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Higher-order symmetric duality for a class of multiobjective fractional programming problems

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Abstract

In this paper, a pair of nondifferentiable multiobjective fractional programming problems is formulated. For a differentiable function, we introduce the definition of higher-order (F, α, ρ, d)-convexity, which extends some kinds of generalized convexity, such as second order F-convexity and higher-order F -convexity. Under the higher-order (F, α, ρ, d)-convexity assumptions, we prove the higher-order weak, higher-order strong and higher-order converse duality theorems.

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Keywords: Higher-order symmetric duality, multiobjective fractional programming, higher-order (F, α, ρ, d)-convexity.

Introduction

Symmetric duality in nonlinear programming in which the dual of the dual is the primal was introduced by Dorn [1]. The notion of symmetric duality was developed significantly by Dantzig et al. [2], and the Wolfe dual models presented in [2]. Mond [3] presented a slightly different pair of symmetric dual nonlinear programs and obtained more generalized duality results than that of Dantzig et al. [2]. Mond and Weir [4] then gave another pair of symmetric dual nonlinear programs in which a weaker convexity assumption was imposed on involved functions. Later, Mond and Weir [5], Weir and Mond [6] as well as Gulati et al. [7] generalized single objective symmetric duality to multiobjective case.

Chandra et al. [8] first formulated a pair of symmetric dual fractional programs with certain convexity hypothesis. Pandey [9] introduced second-order η-invex function for multiobjective fractional programming problem and established weak and strong duality theorems. Yang et al. [10] discussed a class of nondifferentiable multiobjective fractional programming problems, and proved duality theorems under the assumptions of invex (pseudoinvex, pseudoincave) functions. Higher-order duality in nonlinear programs have been studied by some researchers. Mangasarian [11] formulated a class of higher-order dual problems for the nonlinear programming problem by introducing twice differentiable functions. Mond and Zhang [12] obtained duality results for various higher-order dual programming problems under higher-order invexity assumptions. Under invexity-type conditions, such as higher-order type I, higher-order pseudo-type I, and higher-order quasi-type I conditions, Mishra and Rueda [13] gave various duality results. Recently, Chen [14] also discussed the duality theorems under higher-order F-convexity (F-pseudo-convexity, F-quasi-convexity) for a pair of multiobjective nondifferentiable program. But, up to now, there is not sufficient literatures dealing with higher-order fractional symmetric duality.

In this paper, we first formulate a pair of nondifferentiable multiobjective fractional pro-gramming problems. For a differentiable function h: $R^n \times R^n \to R$, we introduce the definition of higher-order (F, α, ρ, d)-convexity, which extends some kinds of generalized convexity, such as second order F-convexity in [15] and higher-order F -convexity in [14]. Under the higher-order (F, α, ρ, d)- convexity assumptions, we prove the higher-order weak, higher-order strong and higher-order converse duality theorems.

Preliminaries

Let R^n be the n-dimensional Euclidean space and let R^n_+ be its non-negative orthant. The following conventions for vectors in R^n will be used:

$$\begin{aligned}
&x < y \quad \text{if and only if} \quad y - x \in \operatorname{int} R^n;\\
&x \le y \quad \text{if and only if} \quad y - x \in R^n_+ \backslash \{0\};\\
&x \leqq y \quad \text{if and only if} \quad y - x \in R^n_+;\\
&x \nleq y \quad \text{is the negation of} \quad x \le y.
\end{aligned}$$

For a real-valued twice differentiable function $h(x, y)$ defined on an open set in $R^n \times R^m$, denote by $\nabla_x h(\bar{x}, \bar{y})$ the gradient vector of h with respect to x at $(\bar{x}, \bar{y})$, $\nabla_{xx} h(\bar{x}, \bar{y})$ the hessian matrix with respect to x at $(\bar{x}, \bar{y})$. Similarly, $\nabla_y h(\bar{x}, \bar{y})$, $\nabla_{xy} h(\bar{x}, \bar{y})$ and $\nabla_{yy} h(\bar{x}, \bar{y})$ are also defined.

Let C be a compact convex set in R^n. The support function of C is defined by

$$s(x|C) = \max\{x^T y : y \in C\}.$$

A support function, being convex and everywhere finite, has a subdifferential, that is, there exists a $z \in R^n$ such that

$$s(y|C) \geqq s(x|C) + z^T(y - x), \ \forall x \in C.$$

The subdifferential of $s(x|C)$ is given by

$$\partial s(x|C) = \{z \in C \ : \ z^T x = s(x|C)\}.$$

For a convex set $D \subset R^n$, the normal cone to D at a point $x \in D$ is defined by

$$N_D(x) = \{y \in R^n \ : \ y^T(z - x) \leqq 0, \ \forall z \in D\}.$$

When C is a compact convex set, $y \in N_C(x)$ if and only if $s(y|C) = x^T y$, or equivalently, $x \in \partial s(y|C)$.

Consider the following multiobjective programming problem (P):

$$\text{Minimize} \quad f(x) \quad \text{subject to } g(x) \leqq 0, \quad x \in X,$$

where f: $R^n \to R^m$, g: $R^n \to R^l$ and $X \subset R^n$. Denote by S the set of feasible solutions of (P).

Definition 2.1. (a) A feasible solution x_0 is said to be an efficient solution of (P) if there is no other $x \in S$ such that $f(x) \le f(x_0)$.

(b) A feasible solution x_0 is said to be a properly efficient solution of (P) if it is an efficient solution of (P), and there exists a real number $M > 0$ such that for all $i \in \{1, ..., m\}$, $x \in S$, and $f_i(x) < f_i(x_0)$,

$$f_i(x_0) - f_i(x) \leqq M(f_j(x) - f_j(x_0))$$

for some $j \in \{1, ..., m\}$ such that $f_j(x) > f_j(x_0)$.

Definition 2.2. A functional F: $X \times X \times R^n \to R$ (where $X \subset R^n$) is sublinear in its third component if for all $(x, u) \in X \times X$,

$$F(x, u; a_1 + a_2) \leqq F(x, u; a_1) + F(x, u; a_2) \text{for all } a_1, a_2 \in R^n;$$
$$F(x, u; \alpha a) = \alpha F(x, u; a) \text{ for all } \alpha \in R_+ \text{ and for all } a \in R^n.$$

For convenience, we write $F_{x,u}(a) = F(x, u, a)$.

We now introduce higher-order (F, α, ρ, d)-convex function. Where, F: $X \times X \times R^n \to R$ is a sublinear functional, α: $X \times X \to R_+ \setminus \{0\}$, $\rho \in R$ and d: $X \times X \to R$. Let Φ: $X \to R$ and h: $X \times R^n \to R$ be differentiable real valued functions.

Definition 2.3. Φ is said to be higher-order (F, α, ρ, d)-convex at $u \in X$ with respect to h if, $\forall(x, p) \in X \times R^n$,

$$\Phi(x) - \Phi(u) \geqq F_{x,u}(\alpha(\nabla_x\Phi(u) + \nabla_p h(u, p))) + h(u, p) - p^T\nabla_p h(u, p) + \rho d^2(x, u).$$

Remark 2.1. (1) When $\alpha = 1$, and $\rho = 0$ or $d = 0$, the higher-order (F, α, ρ, d)-convexity reduces to higher-order F-convexity in [14].

(2) When $\alpha = 1$, $\rho = 0$ or $d = 0$, and $h(u, p) = \frac{1}{2}p^T\nabla_{xx}\Phi(u)p$, the higher-order (F, α, ρ, d)-convexity reduces to second order F-convexity in [15].

we now give an example of higher-order (F, α, ρ, d)-convex function with respect to $h(u, p)$, which is not higher-order F -convex and second order F-convex.

Example 2.1. Let $X \subset R$, $X = \{x: x \geqq 1\}$, f: $X \to R$, F: $X \times X \times R \to R$, h: $X \times R \to R$ and d: $X \times X \to R$ given as follows

$$f(x) = x + \frac{2}{x+1}, \ F_{x,u}(a) = |a|(x-u)^2, \ h(u, p) = \frac{p}{u+1}, \ d(x, u) = x - u.$$

And let $u = 1$, $\rho = -1$, $\alpha = \frac{3}{4}$. Then for all $(x, p) \in X \times R$

$$f(x) - f(u) = \frac{x^2 - x}{x+1} \geqq F_{x,u}\left(\frac{3}{4}(\nabla_x f(u) + \nabla_p h(u, p))\right)$$
$$+ h(u, p) - p^T\nabla_p h(u, p) - d^2(x, u) = -\frac{1}{4}(x-1)^2.$$

This implies $f(x)$ is a higher-order (F, α, ρ, d)-convex function with respect to h at u. But when we let $x = 2$, $p = 3$ and $x = 6$, $p = 3$ respectively, we have

$$f(2) - f(1) = \frac{2}{3} < F_{x,u}(\nabla_x f(u) + \nabla_p h(u, p)) + h(u, p) - p^T\nabla_p h(u, p) = \frac{3}{4},$$
$$f(6) - f(1) = \frac{30}{7} < F_{x,u}(\nabla_x f(u) + \nabla_{xx} f(u)) - \frac{1}{2}p^T\nabla_{xx} f(u)p = \frac{66}{4}.$$

Hence, f is neither a higher-order F-convex function nor a second order F-convex function. From now on, suppose that the sublinear functional F satisfies the following condition:

$$F_{x,y}(a) + a^T y \geqq 0, \quad \forall a \in R^n_+. \tag{1}$$

Higher-order symmetric duality

In the section, we consider the following multiobjective fractional symmetric dual problems: **(MFP)** Minimize $L(x, y, p) = (L_1(x, y, p_1), ..., L_k(x, y, p_k))^T$ subject to

$$\sum_{i=1}^{k}\lambda_i\left[(\nabla_y f_i(x, y) - z_i + \nabla_{p_i}H_i(x, y, p_i)) - L_i(x, y, p_i)(\nabla_y g_i(x, y) + r_i + \nabla_{p_i}G_i(x, y, p_i))\right] \leqq 0,$$

$$y^T\sum_{i=1}^{k}\lambda_i\left[(\nabla_y f_i(x, y) - z_i + \nabla_{p_i}H_i(x, y, p_i)) - L_i(x, y, p_i)(\nabla_y g_i(x, y) + r_i + \nabla_{p_i}G_i(x, y, p_i))\right] \geqq 0,$$

$$\lambda > 0, \quad \lambda^T e = 1, \quad z_i \in D_i, \quad r_i \in F_i, \quad i = 1 \ldots, k.$$

(MFD) Maximize $M(u, v, q) = (M_1(u, v, q_1),..., M_k(u, v, q_k))^T$ subject to

$$\sum_{i=1}^{k}\lambda_i\left[(\nabla_x f_i(u, v) + w_i + \nabla_{q_i}\Phi_i(u, v, q_i)) - M_i(u, v, q_i)(\nabla_x g_i(u, v) - t_i + \nabla_{q_i}\Psi_i(u, v, q_i))\right] \geqq 0,$$

$$u^T\sum_{i=1}^{k}\lambda_i\left[(\nabla_x f_i(u, v) + w_i + \nabla_{q_i}\Phi_i(u, v, q_i)) - M_i(u, v, q_i)(\nabla_x g_i(u, v) - t_i + \nabla_{q_i}\Psi_i(u, v, q_i))\right] \leqq 0,$$

$$\lambda > 0, \quad \lambda^T e = 1, \quad w_i \in C_i, \quad t_i \in E_i, \quad i = 1 \ldots, k.$$

where

$$L_i(x, y, p_i) = \frac{f_i(x, y) + s(x|C_i) - y^T z_i + H_i(x, y, p_i) - p_i^T\nabla_{p_i}H_i(x, y, p_i)}{g_i(x, y) - s(x|E_i) + y^T r_i + G_i(x, y, p_i) - p_i^T\nabla_{p_i}G_i(x, y, p_i)},$$

$$M_i(u, v, q_i) = \frac{f_i(u, v) - s(v|D_i) + u^T w_i + \Phi_i(u, v, q_i) - q_i^T\nabla_{q_i}\Phi_i(u, v, q_i)}{g_i(u, v) + s(v|F_i) - u^T t_i + \Psi_i(u, v, q_i) - q_i^T\nabla_{q_i}\Psi_i(u, v, q_i)},$$

f_i: $R_n \times R_m \to R$; g_i: $R^n \times R^m \to R$; H_i, G_i: $R^n \times R^m \to R$ and Φ_i, Ψ_i: $R_n \times R_m \times R_n \to R$ are twice differentiable functions for all $i = 1 ..., k$. C_i, E_i are compact convex sets in R^n, and D_i, F_i are compact convex sets in R^m, $i = 1, ..., k$. $e = (1, ..., 1)^T \in R^k$. $p_i \in R^m$, $q_i \in R^n$, $i = 1, ..., k$, $p = (p_1, ..., p_k)$, $q = (q_1, ..., q_k)$. It is assumed that in the feasible regions the numerators are nonnegative and denominators are positive.

We let $S = (S_1, ..., S_k)^T$, $W = (W_1, ..., W_k)^T \in R^k$. Then we can express the programs (MFP) and (MFD) equivalently as:

(MFP)$_{\mathbf{S}}$ Minimize S subject to

$$(f_i(x, y) + s(x|C_i) - y^T z_i + H_i(x, y, p_i) - p_i^T\nabla_{p_i}H_i(x, y, p_i)) - S_i(g_i(x, y) - s(x|E_i) + y^T r_i + G_i(x, y, p_i) - p_i^T\nabla_{p_i}G_i(x, y, p_i)) = 0, \ i = 1, \ldots, k, \tag{2}$$

$$\sum_{i=1}^{k}\lambda_i\left[(\nabla_y f_i(x, y) - z_i + \nabla_{p_i}H_i(x, y, p_i)) - S_i(\nabla_y g_i(x, y) + r_i + \nabla_{p_i}G_i(x, y, p_i))\right] \leqq 0, \tag{3}$$

$$\begin{aligned} & y^T \sum_{i=1}^{k} \lambda_i \left[(\nabla_y f_i(x, y) - z_i + \nabla_{p_i} H_i(x, y, p_i)) \right. \\ & \quad \left. - S_i(\nabla_y g_i(x, y) + r_i + \nabla_{p_i} G_i(x, y, p_i)) \right] \geqq 0, \\ & \lambda > 0, \quad \lambda^T e = 1, \quad z_i \in D_i, \quad r_i \in F_i, \quad i = 1 \ldots, k. \end{aligned} \tag{4}$$

$(\mathbf{MFD})_{\mathbf{W}}$ Maximize W subject to

$$\begin{aligned} & (f_i(u, v) - s(v|D_i) + u^T w_i + \Phi_i(u, v, q_i) - q_i^T \nabla_{q_i} \Phi_i(u, v, q_i)) \\ & \quad - W_i(g_i(u, v) + s(v|F_i) - u^T t_i + \Psi_i(u, v, q_i) - q_i^T \nabla_{q_i} \Psi_i(u, v, q_i)) = 0, \quad i = 1, \ldots, k, \end{aligned} \tag{5}$$

$$\begin{aligned} & \sum_{i=1}^{k} \lambda_i \left[(\nabla_x f_i(u, v) + w_i + \nabla_{q_i} \Phi_i(u, v, q_i)) \right. \\ & \quad \left. - W_i(\nabla_x g_i(u, v) - t_i + \nabla_{q_i} \Psi_i(u, v, q_i)) \right] \geqq 0, \end{aligned} \tag{6}$$

$$\begin{aligned} & u^T \sum_{i=1}^{k} \lambda_i \left[(\nabla_x f_i(u, v) + w_i + \nabla_{q_i} \Phi_i(u, v, q_i)) \right. \\ & \quad \left. - W_i(\nabla_x g_i(u, v) - t_i + \nabla_{q_i} \Psi_i(u, v, q_i)) \right] \leqq 0, \\ & \lambda > 0, \quad \lambda^T e = 1, \quad w_i \in C_i, \quad t_i \in E_i, \quad i = 1 \ldots, k. \end{aligned} \tag{7}$$

Now we can prove weak, strong and converse duality theorems for $(\text{MFP})_S$ and $(\text{MFD})_W$, but equally apply to (MFP) and (MFD).

Theorem 3.1 (*Weak duality*). Let $(x, y, S, z_1, \ldots, z_k, r_1, \ldots, r_k, \lambda, p)$ be feasible for $(\text{MFD})_S$ and let $(u, v, W, w_1, \ldots, w_k, t_1 \ldots, t_k, \lambda, q)$ be feasible for $(\text{MFD})_W$. Let $\forall i \in \{1, \ldots, k\}$, $f_i(., v) + (.)^T w_i$ be higher-order (F, α, ρ_i, d_i)-convex at u with respect to $\Phi_i(u, v, q_i)$, - $(g_i(., v) - (.)^T t_i)$ be higher-order (F, α, ρ, d_i)-convex at u with respect to -$\Psi_i(u, v, q_i)$, - $(f_i(x, .) - (.)^T z_i)$ be higher-order $(K, \bar{\alpha}, \bar{\rho}_i, \bar{d}_i)$-convex at y with respect to -$H_i(x, y, p_i)$, $g_i(x, .) + (.)^T r_i$ be higher-order $(K, \bar{\alpha}, \bar{\rho}_i, \bar{d}_i)$-convex at y with respect to $G_i(x, y, p_i)$, where sublinear functional F: $R^n \times R^n \times R^n \to R$ and K: $R^m \times R^m \times R^m \to R$ satisfy the condition (1). If the following conditions hold:

$$g_i(x, v) + v^T r_i - s(x|E_i) > 0, i = 1, \ldots, k, \tag{8}$$

$$\sum_{i=1}^{k} \lambda_i ((1 + W_i)\rho_i d_i^2(x, u) + (1 + S_i)\bar{\rho}_i \bar{d}_i^2(v, y)) \geqq 0. \tag{9}$$

Then $S \nleq W$.

Proof. Since $(u, v, W, w_1, \ldots, w_k, t_1 \ldots, t_k, \lambda, q)$ is feasible for $(\text{MFD})_W$, from (6), (7) and F satisfies condition (1), it follows that

$$F_{x,u}\left(\sum_{i=1}^{k} \lambda_i [(\nabla_x f_i(u, v) + w_i + \nabla_{q_i} \Phi_i(u, v, q_i)) - W_i(\nabla_x g_i(u, v) - t_i + \nabla_{q_i} \Psi_i(u, v, q_i))]\right) \geqq 0. \tag{10}$$

Using the convexity assumptions of $f_i(., v) + (.)^T w_i$ and -$(g_i(., v) - (.)^T t_i)$ at u, we have

$$
\begin{aligned}
&f_i(x,\ v) + x^T w_i - f_i(u,\ v) - u^T w_i\\
&\quad \geqq F_{x,u}(\alpha(\nabla_x f_i(u,\ v) + w_i + \nabla_{q_i}\Phi_i(u,\ v,\ q_i))) + \Phi_i(u,\ v,\ q_i) - q_i^T\nabla_{q_i}\Phi_i(u,\ v,\ q_i) + \rho_i d_i^2(x,\ u),\\
&-g_i(x,\ v) + x^T t_i + g_i(u,\ v) - u^T t_i\\
&\quad \geqq F_{x,u}(\alpha(-\nabla_x g_i(u,\ v) + t_i - \nabla_{q_i}\Psi_i(u,\ v,\ q_i))) - \Psi_i(u,\ v,\ q_i) + q_i^T\nabla_{q_i}\Phi_i(u,\ v,\ q_i) + \rho_i d_i^2(x,\ u).
\end{aligned}
$$

Since F is a sublinear functional and $\lambda > 0$, $W \geqq 0$, $\alpha > 0$, from (10) and the above two inequalities, we have

$$
\begin{aligned}
&\sum_{i=1}^{k}\lambda_i(f_i(x,\ v) + x^T w_i - f_i(u,\ v) - u^T w_i - \Phi_i(u,\ v,\ q_i) + q_i^T\nabla_{q_i}\Phi_i(u,\ v,\ q_i))\\
&\quad + \sum_{i=1}^{k}\lambda_i W_i(g_i(u,\ v) + v^T r_i - u^T t_i + \Psi_i(u,\ v,\ q_i) - q_i^T\nabla_{q_i}\Psi_i(u,\ v,\ q_i)) \qquad (11)\\
&\quad + \sum_{i=1}^{k}\lambda_i W_i(x^T t_i - g_i(x,\ v) - v^T r_i) \geqq \sum_{i=1}^{k}\lambda_i(1 + W_i)\rho_i d_i^2(x,\ u).
\end{aligned}
$$

Since $v^T r_i \leqq s(v|F_i)$, from (5) and (11), we have

$$
\sum_{i=1}^{k}\lambda_i[(f_i(x,\ v) + x^T w_i - s(v|D_i)) + W_i(x^T t_i - v^T r_i - g_i(x,\ v))] \geqq \sum_{i=1}^{k}\lambda_i(1 + W_i)\rho_i d_i^2(x,\ u). \qquad (12)
$$

On the other hand, from (3), (4) and sublinear functional K satisfies condition (1), we obtain

$$
\begin{aligned}
&K_{v,y}\Bigg(-\sum_{i=1}^{k}\lambda_i\left((\nabla_y f_i(x,\ y) - z_i + \nabla_{p_i}H_i(x,\ y, p_i))\right.\\
&\quad \left.-S_i(\nabla_y g_i(x,\ y) + r_i + \nabla_{p_i}G_i(x,\ y, p_i))\right)\Bigg) \geqq 0. \qquad (13)
\end{aligned}
$$

Using the convexity assumptions of $-f_i(x, .) + (.)^T z_i$ and $g_i(x, .) + (.)^T r_i$ at y, we have

$$
\begin{aligned}
-f_i(x,v) + v^T z_i + f_i(x,y) - y^T z_i \quad &\geqq \quad K_{v,y}(\bar{\alpha}(-\nabla_y f_i(x,\ y) + z_i - \nabla_{p_i}H_i(x,\ y, p_i)))\\
&\quad - H_i(x,\ y, p_i) + p_i^T\nabla_{p_i}H_i(x,\ y, p_i) + \bar{\rho}_i\bar{d}_i^2(v,\ y),\\
g_i(x,\ v) + v^T r_i - g_i(x,y) - y^T r_i \quad &\geqq \quad K_{v,y}(\bar{\alpha}(\nabla_y g_i(x,\ y) + r_i + \nabla_{p_i}G_i(x,\ y, p_i)))\\
&\quad + G_i(x,\ y, p_i) - p_i^T\nabla_{p_i}G_i(x,\ y, p_i) + \bar{\rho}_i\bar{d}_i^2(v,\ y).
\end{aligned}
$$

Since K is a sublinear functional, and $\lambda > 0$, $S \geqq 0$, $\bar{\alpha} > 0$, from (13) and the above two inequalities, it holds

$$
\begin{aligned}
&\sum_{i=1}^{k}\lambda_i(-f_i(x,\ v) + v^T z_i + f_i(x,\ y) - y^T z_i + H_i(x,\ y, p_i) - p_i^T\nabla_{p_i}H_i(x,\ y, p_i))\\
&\quad + \sum_{i=1}^{k}\lambda_i S_i(-g_i(x,\ y) + x^T t_i - y^T r_i - G_i(x,\ y, p_i) + p_i^T\nabla_{p_i}G_i(x,\ y, p_i)) \qquad (14)\\
&\quad + \sum_{i=1}^{k}\lambda_i S_i(g_i(x,\ v) + v^T r_i - x^T t_i) \geqq \sum_{i=1}^{k}\lambda_i(1 + S_i)\bar{\rho}_i\bar{d}_i^2(v,\ y).
\end{aligned}
$$

Since $x^T t_i \leqq s(x|E_i)$, from (2) and (14) we have

$$\sum_{i=1}^{k} \lambda_i[(-f_i(x,\ v) + v^T z_i - s(x|C_i)) + S_i(g_i(x,\ v) + v^T r_i - x^T t_i)] \geqq \sum_{i=1}^{k} \lambda_i(1 + S_i)\bar{\rho}_i \bar{d}_i^2(v,\ y).$$

Adding the above inequality and (12), we get

$$\sum_{i=1}^{k} \lambda_i(v^T z_i - s(v|D_i) + x^T w_i - s(x|C_i)) + \sum_{i=1}^{k} \lambda_i(S_i - W_i)(g_i(x,\ v) + v^T r_i - x^T t_i)$$
$$\geqq \sum_{i=1}^{k} \lambda_i(\rho_i d_i^2(x,\ u)(1 + W_i) + \bar{\rho}_i \bar{d}_i^2(v,\ y)(1 + S_i)).$$

Since $\lambda_i > 0$, $v^T z_i - s(v|D_i) + x^T w_i - s(x|C_i) \leqq 0$, $i = 1, ..., k$, by (9) it yields

$$\sum_{i=1}^{k} \lambda_i(S_i - W_i)(g_i(x,\ v) + v^T r_i - x^T t_i) \geqq 0.$$

By assumptions (8), we have $g_i(x, v)+v^T r_i - x^T t_i > 0$, $i = 1, ..., k$. Since $\lambda > 0$, it follows that $S \nleq W$. □

Theorem 3.2 (*Strong duality*). Let $(\bar{x}, \bar{y}, \bar{S}, \bar{z}_1, \ldots, \bar{z}_k, \bar{r}_1, \ldots, \bar{r}_k, \bar{\lambda}, \bar{p})$ be a properly efficient solution of $(\text{MFP})_S$, and fix $\lambda = \bar{\lambda}$ in $(\text{MFD})_W$. Suppose that

(a) $\nabla_x H_i(\bar{x}, \bar{y},\ 0) = \nabla_x G_i(\bar{x}, \bar{y},\ 0) = 0,\ \nabla_{q_i} \Phi_i(\bar{x}, \bar{y},\ 0) = \nabla_{q_i} \Psi_i(\bar{x}, \bar{y},\ 0) = 0,$
$H_i(\bar{x}, \bar{y},\ 0) = G_i(\bar{x}, \bar{y}, 0) = 0,\ \Phi_i(\bar{x}, \bar{y},\ 0) = \Psi_i(\bar{x}, \bar{y},\ 0) = 0,\ \nabla_y H_i(\bar{x}, \bar{y},\ 0) = \nabla_y G_i(\bar{x}, \bar{y},\ 0) = 0,$
$\nabla_{p_i} H_i(\bar{x}, \bar{y},\ 0) = \nabla_{p_i} G_i(\bar{x}, \bar{y},\ 0) = 0, i = 1, \ldots, k.$

(b) For all $i \in \{1, ..., k\}$,

$$f_i(\bar{x}, \bar{y}) + s(\bar{x}|C_i) - \bar{y}^T \bar{z}_i + H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{p}_i^T \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) > 0.$$

(c) (i) $\nabla_{p_i p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i \nabla_{p_i p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i) \neq 0$ for $\bar{p}_i = 0$, $i = 1, ..., k$ and $\nabla_{p_i p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i \nabla_{p_i p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)$ is nonsingular for all $i = 1, ..., k$,

(ii) $\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_{yy} f_i(\bar{x}, \bar{y}) - \bar{S}_i \nabla_{yy} g_i(\bar{x}, \bar{y}))$ is positive definite and $\bar{p}_i^T((\nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) - (\nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) \geqq 0$ for all $i = 1, ..., k$, or $\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_{yy} f_i(\bar{x}, \bar{y}) - \bar{S}_i \nabla_{yy} g_i(\bar{x}, \bar{y}))$ is negative definite and $\bar{p}_i^T((\nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) - (\nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) \leqq 0$ for all $i = 1, ..., k$.

(iii) $\{\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)) : i = 1, \ldots, k\}$ is linearly independent.

Then $\bar{p} = 0$, and there exist $\bar{w}_i \in C_i$ and $\bar{t}_i \in E_i$, $i = 1, ..., k$ such that $(\bar{x}, \bar{y}, \bar{S}, \bar{w}_1, ..., \bar{w}_k, \bar{t}_1, \ldots, \bar{t}_k, \bar{\lambda}, \bar{q} = 0)$ is a feasible solution of $(\text{MFD})_W$. Furthermore, if the hypotheses in Theorem 3.1 are satisfied, then $(\bar{x}, \bar{y}, \bar{S}, \bar{w}_1, ..., \bar{w}_k, \bar{t}_1, \ldots, \bar{t}_k, \bar{\lambda}, \bar{q} = 0)$ is a properly efficient solution of $(\text{MFD})_W$, and the two objective values are equal.

Proof. Since $(\bar{x}, \bar{y}, \bar{S}, \bar{z}_1, \ldots, \bar{z}_k, \bar{r}_1, \ldots, \bar{r}_k, \bar{\lambda}, \bar{p})$ is a properly efficient solution of $(\text{MFP})_S$, by the Fritz John type necessary optimality conditions [16], there exist $\alpha \in R^k$, $\beta \in R^k$, $\gamma \in R^m$, $\delta \in R$, $\mu \in R^k$ and $\bar{w}_i \in R^n, \bar{t}_i \in R^n$, $i = 1, ..., k$ such that

$$\begin{aligned}&\sum_{i=1}^{k}\beta_i\big((\nabla_x f_i(\bar{x},\bar{y})+\bar{w}_i+\nabla_x H_i(\bar{x},\bar{y},\bar{p}_i))-\bar{S}_i(\nabla_x g_i(\bar{x},\bar{y})-\bar{t}_i+\nabla_x G_i(\bar{x},\bar{y},\bar{p}_i))\big)\\&\quad+(\gamma-\delta\bar{y})^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{yx}f_i(\bar{x},\bar{y})-\bar{S}_i\nabla_{yx}g_i(\bar{x},\bar{y}))\\&\quad+\sum_{i=1}^{k}(\nabla_{p_i x}H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{S}_i\nabla_{p_i x}G_i(\bar{x},\bar{y},\bar{p}_i))^T((\gamma-\delta\bar{y})\bar{\lambda}_i-\beta_i\bar{p}_i)=0,\end{aligned}\tag{15}$$

$$\begin{aligned}&\sum_{i=1}^{k}(\beta_i-\delta\bar{\lambda}_i)\big((\nabla_y f_i(\bar{x},\bar{y})-z_i+\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i))-\bar{S}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))\big)\\&\quad+\sum_{i=1}^{k}\beta_i\big((\nabla_y H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{S}_i\nabla_y G_i(\bar{x},\bar{y},\bar{p}_i))-(\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{S}_i\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))\big)\\&\quad+\sum_{i=1}^{k}\bar{\lambda}_i((\nabla_{yy}f_i(\bar{x},\bar{y})-\bar{S}_i\nabla_{yy}g_i(\bar{x},\bar{y}))^T(\gamma-\delta\bar{y}))\\&\quad+\sum_{i=1}^{k}(\nabla_{p_i y}H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{S}_i\nabla_{p_i y}G_i(\bar{x},\bar{y},\bar{p}_i))^T(-\beta_i\bar{p}_i+(\gamma-\delta\bar{y})\bar{\lambda}_i)=0,\end{aligned}\tag{16}$$

$$\begin{aligned}&\alpha_i-\beta_i(g_i(\bar{x},\bar{y})-s(\bar{x}|E_i)+\bar{y}^T\bar{r}_i+G_i(\bar{x},\bar{y},\bar{p}_i)-\bar{p}_i^T\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))\\&\quad-(\gamma-\delta\bar{y})^T(\bar{\lambda}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i)))=0,\quad i=1,\dots,k,\end{aligned}\tag{17}$$

$$\begin{aligned}&(\gamma-\delta\bar{y})^T\big((\nabla_y f_i(\bar{x},\bar{y})-\bar{z}_i+\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{S}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))\big)\\&\quad-\mu_i=0,\quad i=1,\dots,k,\end{aligned}\tag{18}$$

$$(\bar{\lambda}_i(\gamma-\delta\bar{y})-\beta_i\bar{p}_i)^T(\nabla_{p_ip_i}H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{S}_i\nabla_{p_ip_i}G_i(\bar{x},\bar{y},\bar{p}_i))=0,\quad i=1,\dots,k,\tag{19}$$

$$\beta_i\bar{y}+(\gamma-\delta\bar{y})\bar{\lambda}_i\in N_{D_i}(\bar{z}_i),\quad i=1,\dots,k,\tag{20}$$

$$\beta_i\bar{S}_i\bar{y}+\bar{\lambda}_i\bar{S}_i(\gamma-\delta\bar{y})\in N_{F_i}(\bar{r}_i),\quad i=1,\dots,k,\tag{21}$$

$$\begin{aligned}&\gamma^T\sum_{i=1}^{k}\bar{\lambda}_i\big((\nabla_y f_i(\bar{x},\bar{y})-\bar{z}_i+\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i))\\&\quad-\bar{S}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))\big)=0,\end{aligned}\tag{22}$$

$$\begin{aligned}&\delta\bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i\big((\nabla_y f_i(\bar{x},\bar{y})-\bar{z}_i+\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i))\\&\quad-\bar{S}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))\big)=0,\end{aligned}\tag{23}$$

$$\mu^T\bar{\lambda}=0,\tag{24}$$

$$\bar{w}_i\in C_i,\bar{t}_i\in E_i,\bar{x}^T\bar{t}_i=s(\bar{x}|E_i),\bar{x}^T\bar{w}_i=s(\bar{x}|C_i),\quad i=1,\dots,k,\tag{25}$$

$$(\alpha,\ \beta,\ \gamma,\ \delta,\ \mu) \neq 0, (\alpha,\ \gamma,\ \delta,\ \mu) \geqq 0. \tag{26}$$

Since $\bar{\lambda} > 0$, and $\mu \geqq 0$, (24) implies $\mu = 0$. Consequently, (18) yields

$$\begin{aligned}&(\gamma - \delta\bar{y})^T \big((\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x},\bar{y},\bar{p}_i)\\ &\quad -\bar{S}_i(\nabla_y g_i(\bar{x},\bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p}_i))\big) = 0,\ i = 1,\dots,k.\end{aligned} \tag{27}$$

By assumption (i) and (19), we have

$$\bar{\lambda}_i(\gamma - \delta\bar{y}) = \beta_i\bar{p}_i, \quad i = 1, ..., k. \tag{28}$$

Multiplying (16) $(\gamma - \delta\bar{y})$ by left, from (27) and (28) we have

$$\begin{aligned}&(\gamma - \delta\bar{y})^T\sum_{i=1}^{k}\beta_i((\nabla_y H_i(\bar{x},\bar{y},\bar{p}_i) - \bar{S}_i\nabla_y G_i(\bar{x},\bar{y},\bar{p}_i)) - (\nabla_{p_i} H_i(\bar{x},\bar{y},\bar{p}_i) - \bar{S}_i\nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p}_i)))\\ &\quad +(\gamma - \delta\bar{y})^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{yy} f_i(\bar{x},\bar{y}) - \bar{S}_i\nabla_{yy} g_i(\bar{x},\bar{y}))(\gamma - \delta\bar{y}) = 0.\end{aligned}$$

Since $\bar{\lambda} > 0$, from (28) and the above equation, we have

$$\begin{aligned}&\sum_{i=1}^{k}\frac{\beta_i^2}{\bar{\lambda}_i}\bar{p}_i^T((\nabla_y H_i(\bar{x},\bar{y},\bar{p}_i) - \bar{S}_i\nabla_y G_i(\bar{x},\bar{y},\bar{p}_i)) - (\nabla_{p_i} H_i(\bar{x},\bar{y},\bar{p}_i) - \bar{S}_i\nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p}_i)))\\ &\quad +(\gamma - \delta\bar{y})^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{yy} f_i(\bar{x},\bar{y}) - \bar{S}_i\nabla_{yy} g_i(\bar{x},\bar{y}))(\gamma - \delta\bar{y}) = 0.\end{aligned}$$

Which by assumption (ii), we can obtain

$$\gamma - \delta\bar{y} = 0. \tag{29}$$

Using (29) in (28), we have $\beta_i\bar{p}_i = 0$, $i = 1, ..., k$. This implies that $\bar{p}_i = 0$ when $\beta_i \neq 0$, for all $i \in \{1, ..., k\}$. Hence, by assumption (1), we get

$$\sum_{i=1}^{k}\beta_i((\nabla_y H_i(\bar{x},\bar{y},\bar{p}_i) - \bar{S}_i\nabla_y G_i(\bar{x},\bar{y},\bar{p}_i)) - (\nabla_{p_i} H_i(\bar{x},\bar{y},\bar{p}_i) - \bar{S}_i\nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p}_i))) = 0.$$

Combining this with (16), (28) and (29), it follows that

$$\sum_{i=1}^{k}(\beta_i - \delta\bar{\lambda}_i)(\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x},\bar{y},\bar{p}_i) - \bar{S}_i(\nabla_y g_i(\bar{x},\bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p}_i))) = 0,$$

which by assumption (iii), it yields

$$\beta_i - \delta\bar{\lambda}_i = 0, \quad i = 1,\dots,k. \tag{30}$$

We claim that $\delta \neq 0$, otherwise, from (29) and (30) we get $\beta = 0$, $\gamma = 0$. Using (29) in (17), we get $\alpha = 0$. This contradicts with (26). Hence $\delta = 0$. Since $\bar{\lambda} > 0$, from (30) we get $\beta > 0$. Hence $\beta_i\bar{p}_i = 0$, $i = 1, ..., k$ implies $\bar{p}_i = 0$, $i = 1, ..., k$. Using (28), (29) and the fact $\bar{p}_i = 0$, $i = 1, ..., k$ in (15), by assumption (a), we get

$$\sum_{i=1}^{k}\beta_i((\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i) - \bar{S}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i)) = 0,$$

combining this with (30) and $\delta > 0$, $\bar{\lambda} > 0$, it holds

$$\sum_{i=1}^{k} \bar{\lambda}_i((\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i) - \bar{S}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i)) = 0, \tag{31}$$

which yields

$$\bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i((\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i) - \bar{S}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i)) = 0. \tag{32}$$

On the other hand, by assumption (a) and (2) we get

$$(f_i(\bar{x}, \bar{y}) + s(\bar{x}|C_i) - \bar{y}^T \bar{z}_i) - \bar{S}_i(g_i(\bar{x}, \bar{y}) - s(\bar{x}|E_i) + \bar{y}^T \bar{r}_i) = 0, \quad i = 1, \ldots, k. \tag{33}$$

Since $\beta > 0$, by (20) and (29) we get $\bar{y} \in N_{D_i}(\bar{z}_i)$, $i = 1, ..., k$. This implies

$$\bar{y}^T \bar{z}_i = s(\bar{y}|D_i), \quad i = 1, \ldots, k. \tag{34}$$

Assumption (b) implies $\bar{S} > 0$. By (21), we similarly have $\bar{y} \in N_{F_i}(\bar{r}_i)$, $i = 1, ..., k$. This implies

$$\bar{y}^T \bar{r}_i = s(\bar{y}|F_i), \quad i = 1, \ldots, k. \tag{35}$$

Combining (25), (33), (34) and (35), we get

$$(f_i(\bar{x}, \bar{y}) + \bar{x}^T \bar{w}_i - s(\bar{y}|D_i)) - \bar{S}_i\left(g_i(\bar{x}, \bar{y}) - \bar{x}^T \bar{t}_i + s(\bar{y}|F_i) = 0, \quad i = 1, \ldots, k,\right.$$

combining this with (31) and (32), by assumption (a), $(\bar{x}, \bar{y}, \bar{S}, \bar{w}_1, ..., \bar{w}_k, \bar{t}_1, \ldots, \bar{t}_k, \bar{\lambda}, \bar{q} = 0)$ is a feasible solution of $(\text{MFD})_W$.

Under the assumptions of Theorem 3.1, if $(\bar{x}, \bar{y}, \bar{S}, \bar{w}_1, ..., \bar{w}_k, \bar{t}_1, \ldots, \bar{t}_k, \bar{\lambda}, \bar{q} = 0)$ is not an efficient solution of $(\text{MFD})_W$, then there exists other feasible solution $(u, v, W, w_1, \ldots, w_k, t_1, \ldots, t_k, \bar{\lambda}, q)$, of $(\text{MFD})_W$ such that $\bar{S} \leq W$. Since $(\bar{x}, \bar{y}, \bar{S}, \bar{z}_1, \ldots, \bar{z}_k, \bar{r}_1, \ldots, \bar{r}_k, \bar{\lambda}, \bar{p})$ is a feasible solution of $(\text{MFP})_S$, by Theorem 3.1, we have $\bar{S} \nleq W$, hence the contradiction implies $(\bar{x}, \bar{y}, \bar{S}, \bar{w}_1, ..., \bar{w}_k, \bar{t}_1, \ldots, \bar{t}_k, \bar{\lambda}, \bar{q} = 0)$ is an efficient solution of $(\text{MFD})_W$.

If $(\bar{x}, \bar{y}, \bar{S}, \bar{w}_1, ..., \bar{w}_k, \bar{t}_1, \ldots, \bar{t}_k, \bar{\lambda}, \bar{q} = 0)$ is not a properly efficient solution of $(\text{MFD})_W$, then there exists other feasible solution $(u, v, W, w_1, \ldots, w_k, t_1, \ldots, t_k, \bar{\lambda}, q)$ of $(\text{MFD})_W$ such that for an index $i \in \{1, ..., k\}$ and any real number $M > 0$, $W_i - \bar{S}_i > M(\bar{S}_j - W_j)$ for j satisfying $\bar{S}_j > W_j$ whenever $W_i > \bar{S}_i$ This implies $W_i > \bar{S}_i$ can be made arbitrarily large and this contradicts with Theorem 3.1. And it is easy to find that the two objective values are equal. □

Theorem 3.3 (*Strict converse duality*). Let $(\bar{u}, \bar{v}, \bar{W}, \bar{w}_1, \ldots, \bar{w}_k, \bar{t}_1, \ldots, \bar{t}_k, \bar{\lambda}, \bar{q})$ be a properly efficient solution of $(\text{MFD})_W$, and fix $\lambda = \bar{\lambda}$ in $(\text{MFP})_S$. Suppose that

(a) $\nabla_x \Phi_i(\bar{u}, \bar{v}, 0) = \nabla_x \Psi_i(\bar{u}, \bar{v}, 0) = 0$, $\nabla_{q_i} \Phi_i(\bar{u}, \bar{v}, 0) = \nabla_{q_i} \Psi_i(\bar{u}, \bar{v}, 0) = 0$,
$H_i(\bar{u}, \bar{v}, 0) = G_i(\bar{u}, \bar{v}, 0) = 0$, $\Phi_i(\bar{u}, \bar{v}, 0) = \Psi_i(\bar{u}, \bar{v}, 0) = 0$, $\nabla_y \Phi_i(\bar{u}, \bar{v}, 0) = \nabla_y \Psi_i(\bar{u}, \bar{v}, 0) = 0$,
$\nabla_{p_i} H_i(\bar{u}, \bar{v}, 0) = \nabla_{p_i} G_i(\bar{u}, \bar{v}, 0) = 0, \quad i = 1, \ldots, k.$

(b) For all $i \in \{1, ..., k\}$,

$$f_i(\bar{u}, \bar{v}) - s(\bar{v}|D_i) + \bar{u}^T \bar{w}_i + \Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{q}_i^T \nabla_{q_i} \Phi_i(\bar{u}, \bar{v}, \bar{q}_i) > 0.$$

(c) (i) $\nabla_{q_iq_i}\Phi_i(\bar{u},\bar{v},\bar{q}_i) - \bar{W}_i\nabla_{q_iq_i}\Psi_i(\bar{u},\bar{v},\bar{q}_i) \neq 0$, for $\bar{q}_i = 0$, $i = 1, \ldots, k$, and $\nabla_{q_iq_i}\Phi_i(\bar{u},\bar{v},\bar{q}_i) - \bar{W}_i\nabla_{q_iq_i}\Psi_i(\bar{u},\bar{v},\bar{q}_i)$ is nonsingular for all $i = 1, \ldots, k$, and

(ii) $\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{xx}f_i(\bar{u},\bar{v}) - \bar{W}_i\nabla_{xx}g_i(\bar{u},\bar{v}))$ is positive definite and $\bar{q}_i^T((\nabla_x\Phi_i(\bar{u},\bar{v},\bar{q}_i) - \bar{W}_i\nabla_x\Psi_i(\bar{u},\bar{v},\bar{q}_i)) - (\nabla_{q_i}\Phi_i(\bar{u},\bar{v},\bar{q}_i) - \bar{W}_i\nabla_{q_i}\Psi_i(\bar{u},\bar{v},\bar{q}_i))) \geqq 0$ for all $i = 1, \ldots, k$, or $\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{xx}f_i(\bar{u},\bar{v}) - \bar{W}_i\nabla_{xx}g_i(\bar{u},\bar{v}))$ is negative definite and $\bar{q}_i^T((\nabla_x\Phi_i(\bar{u},\bar{v},\bar{q}_i) - \bar{W}_i\nabla_x\Psi_i(\bar{u},\bar{v},\bar{q}_i)) - (\nabla_{q_i}\Phi_i(\bar{u},\bar{v},\bar{q}_i) - \bar{W}_i\nabla_{q_i}\Psi_i(\bar{u},\bar{v},\bar{q}_i))) \leqq 0$ for all $i = 1, \ldots, k$.

(iii) $\{\nabla_x f_i(\bar{u},\bar{v}) + \bar{w}_i + \nabla_{q_i}\Phi_i(\bar{u},\bar{v},\bar{q}_i) - \bar{W}_i(\nabla_x g_i(\bar{u},\bar{v}) - \bar{t}_i + \nabla_{q_i}\Psi_i(\bar{u},\bar{v},\bar{q}_i)) : i = 1, \ldots, k\}$ is linearly independent.

Then $\bar{q} = 0$, and there exist $\bar{z}_i \in D_i$ and $\bar{r}_i \in F_i$, $i = 1, \ldots, k$ such that $(\bar{u},\bar{v},\bar{W},\bar{z}_1,\ldots,\bar{z}_k,\bar{r}_1,\ldots,\bar{r}_k,\bar{\lambda},\bar{p} = 0)$ is a feasible solution of $(\text{MFP})_S$. Furthermore, if the hypotheses in Theorem 3.1 are satisfied, then $(\bar{u},\bar{v},\bar{W},\bar{z}_1,\ldots,\bar{z}_k,\bar{r}_1,\ldots,\bar{r}_k,\bar{\lambda},\bar{p} = 0)$ is a properly efficient solution of $(\text{MFP})_S$, and the two objective values are equal. □

Remark 3.1.(1) If $k = 1$, $H_1(x,y,p_1) = \frac{1}{2}p_1^T\nabla_{yy}f_1(x,y)p_1$, $\Phi_1(u, v, q_1) = \frac{1}{2}q_1^T\nabla_{xx}f_1(u, v)q_1$, $\Phi_1(u, v, q_1) = \frac{1}{2}q_1^T\nabla_{xx}f_1(u, v)q_1$, and $g_1(u,v) + s(v|F_1) - u^Tt_1 + \Psi_1(u,v,q_1) - q_1^T\nabla_{q1}\Psi_1(u,v,q_1) = 1$, then $(\text{MFP})_S$ and $(\text{MFD})_W$ becomes the problems considered by Hou and Yang [17].

(2) If $k = 1$, $g_1(x,y) - s(x|E_1) + y^Tr_1 + G_1(x,y,p_1) - p_1^T\nabla_{p_1}G_1(x,y,p_1) = 1$, and $g_1(u,v) + s(v|F_1) - u^Tt_1 + \Psi_1(u,v,q_1) - q_1^T\nabla_{q1}\Psi_1(u,v,q_1) = 1$, then $(\text{MFP})_S$ and $(\text{MFD})_W$ becomes the problems considered by Mishra [18].

(3) If $g_i(x, y) - s(x|E_i) + y^Tr_i + G_i(x, y,p_i) - p_i^T\nabla_{p_i}G_i(x, y,p_i) = 1$, and $g_i(u, v) + s(v|F_i) - u^Tt_i + \Psi_i(u, v, q_i) - q_i^T\nabla_{q_i}\Psi_i(u, v, q_i) = 1$ for all i {1, ..., k}, then $(\text{MFP})_S$ and $(\text{MFD})_W$ becomes the problems considered by Chen [14].

(4) If $g_i(x, y) - s(x|E_i) + y^Tr_i + G_i(x, y,p_i) - p_i^T\nabla_{p_i}G_i(x, y,p_i) = 1$, $H_i(x, y,p_i) = \frac{1}{2}p_i^T\nabla_{yy}f_i(x, y)p_i$, $\Phi_i(u, v, q_i) = \frac{1}{2}q_i^T\nabla_{xx}f_i(u, v)q_i$, $H_i(x, y,p_i) = \frac{1}{2}p_i^T\nabla_{yy}f_i(x, y)p_i$, $\Phi_i(u, v, q_i) = \frac{1}{2}q_i^T\nabla_{xx}f_i(u, v)q_i$, for all $i \in$ {1, ..., k}, and there is not the condition $\lambda^T e = 1$ in $(\text{MFP})_S$ and $(\text{MFD})_W$, then the two problems reduce to the problems considered by Yang et al. [19].

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Competing interests

The authors declare that they have no competing interests.

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