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On double Hausdorff summability method

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Abstract

Das (Proc. Camb. Philos. Soc. 67:321-326, 1970) proved that every conservative Hausdorff matrix is absolutely *k*th power conservative. Savaş and Rhoades (Anal. Math. 35:249-256, 2009) proved the result of Das for double Hausdorff summability. In this paper we will consider the double Endl-Jakimovski (E-J) generalization and we will prove the corresponding result of Savaş and Şevli (J. Comput. Anal. Appl. 11:702-710, 2009) for double E-J generalized Hausdorff matrices.

MSC: 40F05; 40G05

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Introduction and background

The basic theory of Hausdorff transformations for double sequences was developed by Adams [1] in 1933. Later a few authors studied double Hausdorff matrices; see *e.g.* Ramanujan [2] and Ustina [3].

Several generalizations of Hausdorff matrices have been made. One of them is the Endl-Jakimovski, or E-J generalization defined independently by Endl [4] and Jakimovski [5] as follows.

Let β be a real number, let (μ_n) be a real sequence, and let Δ be the forward difference operator defined by $\Delta\mu_k = \mu_k - \mu_{k+1}$, $\Delta^n(\mu_k) = \Delta(\Delta^{n-1}\mu_k)$. Then the infinite matrix $(H^{(\beta)}, \mu_n^{(\beta)}) = (H^\beta, \mu) = (h_{nk}^{(\beta)})$ is defined by

$$h_{nk}^{(\beta)} = \begin{cases} \binom{n+\beta}{n-k}\Delta^{n-k}\mu_k^{(\beta)}, & 0 \le k \le n, \\ 0, & k > n, \end{cases}$$

and the associated matrix method is called a generalized Hausdorff matrix and generalized Hausdorff method, respectively. The moment sequence $\mu_n^{(\beta)}$ is given by

$$\mu_n^{(\beta)} = \int_0^1 t^{n+\beta}\, d\chi(t),$$

where $\chi(t) \in BV[0,1]$. The case $\beta = 0$ corresponds to ordinary Hausdorff summability.

In a recent paper [6], the first author jointly with Savaş has extended the result of Das [7] to the E-J matrices; *i.e.*, all conservative E-J matrices are absolutely kth power conservative for $k \ge 1$. Thereafter, Savaş and Rhoades [8] proved the result of Das [7] for double Hausdorff summability. In this paper we will consider double E-J generalization and we will prove the corresponding result of [6] for double E-J generalized Hausdorff matrices.

Let $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}$ be an infinite double series with real or complex numbers, with partial sums

$$s_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}.$$

For any double sequence (u_{mn}) we shall define

$$\Delta_{11}u_{mn} = u_{mn} - u_{m+1,n} - u_{m,n+1} + u_{m+1,n+1}.$$

Denote by $\mathcal{A}_k^2$ the sequence space defined by

$$\mathcal{A}_k^2 = \left\{(s_{mn})_{m,n=0}^{\infty} : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|a_{mn}|^k < \infty; a_{mn} = \Delta_{11}s_{m-1,n-1}\right\}$$

for $k \geq 1$.

A four-dimensional matrix $T = (t_{mnij} : m, n, i, j = 0, 1, \ldots)$ is said to be absolutely kth power conservative for $k \geq 1$, if $T \in B(\mathcal{A}_k^2)$; *i.e.*, if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left|\Delta_{11}s_{m-1,n-1}\right|^k < \infty,$$

then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left|\Delta_{11}t_{m-1,n-1}\right|^k < \infty,$$

where

$$t_{mn} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} t_{mnij}s_{ij} \quad (m, n = 0, 1, \ldots),$$

see *e.g.* [9, 10] and the references contained therein.

A double Hausdorff matrix has entries

$$h_{mnij} = \binom{m}{i}\binom{n}{j}\Delta_1^{m-i}\Delta_2^{n-j}\mu_{ij},$$

where $\{\mu_{ij}\}$ is any real or complex sequence and

$$\Delta_1^{m-i}\Delta_2^{n-j}\mu_{ij} = \sum_{s=0}^{m-i}\sum_{t=0}^{n-j}(-1)^{i+j}\binom{m-i}{s}\binom{n-j}{t}\mu_{i+s,j+t}.$$

For double Hausdorff matrices, the necessary and sufficient condition for H to be conservative is the existence of a function $\chi(s,t) \in BV[0,1] \times [0,1]$ such that

$$\int_0^1\int_0^1 \left|d\chi(s,t)\right| < \infty,$$

and

$$\mu_{mn} = \int_0^1 \int_0^1 s^m t^n \, d\chi(s,t).$$

Quite recently, Savaş and Rhoades [8] extended the result of Das [7] to double Hausdorff summability. Their theorem is as follows.

Theorem 1 [8] *Let H be a conservative double Hausdorff matrix. Then $H \in B(\mathcal{A}_k^2)$.*

Our purpose is to achieve the result established in [7] for double E-J Hausdorff summability.

Main results

The matrix $\delta^{(\alpha,\beta)} = (\delta^{(\alpha,\beta)}_{mnij})$, whose elements are defined by

$$\delta^{(\alpha,\beta)}_{mnij} = \begin{cases} (-1)^{i+j}\binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}, & i \le m, j \le n, \\ 0, & \text{otherwise}, \end{cases}$$

is called a difference matrix, where α and β are real numbers.

Theorem 2 *The difference matrix $\delta^{(\alpha,\beta)} = (\delta^{(\alpha,\beta)}_{mnij})$ is its own inverse.*

Proof Let

$$a_{mnkl} = \sum_{i=0}^{m} \sum_{j=0}^{n} \delta^{(\alpha,\beta)}_{mnij} \delta^{(\alpha,\beta)}_{ijkl},$$

thus $A = \delta^{(\alpha,\beta)}\delta^{(\alpha,\beta)}$. For any double sequence (u_{rs})

$$\begin{aligned}
&\sum_{r=0}^{m} \sum_{s=0}^{n} a_{mnrs} u_{rs} \\
&\quad = \sum_{r=0}^{m} \sum_{s=0}^{n} \sum_{i=0}^{m} \sum_{j=0}^{n} \delta^{(\alpha,\beta)}_{mnij} \delta^{(\alpha,\beta)}_{ijrs} u_{rs} \\
&\quad = \sum_{r=0}^{m} \sum_{s=0}^{n} (-1)^{r+s} u_{rs} \sum_{i=r}^{m} \sum_{j=s}^{n} (-1)^{i+j} \binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}\binom{i+\alpha}{i-r}\binom{j+\beta}{j-s} \\
&\quad = \sum_{r=0}^{m} \sum_{s=0}^{n} (-1)^{r+s} u_{rs} \binom{m+\alpha}{m-r}\binom{n+\beta}{n-s} \sum_{i=r}^{m} \sum_{j=s}^{n} (-1)^{i+j} \binom{m-r}{m-i}\binom{n-s}{n-j} \\
&\quad = u_{rs},
\end{aligned}$$

since

$$\sum_{i=r}^{m} \sum_{j=s}^{n} (-1)^{i+j} \binom{m-r}{m-i}\binom{n-s}{n-j} = \begin{cases} (-1)^{r+s}, & m = r, n = s, \\ 0, & \text{otherwise}. \end{cases}$$

□

Let $(\mu_{mn}^{(\alpha,\beta)})$ be a given sequence and $\mu^{(\alpha,\beta)} = (\mu_{mnij}^{(\alpha,\beta)})$ be a diagonal matrix whose only non-zero entries are $\mu_{mn}^{(\alpha,\beta)} = \mu_{mnmn}^{(\alpha,\beta)}$. The transformation matrix

$$H^{(\alpha,\beta)} = \delta^{(\alpha,\beta)}\mu^{(\alpha,\beta)}\delta^{(\alpha,\beta)}$$

is called a double E-J generalized Hausdorff matrix corresponding to the sequence $(\mu_{mn}^{(\alpha,\beta)})$.

Theorem 3 *A matrix $H^{(\alpha,\beta)} = (h_{mnij}^{(\alpha,\beta)})$ is a double E-J generalized Hausdorff matrix corresponding to the sequence $(\mu_{mn}^{(\alpha,\beta)})$ if and only if its elements have the form*

$$h_{mnij}^{(\alpha,\beta)} = \binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}\Delta_1^{m-i}\Delta_2^{n-j}\mu_{ij}^{(\alpha,\beta)},$$

where

$$\Delta_1^{m-i}\Delta_2^{n-j}\mu_{ij}^{(\alpha,\beta)} := \sum_{r=0}^{m-i}\sum_{s=0}^{n-j}(-1)^{r+s}\binom{m-i}{r}\binom{n-j}{s}\mu_{i+r,j+s}^{(\alpha,\beta)}.$$

Proof Let $H^{(\alpha,\beta)} = \delta^{(\alpha,\beta)}\mu^{(\alpha,\beta)}\delta^{(\alpha,\beta)}$ be a double E-J Hausdorff matrix. Applying this to a double sequence (s_{mn}) we have

$$\begin{aligned}
t_{mn} &= \sum_{i=0}^{m}\sum_{j=0}^{n} h_{mnij}^{(\alpha,\beta)} s_{ij} \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{r=0}^{m}\sum_{s=0}^{n} \delta_{mnrs}^{(\alpha,\beta)}\mu_{rs}^{(\alpha,\beta)}\delta_{rsij}^{(\alpha,\beta)} s_{ij} \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}\binom{m+\alpha}{m-r}\binom{n+\beta}{n-s}\mu_{rs}^{(\alpha,\beta)}(-1)^{i+j}\binom{r+\alpha}{r-i}\binom{s+\beta}{s-j}s_{ij} \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}\sum_{r=i}^{m}\sum_{s=j}^{n}(-1)^{r+s}\binom{m-i}{m-r}\binom{n-j}{n-s}\mu_{rs}^{(\alpha,\beta)}s_{ij} \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}\sum_{r=0}^{m-i}\sum_{s=0}^{n-j}(-1)^{r+s}\binom{m-i}{r}\binom{n-j}{s}\mu_{i+r,j+s}^{(\alpha,\beta)}s_{ij}.
\end{aligned}$$

Hence

$$h_{mnij}^{(\alpha,\beta)} = \binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}\sum_{r=0}^{m-i}\sum_{s=0}^{n-j}(-1)^{r+s}\binom{m-i}{r}\binom{n-j}{s}\mu_{i+r,j+s}^{(\alpha,\beta)}. \qquad \square$$

For double E-J Hausdorff matrices, the necessary and sufficient condition for $H^{(\alpha,\beta)}$ to be conservative is the existence of a function $\chi(s,t) \in BV[0,1]\times[0,1]$ such that

$$\int_0^1\int_0^1 \left|d\chi(s,t)\right| < \infty,$$

and

$$\mu_{mn}^{(\alpha,\beta)} = \int_0^1\int_0^1 s^{m+\alpha}t^{n+\beta}\,d\chi(s,t).$$

Theorem 4 *Given a function $\chi(s,t) \in BV[0,1] \times [0,1]$, a bounded variation in the unit square, the corresponding double E-J Hausdorff transformation (t_{mn}), of a sequence (s_{mn}), may be defined by*

$$t_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}s_{ij}\int_0^1\int_0^1 s^{i+\alpha}(1-s)^{m-i}t^{j+\beta}(1-t)^{n-j}\,d\chi(s,t).$$

Proof For $i \le m$ and $j \le n$,

$$\begin{aligned}
h_{mnij}^{(\alpha,\beta)} &= \sum_{k=i}^{m}\sum_{l=j}^{n}\delta_{mnkl}^{(\alpha,\beta)}\mu_{kl}^{(\alpha,\beta)}\delta_{klij}^{(\alpha,\beta)} \\
&= \sum_{k=i}^{m}\sum_{l=j}^{n}\delta_{mnkl}^{(\alpha,\beta)}\int_0^1\int_0^1 s^{k+\alpha}t^{l+\beta}\,d\chi(s,t)\cdot\delta_{klij}^{(\alpha,\beta)} \\
&= \int_0^1\int_0^1\sum_{k=i}^{m}\sum_{l=j}^{n}(-1)^{k+l}\binom{m+\alpha}{m-k}\binom{n+\beta}{n-l}(-1)^{i+j}\binom{k+\alpha}{k-i}\binom{l+\beta}{l-j}s^{k+\alpha}t^{l+\beta}\,d\chi(s,t) \\
&= \int_0^1\int_0^1\sum_{k=i}^{m}\sum_{l=j}^{n}(-1)^{k+l+i+j}\binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}\binom{m-i}{m-k}\binom{n-j}{n-l}s^{k+\alpha}t^{l+\beta}\,d\chi(s,t) \\
&= \int_0^1\int_0^1\binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}\sum_{k=0}^{m-i}\sum_{l=0}^{n-j}(-1)^{k+l}\binom{m-i}{k}\binom{n-j}{l}s^{k+i+\alpha}t^{l+j+\beta}\,d\chi(s,t) \\
&= \int_0^1\int_0^1\binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}s^{i+\alpha}t^{j+\beta}\left(\sum_{k=0}^{m-i}\sum_{l=0}^{n-j}(-1)^{k+l}\binom{m-i}{k}\binom{n-j}{l}s^k t^l\right)d\chi(s,t) \\
&= \int_0^1\int_0^1\binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}s^{i+\alpha}t^{j+\beta}(1-s)^{m-i}(1-t)^{n-j}\,d\chi(s,t).
\end{aligned}$$

□

Theorem 5 *Let $H^{(\alpha,\beta)}$ be a conservative double E-J Hausdorff matrix. Then $H^{(\alpha,\beta)} \in B(\mathcal{A}_k^2)$, $\alpha,\beta \ge 0$.*

As tools to prove our result, we need to the following lemmas.

Lemma 1 [6] *Let $k \ge 1$, $n \ge \nu$ and $\alpha \ge 0$. Then*

$$E_{m+\alpha}^{k-1}E_{m-\mu}^{\mu+\alpha-1} \le E_{\mu+\alpha}^{k-1}E_{m-\mu}^{\mu+\alpha+k-2}.$$

The following lemma is a double version of [11].

Lemma 2 *For $0 \le s \le 1$, $0 \le t \le 1$, $\alpha \ge 0$ and $\beta \ge 0$*

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m+\alpha}{i}\binom{n+\beta}{j}(1-s)^m(1-t)^n s^{m+\alpha-i}t^{n+\beta-j} \le 1.$$

Proof of Theorem 5 Let (t_{mn}) be the double E-J transform of a double sequence (s_{mn}); *i.e.*,

$$t_{mn} = \sum_{\mu=0}^{m}\sum_{\nu=0}^{n}h_{mn\mu\nu}^{(\alpha,\beta)}s_{\mu\nu}.$$

We will demonstrate that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|a_{mn}|^k<\infty \quad \Rightarrow \quad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|\Delta_{11}t_{m-1,n-1}|^k<\infty. \tag{1}$$

Write

$$t_{mn}=\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}b_{\mu\nu}.$$

Then $b_{mn}=\Delta_{11}t_{m-1,n-1}$. For $k\geq 1$

$$E_m^{k-1}=\binom{m+k-1}{m}=\binom{m+k-1}{k-1}=\frac{(m+k-1)!}{m!(k-1)!}=\frac{\Gamma(m+k)}{\Gamma(m+1)\Gamma(k)}.$$

Then

$$E_m^{k-1}\approx\frac{m^{k-1}}{\Gamma(k)}\approx m^{k-1},$$
$$m^{k-1}\approx E_m^{k-1}\approx E_{m+\alpha}^{k-1}.$$

Due to this (1) is equivalent to

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}E_{m+\alpha}^{k-1}E_{n+\beta}^{k-1}|a_{mn}|^k<\infty \quad \Rightarrow \quad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}E_{m+\alpha}^{k-1}E_{n+\beta}^{k-1}|b_{mn}|^k<\infty. \tag{2}$$

For $s\in[0,1]$ and $t\in[0,1]$ define

$$\phi_{mn}(s,t)=\sum_{\mu=1}^{m}\sum_{\nu=1}^{n}E_{m-\mu}^{\mu+\alpha-1}E_{n-\nu}^{\nu+\beta-1}s^{\mu+\alpha}t^{\nu+\beta}(1-s)^{m-\mu}(1-t)^{n-\nu}a_{\mu\nu}. \tag{3}$$

It follows from the Hölder inequality that

$$\begin{aligned}\left|\phi_{mn}(s,t)\right|^k&=\left|\sum_{\mu=1}^{m}\sum_{\nu=1}^{n}E_{m-\mu}^{\mu+\alpha-1}E_{n-\nu}^{\nu+\beta-1}s^{\mu+\alpha}t^{\nu+\beta}(1-s)^{m-\mu}(1-t)^{n-\nu}a_{\mu\nu}\right|^k\\&\leq\sum_{\mu=1}^{m}\sum_{\nu=1}^{n}E_{m-\mu}^{\mu+\alpha-1}E_{n-\nu}^{\nu+\beta-1}s^{\mu+\alpha}t^{\nu+\beta}(1-s)^{m-\mu}(1-t)^{n-\nu}|a_{\mu\nu}|^k\\&\quad\times\left\{\sum_{\mu=1}^{m}\sum_{\nu=1}^{n}E_{m-\mu}^{\mu+\alpha-1}E_{n-\nu}^{\nu+\beta-1}s^{\mu+\alpha}t^{\nu+\beta}(1-s)^{m-\mu}(1-t)^{n-\nu}\right\}^{k-1}.\end{aligned}$$

From Lemma 2

$$\begin{aligned}&\sum_{\mu=1}^{m}\sum_{\nu=1}^{n}E_{m-\mu}^{\mu+\alpha-1}E_{n-\nu}^{\nu+\beta-1}s^{\mu+\alpha}t^{\nu+\beta}(1-s)^{m-\mu}(1-t)^{n-\nu}\\&\quad=\sum_{\mu=1}^{m}\sum_{\nu=1}^{n}\binom{m+\alpha-1}{m-\mu}\binom{n+\beta-1}{n-\nu}s^{\mu+\alpha}t^{\nu+\beta}(1-s)^{m-\mu}(1-t)^{n-\nu}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1}\binom{m+\alpha-1}{m-\mu-1}\binom{n+\beta-1}{n-\nu-1}s^{\mu+\alpha+1}t^{\nu+\beta+1}(1-s)^{m-\mu-1}(1-t)^{n-\nu-1} \\
&= st\sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1}\binom{m+\alpha-1}{m-\mu-1}\binom{n+\beta-1}{n-\nu-1}s^{\mu+\alpha}t^{\nu+\beta}(1-s)^{m-\mu-1}(1-t)^{n-\nu-1} \\
&= O(st).
\end{aligned}
$$

Hence

$$
\left|\phi_{mn}(s,t)\right|^k = O(1)(st)^{k-1}\sum_{\mu=1}^{m}\sum_{\nu=1}^{n}E_{m-\mu}^{\mu+\alpha-1}E_{n-\nu}^{\nu+\beta-1}s^{\mu+\alpha}t^{\nu+\beta}(1-s)^{m-\mu}(1-t)^{n-\nu}|a_{\mu\nu}|^k
$$

and from Lemma 1

$$
\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}E_{m+\alpha}^{k-1}E_{n+\beta}^{k-1}\left|\phi_{mn}(s,t)\right|^k \\
&\quad= O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}E_{m+\alpha}^{k-1}E_{n+\beta}^{k-1}(st)^{k-1} \\
&\qquad\times\sum_{\mu=1}^{m}\sum_{\nu=1}^{n}E_{m-\mu}^{\mu+\alpha-1}E_{n-\nu}^{\nu+\beta-1}s^{\mu+\alpha}t^{\nu+\beta}(1-s)^{m-\mu}(1-t)^{n-\nu}|a_{\mu\nu}|^k \\
&\quad= O(1)(st)^{k-1}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}s^{\mu+\alpha}t^{\nu+\beta}|a_{\mu\nu}|^k \\
&\qquad\times\sum_{m=\mu}^{\infty}\sum_{n=\nu}^{\infty}E_{m+\alpha}^{k-1}E_{n+\beta}^{k-1}E_{m-\mu}^{\mu+\alpha-1}E_{n-\nu}^{\nu+\beta-1}(1-s)^{m-\mu}(1-t)^{n-\nu} \\
&\quad= O(1)(st)^{k-1}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}s^{\mu+\alpha}t^{\nu+\beta}|a_{\mu\nu}|^kE_{\mu+\alpha}^{k-1}E_{\nu+\beta}^{k-1} \\
&\qquad\times\sum_{m=\mu}^{\infty}\sum_{n=\nu}^{\infty}E_{m-\mu}^{\mu+\alpha+k-2}E_{n-\nu}^{\nu+\beta+k-2}(1-s)^{m-\mu}(1-t)^{n-\nu} \\
&\quad= O(1)(st)^{k-1}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}s^{\mu+\alpha}t^{\nu+\beta}|a_{\mu\nu}|^kE_{\mu+\alpha}^{k-1}E_{\nu+\beta}^{k-1}s^{-\mu-\alpha-k+1}t^{-\nu-\beta-k+1} \\
&\quad= O(1)\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}E_{\mu+\alpha}^{k-1}E_{\nu+\beta}^{k-1}|a_{\mu\nu}|^k.
\end{aligned}
$$

From Lemma of [8], if (t_{mn}) and (τ_{mn}) are the (H,μ_{mn}) transformation of (s_{mn}) and (mna_{mn}), respectively, then

$$
\tau_{mn} = mn\Delta_{11}t_{m-1,n-1}.
$$

A similar consequence can be proved for $(H^{(\alpha,\beta)},\mu^{(\alpha,\beta)})$, see [6]; *i.e.*,

$$
\tau_{mn} = (m+\alpha)(n+\beta)\Delta_{11}t_{m-1,n-1}.
$$

Hence

$$\begin{aligned}
b_{mn} &= \frac{1}{(m+\alpha)(n+\beta)}\tau_{mn} \\
&= \frac{1}{(m+\alpha)(n+\beta)}\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}\Delta_1^{m-i}\Delta_2^{n-j}\mu_{ij}^{(\alpha,\beta)}(i+\alpha)(j+\beta)a_{ij} \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m+\alpha-1}{m-i}\binom{n+\beta-1}{n-j}\Delta_1^{m-i}\Delta_2^{n-j}\mu_{ij}^{(\alpha,\beta)}a_{ij} \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n}E_{m-i}^{i+\alpha-1}E_{n-j}^{j+\beta-1}\Delta_1^{m-i}\Delta_2^{n-j}\mu_{ij}^{(\alpha,\beta)}a_{ij}.
\end{aligned}$$

Since $H^{(\alpha,\beta)}$ is conservative, $\mu_n^{(\alpha,\beta)}$ is a moment sequence,

$$\mu_{mn}^{(\alpha,\beta)} = \int_0^1\int_0^1 s^{m+\alpha}t^{n+\beta}\,d\chi(s,t),$$

and

$$\Delta_1^{m-i}\Delta_2^{n-j}\mu_{ij}^{(\alpha,\beta)} = \int_0^1\int_0^1 s^{i+\alpha}(1-s)^{m-i}t^{j+\beta}(1-t)^{n-j}\,d\chi(s,t)$$

from Theorem 4. In view of (3) we can deduce that

$$\begin{aligned}
b_{mn} &= \sum_{i=0}^{m}\sum_{j=0}^{n}E_{m-i}^{i+\alpha-1}E_{n-j}^{j+\beta-1}\int_0^1\int_0^1 s^{i+\alpha}(1-s)^{m-i}t^{j+\beta}(1-t)^{n-j}\,d\chi(s,t)a_{ij} \\
&= \int_0^1\int_0^1\left(\sum_{i=0}^{m}\sum_{j=0}^{n}E_{m-i}^{i+\alpha-1}E_{n-j}^{j+\beta-1}s^{i+\alpha}t^{j+\beta}(1-s)^{m-i}(1-t)^{n-j}a_{ij}\right)d\chi(s,t) \\
&= \int_0^1\int_0^1 \phi_{mn}(s,t)\,d\chi(s,t).
\end{aligned}$$

Using Minkowski's inequality we get

$$\begin{aligned}
\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}E_{m+\alpha}^{k-1}E_{n+\beta}^{k-1}|b_{mn}|^k\right\}^{1/k} &= \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}E_{m+\alpha}^{k-1}E_{n+\beta}^{k-1}\left|\int_0^1\int_0^1\phi_{mn}(s,t)\,d\chi(s,t)\right|^k\right\}^{1/k} \\
&\le \int_0^1\int_0^1\left|d\chi(s,t)\right|\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}E_{m+\alpha}^{k-1}E_{n+\beta}^{k-1}\left|\phi_{mn}(s,t)\right|^k\right\}^{1/k} \\
&= O(1)\int_0^1\int_0^1\left|d\chi(s,t)\right|\left\{\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}E_{\mu+\alpha}^{k-1}E_{\nu+\beta}^{k-1}|a_{\mu\nu}|^k\right\}^{1/k}.
\end{aligned}$$

Therefore the proof of Theorem 5 is complete. □

Specially, if we take $\alpha = 0$ and $\beta = 0$ in Theorem 5, we get Theorem 1 as a corollary.

The following is an example of a double E-J Hausdorff matrix.

A doubly infinite Cesàro matrix (C,γ,δ) is a doubly infinite Hausdorff matrix with entries

$$h_{mnij}=\frac{\binom{m+\gamma-i-1}{n-i}\binom{n+\delta-j-1}{n-j}}{\binom{m+\gamma}{\gamma}\binom{n+\delta}{\delta}},\quad \gamma,\delta\geq 0.$$

We use the following to denote the corresponding E-J generalizations of the (C,γ,δ). $(C^{(\alpha,\beta)},\gamma,\delta)$ has moment sequence

$$\mu_{mn}^{(\alpha,\beta)}=\int_0^1\int_0^1 u^{m+\alpha}v^{n+\beta}\gamma\delta(1-u)^{\gamma-1}(1-v)^{\delta-1}\,du\,dv,$$

where

$$\chi(u,v)=\gamma\delta\int_0^u\int_0^v(1-s)^{\gamma-1}(1-t)^{\delta-1}\,ds\,dt.$$

For $i\leq m$ and $j\leq n$,

$$\begin{aligned}
h_{mnij}^{(\alpha,\beta)}&=\int_0^1\int_0^1\binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}u^{i+\alpha}v^{j+\beta}(1-u)^{m-i}(1-v)^{n-j}\,d\chi(u,v)\\
&=\int_0^1\int_0^1\binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}\\
&\quad\times u^{i+\alpha}v^{j+\beta}(1-u)^{m-i}(1-v)^{n-j}\gamma\delta(1-u)^{\gamma-1}(1-v)^{\delta-1}\,du\,dv\\
&=\gamma\delta\binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}\int_0^1\int_0^1 u^{i+\alpha}(1-u)^{m-i+\gamma-1}v^{j+\beta}(1-v)^{n-j+\delta-1}\,du\,dv\\
&=\gamma\delta\binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}B(i+\alpha+1,m-i+\gamma)B(j+\beta+1,n-j+\delta)\\
&=\frac{\gamma\Gamma(m+\alpha+1)\Gamma(m-i+\gamma)}{\Gamma(m-i+1)\Gamma(m+\alpha+\gamma+1)}\frac{\delta\Gamma(n+\beta+1)\Gamma(n-j+\delta)}{\Gamma(n-j+1)\Gamma(n+\beta+\delta+1)}\\
&=\frac{E_{m-i}^{\gamma-1}E_{n-j}^{\delta-1}}{E_{m+\alpha}^{\gamma}E_{n+\beta}^{\delta}}.
\end{aligned}$$

For the special case $\gamma,\delta=1$,

$$\left(C^{(\alpha,\beta)},1,1\right)=\begin{cases}\frac{1}{(m+\alpha+1)(n+\beta+1)}, & i\leq m\text{ and }j\leq n,\\ 0, & \text{otherwise}\end{cases}$$

is a double E-J Hausdorff matrix.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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References

1. Adams, CR: Hausdorff transformations for double sequences. Bull. Am. Math. Soc. **39**, 303-312 (1933)
2. Ramanujan, MS: On Hausdorff transformations for double sequences. Proc. Indian Acad. Sci. Sect. A. **42**, 131-135 (1955)
3. Ustina, F: The Hausdorff means for double sequences. Can. Math. Bull. **10**, 347-352 (1967)
4. Endl, K: Untersuchungen über momentenprobleme bei verfahren vom Hausdorffschen typus. Math. Ann. **139**, 403-432 (1960)
5. Jakimovski, A: The product of summability methods; part 2. Technical Report 8, Jerusalem (1959)
6. Savaş, E, Şevli, H: Generalized Hausdorff matrices as bounded operators over $\mathcal{A}_k$. J. Comput. Anal. Appl. **11**, 702-710 (2009)
7. Das, G: A Tauberian theorem for absolute summability. Proc. Camb. Philos. Soc. **67**, 321-326 (1970)
8. Savaş, E, Rhoades, BE: Every conservative double Hausdorff matrix is a k-th absolutely summable operator. Anal. Math. **35**, 249-256 (2009)
9. Savaş, E, Şevli, H: On absolute summability for double triangle matrices. Math. Slovaca **60**, 495-506 (2010)
10. Şevli, H, Savaş, E: Some further extensions of absolute Cesàro summability for double series. J. Inequal. Appl. **2013**, 144 (2013)
11. Jakimovski, A, Ramanujan, MS: A uniform approximation theorem and its application to moment problems. Math. Z. **84**, 143-153 (1964)

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