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# On a certain new subclass of meromorphic close-to-convex functions

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**Abstract**
In this paper, we introduce and investigate a new subclass $MK^{(k)}(\beta,\gamma)$ of meromorphic close-to-convex functions. For functions belonging to the class $MK^{(k)}(\beta,\gamma)$, we obtain some coefficient inequalities and a distortion theorem. The results presented here would unify and extend some recent work of Wang *et al.* (Appl. Math. Lett. 25:454-460, 2012).
**MSC:** 30C45

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## 1 Introduction

Let $\Sigma$ be the class of functions $f$ of the form:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the punctured open unit disk $U^* = \{z \in C : 0 < |z| < 1\} = U \setminus \{0\}$.

Let $P$ denote the class of functions $p$ given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (z \in U), \tag{1.2}$$

which are analytic and convex in $U$ and satisfy the condition $\Re(p(z)) > 0$ $(z \in U)$.

A function $f \in \Sigma$ is said to be in the class $MS^*(\alpha)$ of meromorphic starlike functions of order $\alpha$ if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \quad \left(z \in U^*; 0 \le \alpha < 1\right).$$

In addition, a function $f \in \Sigma$ is said to be in the class $MC$ of meromorphic close-to-convex functions if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{g(z)}\right) < 0 \quad \left(z \in U^*; g \in MS^*(0) = MS^*\right).$$

Recently, Srivastava *et al.* [1] (see also [2, 3]) introduced and studied the class $MS_s^*$ of meromorphic starlike functions with respect to symmetric points, which satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)-f(-z)}\right)<0 \quad \left(z\in U^*\right).$$

More recently, Wang *et al.* [4] discussed a class $MK$ of meromorphic close-to-convex functions, that is, a function $f\in\Sigma$ is said to be in the class $MK$ if it satisfies the inequality

$$\Re\left(\frac{f'(z)}{g(z)g(-z)}\right)>0 \quad \left(z\in U^*\right),$$

where $g\in MS^*(\frac{1}{2})$.

Let $f(z)=z+a_2z^2+\cdots$ be analytic in $U$. If there exists a function $g\in S^*(\frac{1}{2})$, such that

$$\left|\frac{z^2f'(z)}{g(z)g(-z)}+1\right|<\left|\frac{z^2f'(z)}{g(z)g(-z)}-1+2\gamma\right| \quad (z\in U),$$

then we say that $f\in K_s(\gamma)$, $0\le\gamma<1$, where $S^*(\frac{1}{2})$ denotes the usual class of starlike functions of order 1/2. The function class $K_s(\gamma)$ was introduced and studied recently by Kowalczyk and Les-Bomba [5] (see also [6–9]).

For two functions $f$ and $g$ analytic in $U$, we say that the function $f(z)$ is subordinate to $g(z)$ in $U$, and we write $f(z)\prec g(z)$ $(z\in U)$ if there exists a Schwarz function $w(z)$, analytic in $U$ with $w(0)=0$ and $|w(z)|\le 1$, such that $f(z)=g(w(z))$ $(z\in U)$. In particular, if the function $g$ is univalent in $U$, then we have $f(0)=g(0)$ and $f(U)\subset g(U)$ (see, for example, [10]).

Motivated essentially by the above mentioned function classes $MK$ and $K_s(\gamma)$, we now introduce a new class $MK^{(k)}(\beta,\gamma)$ of meromorphic functions.

**Definition 1** Let $MK^{(k)}(\beta,\gamma)$ denote the class of functions in $\Sigma$ satisfying the inequality

$$\left|\frac{z^{2-k}f'(z)}{g_k(z)}+1\right|<\beta\left|\frac{z^{2-k}f'(z)}{g_k(z)}+2\gamma-1\right| \quad \left(z\in U^*;0<\beta\le 1;0\le\gamma<1\right), \tag{1.3}$$

where $g\in MS^*(\frac{k-1}{k})$, $k\ge 1$, is a fixed positive integer and $g_k(z)$ is defined by the following equality:

$$g_k(z)=\prod_{\nu=0}^{k-1}\varepsilon^{-\nu}g\left(\varepsilon^{\nu}z\right) \quad \left(\varepsilon^k=1\right). \tag{1.4}$$

We note that $MK^{(2)}(1,0)=MK$ (see [4]), so the class $MK^{(k)}(\beta,\gamma)$ is a generation of the class $MK$.

In this paper, we prove that the class $MK^{(k)}(\beta,\gamma)$ is a subclass of meromorphic close-to-convex functions. Moreover, we provide some coefficient inequalities and a distortion theorem for functions in the class $MK^{(k)}(\beta,\gamma)$. Our results unify and extend the corresponding results obtained by Wang *et al.* [4].

## 2 Main results

First of all, we give two meaningful conclusions about the class $MK^{(k)}(\beta,\gamma)$. The proof of Theorem 1 below is much akin to that of Theorem 1 in [11], so we choose to omit the details involved.

**Theorem 1** *A function $f \in MK^{(k)}(\beta,\gamma)$ if and only if there exists $g \in MS^*(\frac{k-1}{k})$ such that*

$$-\frac{z^{2-k}f'(z)}{g_k(z)} \prec \frac{1+(1-2\gamma)\beta z}{1-\beta z} \quad \bigl(z \in U^*\bigr). \tag{2.1}$$

**Remark 1** From Theorem 1, we know that

$$\Re\left(\frac{z^{2-k}f'(z)}{g_k(z)}\right) < 0 \quad \bigl(z \in U^*\bigr), \tag{2.2}$$

because of $\Re(\frac{1+(1-2\gamma)\beta z}{1-\beta z}) > 0$ $(z \in U^*)$.

**Lemma 1** *Let $\varphi_i \in MS^*(\alpha_i)$, where $0 \le \alpha_i < 1$ $(i = 0,1,\ldots,k-1)$. Then for $k-1 \le \sum_{i=0}^{k-1}\alpha_i < k$, we have*

$$z^{k-1}\prod_{i=0}^{k-1}\varphi_i(z) \in MS^*\left(\sum_{i=0}^{k-1}\alpha_i - (k-1)\right).$$

*Proof* Since $\varphi_i \in MS^*(\alpha_i)$ $(i = 0,1,\ldots,k-1)$, by the definition of meromorphic starlike functions, we have

$$\Re\left(\frac{z\varphi_0'(z)}{\varphi_0(z)}\right) < -\alpha_0, \qquad \Re\left(\frac{z\varphi_1'(z)}{\varphi_1(z)}\right) < -\alpha_1, \qquad \ldots, \qquad \Re\left(\frac{z\varphi_{k-1}'(z)}{\varphi_{k-1}(z)}\right) < -\alpha_{k-1}. \tag{2.3}$$

We now let

$$F(z) = z^{k-1}\varphi_0(z)\varphi_1(z)\cdots\varphi_{k-1}(z). \tag{2.4}$$

Differentiating (2.4) with respect to $z$ logarithmically, we easily get

$$\frac{zF'(z)}{F(z)} = \frac{z\varphi_0'(z)}{\varphi_0(z)} + \frac{z\varphi_1'(z)}{\varphi_1(z)} + \cdots + \frac{z\varphi_{k-1}'(z)}{\varphi_{k-1}(z)} + (k-1). \tag{2.5}$$

From (2.5) together with (2.3), we obtain

$$\begin{aligned}\Re\left(\frac{zF'(z)}{F(z)}\right) &= \Re\left(\frac{z\varphi_0'(z)}{\varphi_0(z)}\right) + \Re\left(\frac{z\varphi_1'(z)}{\varphi_1(z)}\right) + \cdots + \Re\left(\frac{z\varphi_{k-1}'(z)}{\varphi_{k-1}(z)}\right) + (k-1)\\ &< -\sum_{i=0}^{k-1}\alpha_i + (k-1) = -\left(\sum_{i=0}^{k-1}\alpha_i - (k-1)\right)\end{aligned}$$

by noting that $0 \le \sum_{i=0}^{k-1}\alpha_i - (k-1) < 1$, which implies that

$$F(z) = z^{k-1}\prod_{i=0}^{k-1}\varphi_i(z) \in MS^*\left(\sum_{i=0}^{k-1}\alpha_i - (k-1)\right).$$

The proof of Lemma 1 is thus completed. □

**Theorem 2** *Let* $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \in MS^*(\frac{k-1}{k})$, *then* $z^{k-1} g_k(z) \in MS^*$.

*Proof* From (1.4), we know

$$\begin{aligned} z^{k-1} g_k(z) &= z^{k-1} \prod_{\nu=0}^{k-1} \varepsilon^{-\nu} g(\varepsilon^{\nu} z) = z^{k-1} \prod_{\nu=0}^{k-1} \varepsilon^{-\nu} \left[ \frac{1}{\varepsilon^{\nu} z} + \sum_{n=1}^{\infty} b_n (\varepsilon^{\nu} z)^n \right] \\ &= z^{k-1} \prod_{\nu=0}^{k-1} \left[ \frac{1}{\varepsilon^{2\nu} z} + \sum_{n=1}^{\infty} b_n \varepsilon^{(n-1)\nu} z^n \right]. \end{aligned} \tag{2.6}$$

Since $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \in MS^*(\frac{k-1}{k})$, by Lemma 1 and (2.6), we can easily get the assertion of Theorem 2. □

**Remark 2** From Theorem 2 and the inequality (2.2), we see that if $f \in MK^{(k)}(\beta, \gamma)$, then $f(z)$ is a meromorphic close-to-convex function. So, $MK^{(k)}(\beta, \gamma)$ is a subclass of the class $MC$ of meromorphic close-to-convex functions.

Next, we give some coefficient inequalities for functions belonging to the class $MK^{(k)}(\beta, \gamma)$.

**Theorem 3** *Let* $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ *and* $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ *be analytic in* $U^*$. *If for* $0 < \beta \le 1$ *and* $0 \le \gamma < 1$, *we have*

$$\sum_{n=1}^{\infty} n(1+\beta)|a_n| + \sum_{n=1}^{\infty} \bigl(\beta|1-2\gamma| + 1\bigr)|B_n| \le 2\beta(1-\gamma), \tag{2.7}$$

*where the coefficients* $B_n$ $(n = 1, 2, \ldots)$ *are given by* (2.9), *then* $f \in MK^{(k)}(\beta, \gamma)$.

*Proof* Suppose that

$$G_k(z) = z^{k-1} g_k(z). \tag{2.8}$$

By Theorem 2, we know that $G_k \in MS^*$. Hence, equality (2.6) can be written as

$$G_k(z) = z^{k-1} g_k(z) = \frac{1}{z} + \sum_{n=1}^{\infty} B_n z^n \in MS^*. \tag{2.9}$$

To prove $f \in MK^{(k)}(\beta, \gamma)$, it suffices to show that

$$\left| \frac{\frac{zf'(z)}{G_k(z)} + 1}{\frac{zf'(z)}{G_k(z)} + 2\gamma - 1} \right| < \beta,$$

where $G_k$ is given by (2.8). From (2.7), we know that

$$\beta(2-2\gamma) - \sum_{n=1}^{\infty} n\beta|a_n| - \sum_{n=1}^{\infty} \beta|1-2\gamma||B_n| \ge \sum_{n=1}^{\infty} n|a_n| + \sum_{n=1}^{\infty} |B_n| > 0. \tag{2.10}$$

Now, by the maximum modulus principle, we deduce from (1.1), (2.9) and (2.10) that

$$\left|\frac{\frac{zf'(z)}{G_k(z)}+1}{\frac{zf'(z)}{G_k(z)}+2\gamma-1}\right| = \left|\frac{\sum_{n=1}^{\infty} na_n z^{n+1}+\sum_{n=1}^{\infty} B_n z^{n+1}}{\sum_{n=1}^{\infty} na_n z^{n+1}-\sum_{n=1}^{\infty}(1-2\gamma)B_n z^{n+1}-(2-2\gamma)}\right|$$

$$< \frac{\sum_{n=1}^{\infty} n|a_n|+\sum_{n=1}^{\infty}|B_n|}{(2-2\gamma)-\sum_{n=1}^{\infty} n|a_n|-\sum_{n=1}^{\infty}|1-2\gamma||B_n|} \le \beta.$$

This evidently completes the proof of Theorem 3. □

**Theorem 4** *Let* $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty} a_n z^n \in MK^{(k)}(\beta,\gamma)$. *Then*

$$1+\sum_{n=1}^{\infty}\frac{n(1-\beta e^{i\theta})}{2(1-\gamma)\beta e^{i\theta}}a_n z^{n+1}+\sum_{n=1}^{\infty}\frac{1+(1-2\gamma)\beta e^{i\theta}}{2(1-\gamma)\beta e^{i\theta}}B_n z^{n+1}\neq 0$$

$$\bigl(z\in U^*;0<\theta<2\pi\bigr), \tag{2.11}$$

*where the coefficients* $B_n$ $(n=1,2,\ldots)$ *are given by* (2.9).

*Proof* Suppose that $f\in MK^{(k)}(\beta,\gamma)$. Then we know that

$$-\frac{zf'(z)}{G_k(z)}\neq\frac{1+(1-2\gamma)\beta e^{i\theta}}{1-\beta e^{i\theta}}\quad\bigl(z\in U^*;0<\theta<2\pi\bigr), \tag{2.12}$$

where $G_k$ is given by (2.8). After a simple computation, the inequality (2.2) is equivalent to

$$zf'(z)\bigl(1-\beta e^{i\theta}\bigr)+G_k(z)\bigl(1+(1-2\gamma)\beta e^{i\theta}\bigr)\neq 0\quad\bigl(z\in U^*;0<\theta<2\pi\bigr). \tag{2.13}$$

By substituting (1.1) and (2.9) into (2.13), we obtain the desired assertion (2.11) of Theorem 4.

Finally, we provide the following distortion theorem for the considered class of functions $MK^{(k)}(\beta,\gamma)$. □

**Theorem 5** *If* $f\in MK^{(k)}(\beta,\gamma)$, *then*

$$\frac{(1-r)^2(1-(1-2\gamma)\beta r)}{r^2(1+\beta r)}\le\bigl|f'(z)\bigr|$$

$$\le\frac{(1+r)^2(1+(1-2\gamma)\beta r)}{r^2(1-\beta r)}\quad\bigl(|z|=r;0<r<1\bigr). \tag{2.14}$$

*Proof* If $f\in MK^{(k)}(\beta,\gamma)$, then there exists a function $g\in MS^*(\frac{k-1}{k})$ such that (1.3) holds true. It follows from Theorem 2 that the function $G_k$ given by (2.8) is a meromorphic starlike function. Hence, we have (see [12])

$$\frac{(1-r)^2}{r}\le\bigl|G_k(z)\bigr|\le\frac{(1+r)^2}{r}\quad\bigl(|z|=r;0<r<1\bigr). \tag{2.15}$$

Let us define $p(z)$ by

$$-\frac{zf'(z)}{G_k(z)} = p(z) \quad \bigl(z \in U^*\bigr), \tag{2.16}$$

where

$$p(z) \prec \frac{1 + (1 - 2\gamma)\beta z}{1 - \beta z}.$$

Then, by using a similar method as in [13, p.105], we have

$$\frac{1 - (1 - 2\gamma)\beta r}{1 + \beta r} \le \bigl|p(z)\bigr| \le \frac{1 + (1 - 2\gamma)\beta r}{1 - \beta r} \quad \bigl(|z| = r; 0 < r < 1\bigr). \tag{2.17}$$

Thus, from (2.15), (2.16) and (2.17), we readily get the inequality (2.14). The proof of Theorem 5 is thus completed. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors jointly worked on the results and they read and approved the final manuscript.

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