{n-1},x_n,t))}\\&\quad\leq\bigl[t^2+1\bigr]^{\psi(M(x_{n-1},x_n,t))},\end{aligned}$$

and so

$$\psi\bigl(M(x_n,x_{n+1},t)\bigr)\leq\psi\bigl(M(x_{n-1},x_n,t)\bigr)-\phi\bigl(M(x_{n-1},x_n,t)\bigr)\leq\psi\bigl(M(x_{n-1},x_n,t)\bigr). \tag{3.2}$$

Since ψ is decreasing, so $M(x_{n-1}, x_n, t) \leq M(x_n, x_{n+1}, t)$. Hence, $\{M(x_n, x_{n+1}, t)\}$ is an increasing sequence in $(0,1]$. Then there exists $l(t) \in (0,1]$ such that

$$\lim_{n\to+\infty} M(x_n, x_{n+1}, t) = l(t)$$

for all $t > 0$. Let us prove that $l(t) = 1$ for all $t > 0$. Suppose that there exists $t_0 > 0$ such that $0 < l(t_0) < 1$. By taking the limit as $n \to +\infty$ in (3.2), we have

$$\psi\bigl(l(t_0)\bigr) \leq \psi\bigl(l(t_0)\bigr) - \phi\bigl(l(t_0)\bigr).$$

Then $\phi(l(t_0)) = 0$, which is a contradiction, and so $l(t) = 1$ for all $t > 0$. Now, we want to show that $\{x_n\}$ is a Cauchy sequence. Assume that it is not true. Then there exist $\epsilon \in (0,1)$ and $t_0 > 0$ such that for all $k \in \mathbb{N}$ there exist $n(k), m(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ and

$$M(x_{m(k)}, x_{n(k)}, t_0) \leq 1 - \epsilon. \tag{3.3}$$

Assume that $m(k)$ is the least integer exceeding $n(k)$ satisfying the above inequality. Equivalently,

$$M(x_{m(k)-1}, x_{n(k)}, t_0) > 1 - \epsilon, \tag{3.4}$$

and so for all k we get

$$\begin{aligned} 1 - \epsilon &\geq M(x_{m(k)}, x_{n(k)}, t_0) \\ &\geq M(x_{m(k)-1}, x_{m(k)}, t_0) \star M(x_{m(k)-1}, x_{n(k)}, t_0) \\ &> \tau_{m(k)}(t_0) \star (1 - \epsilon). \end{aligned} \tag{3.5}$$

By taking limit as $n \to +\infty$ in (3.5), we deduce that

$$\lim_{n\to+\infty} M(x_{m(k)}, x_{n(k)}, t_0) = 1 - \epsilon$$

for $t > 0$.

From

$$\begin{aligned} M(x_{m(k)+1}, x_{n(k)+1}, t_0) \geq{}& M(x_{m(k)+1}, x_{m(k)}, t_0) \star M(x_{m(k)}, x_{n(k)}, t_0) \\ &\star M(x_{n(k)}, x_{n(k)+1}, t_0) \end{aligned}$$

and

$$\begin{aligned} M(x_{m(k)}, x_{n(k)}, t_0) \geq{}& M(x_{m(k)+1}, x_{m(k)}, t_0) \star M(x_{m(k)+1}, x_{n(k)+1}, t_0) \\ &\star M(x_{n(k)}, x_{n(k)+1}, t_0), \end{aligned}$$

we get

$$\lim_{n\to+\infty} M(x_{m(k)+1}, x_{n(k)+1}, t_0) = 1 - \epsilon.$$

From (iii) with $x = x_{m(k)}$ and $y = x_{n(k)}$, we deduce

$$\begin{aligned}
&\left[t_0^2+1\right]^{\psi(M(x_{m(k)+1},x_{n(k)+1},t_0))+\phi(M(x_{m(k)},x_{n(k)},t_0))} \\
&\quad = \left[t_0^2+1\right]^{\psi(M(fx_{m(k)},fx_{n(k)},t_0))}\left[t_0^2+\frac{t_0+t_0}{2t_0}\right]^{\phi(M(x_{m(k)},x_{n(k)},t_0))} \\
&\quad \le \left[\alpha(x_{m(k)},fx_{m(k)},t_0)\alpha(x_{n(k)},fx_{n(k)},t_0)+1\right]^{\psi(M(fx_{m(k)},fx_{n(k)},t_0))} \\
&\qquad \times \left[t_0^2+\frac{\alpha(x_{m(k)},fx_{m(k)},t_0)+\alpha(x_{n(k)},fx_{n(k)},t_0)}{2t_0}\right]^{\phi(M(x_{m(k)},x_{n(k)},t_0))} \\
&\quad \le \left[t_0^2+1\right]^{\psi(M(x_{m(k)},x_{n(k)},t_0))},
\end{aligned}$$

which implies

$$\psi\bigl(M(x_{m(k)+1},x_{n(k)+1},t_0)\bigr) \le \psi\bigl(M(x_{m(k)},x_{n(k)},t_0)\bigr) - \phi\bigl(M(x_{m(k)},x_{n(k)},t_0)\bigr).$$

Applying the continuity of the functions ϕ and ψ, by taking the limit as $k \to +\infty$ in the above inequality, we get

$$\psi(1-\epsilon) \le \psi(1-\epsilon) - \phi(1-\epsilon),$$

and so $\phi(1-\epsilon) = 0$, which is a contradiction. Then $\{x_n\}$ is a Cauchy sequence. Since $(X, M, \star)$ is a complete non-Archimedean fuzzy metric space, then the sequence $\{x_n\}$ converges to some $z \in X$, that is, for all $t > 0$,

$$\lim_{n\to+\infty} M(x_n, z, t) = 1.$$

Assume that there exists $t_0 > 0$ such that $0 < M(z, fz, t_0) < 1$. Then by (2.11) and (ii) we get

$$\begin{aligned}
&\left[t_0^2+1\right]^{\psi(M(x_{n+1},fz,t_0))+\phi(M(x_n,z,t_0))} \\
&\quad = \left[t_0^2+1\right]^{\psi(M(fx_n,fz,t_0))}\left[t_0^2+\frac{t_0+t_0}{2t_0}\right]^{\phi(M(x_n,z,t_0))} \\
&\quad \le \left[\alpha(x_n,fx_n,t_0)\alpha(z,fz,t_0)+1\right]^{\psi(M(fx_n,fz,t_0))}\left[t_0^2+\frac{\alpha(x_n,fx_n,t_0)+\alpha(z,fz,t_0)}{2t_0}\right]^{\phi(M(x_n,z,t_0))} \\
&\quad \le \left[t_0^2+1\right]^{\psi(M(x_n,z,t_0))},
\end{aligned}$$

and hence

$$\psi\bigl(M(x_{n+1},fz,t_0)\bigr) \le \psi\bigl(M(x_n,z,t_0)\bigr) - \phi\bigl(M(x_n,z,t_0)\bigr).$$

By taking the limit as $n \to +\infty$ in the above inequality, we have

$$\psi\bigl(M(z,fz,t_0)\bigr) \le \psi(1) - \phi(1) \le \psi(1) = 0.$$

Then $\psi(M(z, fz, t_0)) = 0$, *i.e.*, $M(z, fz, t_0) = 1$, which is a contradiction. Hence, $M(z, fz, t) = 1$ for all $t > 0$, that is, $z = fz$. □

If in Theorem 3.1 we take $\alpha(x,y,t) = t$ for all $x, y \in X$ and all $t > 0$, then we deduce the following corollary.

Corollary 3.1 *Let $(X, M, \star, \preceq)$ be a partially ordered complete non-Archimedean fuzzy metric space, $\psi, \phi : [0,1] \to [0,1]$ as in Theorem 3.1 and $f : X \to X$ be an increasing mapping such that*

$$\psi\big(M(fx,fy,t)\big) \le \psi\big(M(x,y,t)\big) - \phi\big(M(x,y,t)\big)$$

holds for all comparable $x, y \in X$. If the following assertions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;*

(ii) *if $\{x_n\}$ is an increasing sequence such that $x_n \to x$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then f has a fixed point.

Theorem 3.2 *Let $(X, M, \star, \preceq)$ be a partially ordered complete non-Archimedean fuzzy metric space and f be an α-admissible and non-increasing mapping such that the following assertions hold:*

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \ge t$ for all $t > 0$ and $x_0 \preceq fx_0$;*

(ii) *if $\{x_n\}$ is an increasing sequence such that $\alpha(x_n, x_{n+1}, t) \ge t$ for all $n \in \mathbb{N} \cup \{0\}$, all $t > 0$ and $x_n \to x$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$ where $\alpha(x, fx, t) \ge t$;*

(iii) *assume that there exists a function $\beta : [0,1] \to [1, +\infty)$ such that for any sequence $\{t_n\} \subseteq [0,1]$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 1$ such that*

$$\begin{aligned} &\big[t^2+1\big]^{M(fx,fy,t)} \\ &\quad \ge \left[\frac{\alpha(x,fx,t)\alpha(y,fy,t)}{t^2}\right]\big[\alpha(x,fx,t)\alpha(y,fy,t)+1\big]^{\beta(M(x,y,t))M(x,y,t)} \end{aligned} \tag{3.6}$$

holds for all comparable $x, y \in X$ and all $t > 0$.

Then f has a fixed point.

Proof Let $x_0 \preceq fx_0$. If $x_0 = fx_0$, then the result is proved. Hence we suppose that $x_0 \prec fx_0$. Define a sequence $\{x_n\}$ by $x_n = f^n x_0 = fx_{n-1}$ for all $n \in \mathbb{N}$. Since f is non-decreasing and $x_0 \prec fx_0$, then

$$x_0 \prec x_1 \preceq x_2 \preceq \cdots, \tag{3.7}$$

and hence $\{x_n\}$ is a non-decreasing sequence. If $x_n = x_{n+1} = fx_n$ for some $n \in \mathbb{N}$, then the result is proved as x_n is a fixed point of f. In what follows we suppose that $0 < M(x_n, x_{n+1}, t) < 1$. Since f is an α-admissible mapping with respect to η and $\alpha(x_0, fx_0, t) = \alpha(x_0, x_1, t) \ge t$, we deduce that $\alpha(x_1, x_2, t) = \alpha(fx_0, fx_1, t) \ge t$. By continuing this process, we get $\alpha(x_n, x_{n+1}, t) \ge t$ for all $n \in \mathbb{N} \cup \{0\}$ and all $t > 0$. From (3.6) we get

$$\begin{aligned} &\big[t^2+1\big]^{M(fx_{n-1},fx_n,t)} \\ &\quad \ge \left[\frac{\alpha(x_{n-1},fx_{n-1},t)\alpha(x_n,fx_n,t)}{t^2}\right] \end{aligned}$$

$$\times \left[\alpha(x_{n-1},fx_{n-1},t)\alpha(x_n,fx_n,t)+1\right]^{\beta(M(x_{n-1},x_n,t))M(x_{n-1},x_n,t)}$$
$$\geq \left[\frac{t\cdot t}{t^2}\right][t\cdot t+1]^{\beta(M(x_{n-1},x_n,t))M(x_{n-1},x_n,t)}$$
$$=\left[t^2+1\right]^{\beta(M(x_{n-1},x_n,t))M(x_{n-1},x_n,t)}.$$

Thus

$$M(fx_{n-1},fx_n,t)\geq \beta\big(M(x_{n-1},x_n,t)\big)M(x_{n-1},x_n,t).$$

Hence,

$$M(x_n,x_{n+1},t)\geq \beta\big(M(x_{n-1},x_n,t)\big)M(x_{n-1},x_n,t)\geq M(x_{n-1},x_n,t). \tag{3.8}$$

That is, $\{S_n=M(x_n,x_{n+1},t)\}$ is an increasing sequence in $(0,1]$. Then there exists $l(t)\in(0,1]$ such that $\lim_{n\to+\infty}M(x_n,x_{n+1},t)=l(t)$ for all $t>0$. We shall prove that $l(t)=1$ for all $t>0$. By (3.8) we deduce

$$\frac{M(x_n,x_{n+1},t)}{M(x_{n-1},x_n,t)}\geq \beta\big(M(x_{n-1},x_n,t)\big)\geq 1,$$

which implies $\lim_{n\to+\infty}\beta(M(x_{n-1},x_n,t))=1$. Regarding the property of the function β, we conclude that

$$\lim_{n\to+\infty}M(x_n,x_{n+1},t)=1.$$

Next, we shall prove that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Proceeding as in the proof of Theorem 3.1, there exist $\epsilon\in(0,1)$ and $t_0>0$ such that for all $k\in\mathbb{N}$ there exist $n(k),m(k)\in\mathbb{N}$ with $m(k)>n(k)\geq k$ such that

$$\lim_{n\to+\infty}M(x_{m(k)},x_{n(k)},t_0)=1-\epsilon$$

and

$$\lim_{n\to+\infty}M(x_{m(k)+1},x_{n(k)+1},t_0)=1-\epsilon.$$

From (3.6) with $x=x_{m(k)}$ and $y=x_{n(k)}$ we deduce

$$\left[t^2+1\right]^{M(fx_{m(k)},fx_{n(k)},t_0)}$$
$$\geq \left[\frac{\alpha(x_{m(k)},fx_{m(k)},t)\alpha(x_{n(k)},fx_{n(k)},t)}{t^2}\right]$$
$$\times \left[\alpha(x_{m(k)},fx_{m(k)},t)\alpha(x_{n(k)},fx_{n(k)},t)+1\right]^{\beta(M(x_{m(k)},x_{n(k)},t))M(x_{m(k)},x_{n(k)},t_0)}$$
$$\geq \left[\frac{t\cdot t}{t^2}\right]\left[t^2+1\right]^{\beta(M(x_{m(k)},x_{n(k)},t))M(x_{m(k)},x_{n(k)},t_0)}$$
$$=\left[t^2+1\right]^{\beta(M(x_{m(k)},x_{n(k)},t))M(x_{m(k)},x_{n(k)},t_0)},$$

which implies

$$M(fx_{m(k)},fx_{n(k)},t_0) \geq \beta\big(M(x_{m(k)},x_{n(k)},t)\big)M(x_{m(k)},x_{n(k)},t_0),$$

and so

$$\frac{M(x_{m(k)+1},x_{n(k)+1},t_0)}{M(x_{m(k)},x_{m(k)},t_0)} \geq \beta\big(M(x_{m(k)},x_{m(k)},t_0)\big) \geq 1.$$

Taking the limit as $k \to +\infty$ in the above inequality, we get

$$\lim_{k\to+\infty} \beta\big(M(x_{m(k)},x_{n(k)},t_0)\big) = 1,$$

which implies

$$1-\epsilon = \lim_{k\to+\infty} M(x_{m(k)},x_{n(k)},t_0) = 1,$$

and so $\epsilon = 0$, which is a contradiction. Then $\{x_n\}$ is a Cauchy sequence. Since $(X,M,\star)$ is a complete space, then the sequence $\{x_n\}$ converges to some $z \in X$ such that for all $t > 0$,

$$\lim_{n\to+\infty} M(x_n,z,t) = 1.$$

By (3.6) and (ii) we get

$$\begin{aligned}
\big[t^2+1\big]^{M(fx_n,fz,t)} &\geq \left[\frac{\alpha(x_n,fx_n,t)\alpha(z,fz,t)}{t^2}\right]\big[\alpha(x_n,fx_n,t)\alpha(z,fz,t)+1\big]^{\beta(M(x_n,z,t))M(x_n,z,t)} \\
&\geq \left[\frac{t\cdot t}{t^2}\right]\big[t^2+1\big]^{M(x_n,z,t)} \\
&= \big[t^2+1\big]^{M(x_n,z,t)},
\end{aligned}$$

and hence

$$M(fx_n,fz,t) \geq M(x_n,z,t).$$

Taking the limit as $n \to +\infty$ in the above inequality, we have $\lim_{n\to+\infty} M(fx_n,fz,t) = 1$ for all $t > 0$ and then

$$M(z,fz,t) \geq \lim_{n\to+\infty} M(fx_n,z,t) \star \lim_{n\to\infty} M(fx_n,fz,t) = 1 \star 1 = 1,$$

that is, $z = fz$. □

Example 3.1 Let $(X,M,\star)$ be the non-Archimedean fuzzy metric space where $M(x,y,t) = \frac{\min\{x,y\}}{\max\{x,y\}}$ for all $t > 0$ and $a \star b = \min\{a,b\}$. Define $f : X \to X$ with

$$fx = \begin{cases} \frac{1}{3}x & \text{if } x \in [1,3], \\ \ln(x-3)+1 & \text{if } x \in (3,+\infty). \end{cases}$$

Also define $\alpha(x,y,t) = \left\{\begin{smallmatrix} t, & \text{if } x,y \in [1,3], \\ 0, & \text{otherwise} \end{smallmatrix}\right.$ and $\beta(t) = 1$. Also, $x \preceq y$ iff $x \leq y$.

Let, $x, y \in [1,3]$ and $x \leq y$. Then

$$\begin{aligned}[t^2+1]^{M(fx,fy,t)} &= 2^{(\frac{x}{y})} \geq 2^{(\frac{x}{y})} \\ &= \left[\frac{\alpha(x,fx,t)\alpha(y,fy,t)}{2t}\right]\left[\alpha(x,fx,t)\alpha(y,fy,t)+1\right]^{\beta(M(x,y,t))M(x,y,t)}.\end{aligned}$$

Otherwise, $\alpha(x,fx,t)\alpha(y,fy,t) = 0$, and so

$$[t^2+1]^{M(fx,fy,t)} \geq 0 = \left[\frac{\alpha(x,fx,t)\alpha(y,fy,t)}{2t}\right]\left[\alpha(x,fx,t)\alpha(y,fy,t)+1\right]^{\beta(M(x,y,t))M(x,y,t)}.$$

Clearly, $\alpha(1,f1,t) \geq t$ and $1 \leq f1$. Now, if $\{x_n\}$ is an increasing sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, then $\{x_n\} \subset [0,1]$ and hence $x \in [0,1]$. This implies that $x_n \to x$ for all $n \in \mathbb{N}$ and $\alpha(x,fx,t) \geq t$. Hence, all the conditions of Theorem 3.2 hold and f has a fixed point. Then, by Theorem 3.2, f has a fixed point.

If in Theorem 3.2 we take $\alpha(x,y,t) = t$ for all $x, y \in X$, then we deduce the following result.

Corollary 3.2 *Let $(X, M, \star, \preceq)$ be a complete non-Archimedean fuzzy metric space, and f be an increasing mapping on X. Assume that there exists a function $\beta : [0,1] \to [1,+\infty)$ such that for any sequence $\{t_n\} \subseteq [0,1]$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 1$ and*

$$M(fx,fy,t) \geq \beta\bigl(M(x,y,t)\bigr)M(x,y,t)$$

for all comparable $x, y \in X$ and all $t > 0$. Also suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;*

(ii) *if $\{x_n\}$ is an increasing sequence such that $x_n \to x$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then f has a fixed point.

4 Application to integral equations

Fixed point theorems for monotone operators in ordered metric spaces are widely investigated and have found various applications in differential and integral equations (see [27–31] and the references therein). Let $X = C([0,T], \mathbb{R})$ be the set of real continuous functions defined on $[0,T]$ and $M : X \times X \times (0,+\infty) \to [0,1]$ be defined by

$$M(x,y,r) = \frac{r}{r + \|x-y\|_\infty}$$

for all $x, y \in X$ and all $r > 0$. Also define $a \star b = \min\{a,b\}$. Then $(M, X, \star)$ is a complete triangular fuzzy metric space.

Consider the integral equation

$$x(t) = p(t) + \int_0^T S(t,s)f\bigl(s, x(s)\bigr)\,ds \tag{4.1}$$

and the mapping $F : X \to X$ defined by

$$Fx(t) = p(t) + \int_0^T S(t,s)f\bigl(s, x(s)\bigr)\,ds, \tag{4.2}$$

where

(A) $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous;

(B) $p : [0, T] \to \mathbb{R}$ is continuous;

(C) $S : [0, T] \times [0, T] \to [0, +\infty)$ is continuous;

(D) there exist $\theta : X \times X \to \mathbb{R}$ and $\lambda \in (0, 1)$ such that if $\theta(x, y) \geq 0$ for $x, y \in X$, then for all $s \in [0, T]$ and all $r > 0$ we have

$$\begin{aligned}\bigl|f\bigl(s, x(s)\bigr) - f\bigl(s, y(s)\bigr)\bigr| &\leq \lambda \max\bigl\{|x - y|, \bigl|x(s) - Fx(s)\bigr|, \bigl|y(s) - Fy(s)\bigr|\bigr\} \\ &\quad + r\bigl|Q^F(x, y, r) - \lambda\bigr| + \lambda r - r;\end{aligned}$$

(F)

$$\int_0^T S(t, s)\, ds \leq 1 \quad \text{for all } t \in [0, T];$$

(G) there exists $x_0 \in X$ such that $\theta(x_0, Fx_0) \geq 0$;

(H)

$$\theta(x, y) \geq 0 \quad \text{for some } x \in X \quad \text{implies} \quad \theta(Fx, Fy) \geq 0;$$

(I) if $\{x_n\}$ is a sequence in X such that $\theta(x_n, x_{n+1}) \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\theta(x_n, x) \geq 0$.

Theorem 4.1 *Under the assumptions* (A)-(I), *the integral equation* (4.1) *has a solution in* $X = C([0, T], \mathbb{R})$.

Proof Let $F : X \to X$ be defined by (4.2) and let $x, y \in X$ be such that $\theta(x, y) \geq 0$. By the condition (D), we deduce that

$$\begin{aligned}\bigl|Fx(t) - Fy(t)\bigr| &= \left|\int_0^T S(t, s)\bigl[f\bigl(s, x(s)\bigr) - f\bigl(s, y(s)\bigr)\bigr]\, ds\right| \\ &\leq \int_0^T S(t, s)\bigl|f\bigl(s, x(s)\bigr) - f\bigl(s, y(s)\bigr)\bigr|\, ds \\ &\leq \int_0^T S(t, s)\bigl[\lambda \max\bigl\{|x - y|, \bigl|x(s) - Fx(s)\bigr|, \bigl|y(s) - Fy(s)\bigr|\bigr\} \\ &\quad + r\bigl|Q^F(x, y, r) - \lambda\bigr| + \lambda r - r\bigr]\, ds \\ &\leq \int_0^T S(t, s)\bigl[\lambda \max\bigl\{\|x - y\|_\infty, \bigl\|x(s) - Fx(s)\bigr\|_\infty, \bigl\|y(s) - Fy(s)\bigr\|_\infty\bigr\} \\ &\quad + r\bigl|Q^F(x, y, r) - \lambda\bigr| + \lambda r - r\bigr]\, ds \\ &= \bigl[\lambda \max\bigl\{\|x - y\|_\infty, \bigl\|x(s) - Fx(s)\bigr\|_\infty, \bigl\|y(s) - Fy(s)\bigr\|_\infty\bigr\} \\ &\quad + r\bigl|Q^F(x, y, r) - \lambda\bigr| + \lambda r - r\bigr]\left(\int_0^T S(t, s)\, ds\right) \\ &\leq \lambda \max\bigl\{\|x - y\|_\infty, \bigl\|x(s) - Fx(s)\bigr\|_\infty, \bigl\|y(s) - Fy(s)\bigr\|_\infty\bigr\} \\ &\quad + r\bigl|Q^F(x, y, r) - \lambda\bigr| + \lambda r - r\end{aligned}$$

$$
\begin{aligned}
&= \lambda r \max\left\{\frac{1}{r}\|x-y\|_\infty, \frac{1}{r}\big\|x(s)-Fx(s)\big\|_\infty, \frac{1}{r}\big\|y(s)-Fy(s)\big\|_\infty\right\} \\
&\quad + r\big|Q^F(x,y,r)-\lambda\big| + \lambda r - r \\
&= \lambda r \max\left\{\frac{1}{r}\|x-y\|_\infty + 1, \frac{1}{r}\|x-Fx\|_\infty + 1, \frac{1}{r}\|y-Fy\|_\infty + 1\right\} \\
&\quad + r\big|Q^F(x,y,r)-\lambda\big| + \lambda r - r - \lambda r \\
&= \lambda r \max\left\{\frac{1}{M(x,y,r)}, \frac{1}{M(x,Fx,r)}, \frac{1}{M(y,Fy,r)}\right\} \\
&\quad + r\big|Q^F(x,y,r)-\lambda\big| - r \\
&\leq \lambda r P^F(x,y,r) + r\big|Q^F(x,y,r)-\lambda\big| - r,
\end{aligned}
$$

and so

$$
\|Fx-Fy\|_\infty \leq \lambda r P^f(x,y,r) + r\big|Q^f(x,y,r)-\lambda\big| - r.
$$

Now we define $\alpha : X \times (0,\infty) \to [0,+\infty)$ by

$$
\alpha(x,y,r) = \begin{cases} r, & \text{if } \theta(x,y) \geq 0, \\ 0, & \text{otherwise.} \end{cases}
$$

Since $\theta(x,y) \geq 0$, so, $\alpha(x,y,r) = r$. Therefore we can write

$$
\begin{aligned}
\frac{\alpha(x,y,r)}{rM(Fx,Fy,r)} &= \frac{r}{rM(Fx,Fy,r)} = \frac{1}{M(Fx,Fy,r)} = \frac{1}{r}\|Fx-Fy\|_\infty + 1 \\
&\leq \frac{1}{r}\big(\lambda r P^F(x,y,r) + r\big|Q^F(x,y,r)-\lambda\big| - r\big) + 1 \\
&= \lambda P^F(x,y,r) + \big|Q^F(x,y,r)-\lambda\big| - 1 + 1 \\
&= \lambda P^F(x,y,r) + \big|Q^F(x,y,r)-\lambda\big|.
\end{aligned}
$$

Thus all of the conditions of Corollary 2.2 are satisfied and hence the mapping F has a fixed point which is a solution of the integral equation (4.1) in $X = C([0,T],\mathbb{R})$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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