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On Lyapunov-type inequalities for (p, q)-Laplacian systems

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Abstract

We establish Lyapunov-type inequalities for a system involving one-dimensional (p_i, q_i)-Laplacian operators ($i = 1, 2$). Next, the obtained inequalities are used to derive some geometric properties of the generalized spectrum associated to the considered problem.

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1 Introduction

In this paper, we are concerned with the following system involving one-dimensional (p_i, q_i)-Laplacian operators ($i = 1, 2$):

$$(S): \begin{cases} -(|u'(x)|^{p_1-2}u'(x))' - (|u'(x)|^{q_1-2}u'(x))' = f(x)|u(x)|^{\alpha-2}|v(x)|^{\beta}u(x), \\ -(|v'(x)|^{p_2-2}v'(x))' - (|v'(x)|^{q_2-2}v'(x))' = g(x)|u(x)|^{\alpha}|v(x)|^{\beta-2}v(x) \end{cases}$$

on the interval (a, b), under Dirichlet boundary conditions

$$(DBC): \quad u(a) = u(b) = v(a) = v(b) = 0.$$

System (S) is investigated under the assumptions

$$\alpha \geq 2, \qquad \beta \geq 2, \qquad p_i \geq 2, \qquad q_i \geq 2, \quad i = 1, 2,$$

and

$$\frac{2\alpha}{p_1 + q_1} + \frac{2\beta}{p_2 + q_2} = 1. \tag{1}$$

We suppose also that f and g are two nonnegative real-valued functions such that $(f, g) \in L^1(a, b) \times L^1(a, b)$. We establish a Lyapunov-type inequality for problem (S)-(DBC). Next, we use the obtained inequality to derive some geometric properties of the generalized spectrum associated to the considered problem.

The standard Lyapunov inequality [1] (see also [2]) states that if the boundary value problem

$$\begin{cases} u''(t) + q(t)u(t) = 0, & a < t < b, \\ u(a) = u(b) = 0, \end{cases}$$

has a nontrivial solution, where $q : [a, b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |q(t)| \, dt > \frac{4}{b-a}. \tag{2}$$

Inequality (2) was successfully applied to oscillation theory, stability criteria for periodic differential equations, estimates for intervals of disconjugacy, and eigenvalue bounds for ordinary differential equations. In [3] (see also [4]), Elbert extended inequality (2) to the one-dimensional p-Laplacian equation. More precisely, he proved that, if u is a nontrivial solution of the problem

$$\begin{cases} (|u'|^{p-2}u')' + h(t)|u|^{p-2}u = 0, & a < t < b, \\ u(a) = u(b) = 0, \end{cases}$$

where $1 < p < \infty$ and $h \in L^1(a, b)$, then

$$\int_a^b |h(t)| \, dt > \frac{2^p}{(b-a)^{p-1}}. \tag{3}$$

Observe that for $p = 2$, (3) reduces to (2). Inequality (3) was extended in [5] to the following problem involving the φ-Laplacian operator:

$$\begin{cases} (\varphi(u'))' + w(t)\varphi(u) = 0, & a < t < b, \\ u(a) = u(b) = 0, \end{cases}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a convex nondecreasing function satisfying a Δ_2 condition. In [6], Nápoli and Pinasco considered the quasilinear system of resonant type

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' = f(x)|u(x)|^{\alpha-2}|v(x)|^{\beta}u(x), \\ -(|v'(x)|^{q-2}v'(x))' = g(x)|u(x)|^{\alpha}|v(x)|^{\beta-2}v(x) \end{cases} \tag{4}$$

on the interval (a, b), with Dirichlet boundary conditions

$$u(a) = u(b) = v(a) = v(b) = 0. \tag{5}$$

Under the assumptions $p, q > 1, f, g \in L^1(a, b), f, g \geq 0, \alpha, \beta \geq 0$, and

$$\frac{\alpha}{p} + \frac{\beta}{q} = 1,$$

it was proved (see [6], Theorem 1.5) that if (4)-(5) has a nontrivial solution, then

$$2^{\alpha+\beta} \le (b-a)^{\frac{\alpha}{p'}+\frac{\beta}{q'}} \left(\int_a^b f(x)\,dx \right)^{\frac{\alpha}{p}} \left(\int_a^b g(x)\,dx \right)^{\frac{\beta}{q}}, \tag{6}$$

where $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$. Some nice applications to generalized eigenvalues are also presented in [6]. Different generalizations and extensions of inequality (6) were obtained by many authors. In this direction, we refer the reader to [7–16] and the references therein. For other results concerning Lyapunov-type inequalities, we refer the reader to [17–29] and the references therein.

2 Lyapunov-type inequalities

A Lyapunov-type inequality for problem (S)-(DBC) is established in this section, and some particular cases are discussed.

Theorem 2.1 *If* (S)-(DBC) *admits a nontrivial solution* $(u,v) \in C^2[a,b] \times C^2[a,b]$, *then*

$$\left[\min\left\{ \frac{2^{p_1}}{(b-a)^{p_1-1}}, \frac{2^{q_1}}{(b-a)^{q_1-1}} \right\} \right]^{\frac{2\alpha}{p_1+q_1}} \left[\min\left\{ \frac{2^{p_2}}{(b-a)^{p_2-1}}, \frac{2^{q_2}}{(b-a)^{q_2-1}} \right\} \right]^{\frac{2\beta}{p_2+q_2}}$$
$$\le \left(\frac{1}{2} \int_a^b f(x)\,dx \right)^{\frac{2\alpha}{p_1+q_1}} \left(\frac{1}{2} \int_a^b g(x)\,dx \right)^{\frac{2\beta}{p_2+q_2}}. \tag{7}$$

Proof Let $(u,v) \in C^2[a,b] \times C^2[a,b]$ be a nontrivial solution to (S)-(DBC). Let $(x_0, y_0) \in (a,b) \times (a,b)$ be such that

$$\left| u(x_0) \right| = \max\left\{ \left| u(x) \right| : a \le x \le b \right\}$$

and

$$\left| v(y_0) \right| = \max\left\{ \left| v(x) \right| : a \le x \le b \right\}.$$

From the boundary conditions (DBC), we can write that

$$2u(x_0) = \int_a^{x_0} u'(x)\,dx - \int_{x_0}^b u'(x)\,dx,$$

which yields

$$2 \left| u(x_0) \right| \le \int_a^b \left| u'(x) \right| dx.$$

Using Hölder's inequality with parameters p_1 and $p_1' = \frac{p_1}{p_1-1}$, we get

$$2 \left| u(x_0) \right| \le (b-a)^{\frac{1}{p_1'}} \left(\int_a^b \left| u'(x) \right|^{p_1} dx \right)^{\frac{1}{p_1}},$$

that is,

$$\frac{2^{p_1}}{(b-a)^{p_1-1}}\big|u(x_0)\big|^{p_1} \leq \int_a^b \big|u'(x)\big|^{p_1}\,dx. \tag{8}$$

Similarly, using Hölder's inequality with parameters q_1 and $q_1' = \frac{q_1}{q_1-1}$, we get

$$\frac{2^{q_1}}{(b-a)^{q_1-1}}\big|u(x_0)\big|^{q_1} \leq \int_a^b \big|u'(x)\big|^{q_1}\,dx. \tag{9}$$

By repeating the same argument for the function v, we obtain

$$\frac{2^{p_2}}{(b-a)^{p_2-1}}\big|v(y_0)\big|^{p_2} \leq \int_a^b \big|v'(x)\big|^{p_2}\,dx \tag{10}$$

and

$$\frac{2^{q_2}}{(b-a)^{q_2-1}}\big|v(y_0)\big|^{q_2} \leq \int_a^b \big|v'(x)\big|^{q_2}\,dx. \tag{11}$$

Now, multiplying the first equation of (S) by u and integrating over (a,b), we obtain

$$\int_a^b \big|u'(x)\big|^{p_1}\,dx + \int_a^b \big|u'(x)\big|^{q_1}\,dx = \int_a^b f(x)\big|u(x)\big|^\alpha\big|v(x)\big|^\beta\,dx. \tag{12}$$

Multiplying the second equation of (S) by v and integrating over (a,b), we obtain

$$\int_a^b \big|v'(x)\big|^{p_2}\,dx + \int_a^b \big|v'(x)\big|^{q_2}\,dx = \int_a^b g(x)\big|u(x)\big|^\alpha\big|v(x)\big|^\beta\,dx. \tag{13}$$

Using (8), (9) and (12), we obtain

$$\big|u(x_0)\big|^\alpha\big|v(y_0)\big|^\beta \int_a^b f(x)\,dx \geq \frac{2^{p_1}}{(b-a)^{p_1-1}}\big|u(x_0)\big|^{p_1} + \frac{2^{q_1}}{(b-a)^{q_1-1}}\big|u(x_0)\big|^{q_1},$$

which yields

$$\big|u(x_0)\big|^\alpha\big|v(y_0)\big|^\beta \int_a^b f(x)\,dx \geq \min\left\{\frac{2^{p_1}}{(b-a)^{p_1-1}}, \frac{2^{q_1}}{(b-a)^{q_1-1}}\right\}\left(\big|u(x_0)\big|^{p_1} + \big|u(x_0)\big|^{q_1}\right).$$

Using the inequality

$$A + B \geq 2\sqrt{A}\sqrt{B}$$

with $A = |u(x_0)|^{p_1}$ and $B = |u(x_0)|^{q_1}$, we get

$$\min\left\{\frac{2^{p_1+1}}{(b-a)^{p_1-1}}, \frac{2^{q_1+1}}{(b-a)^{q_1-1}}\right\} \leq \big|u(x_0)\big|^{\alpha-\frac{p_1+q_1}{2}}\big|v(y_0)\big|^\beta \int_a^b f(x)\,dx. \tag{14}$$

Similarly, using (10), (11) and (13), we obtain

$$\min\left\{\frac{2^{p_2+1}}{(b-a)^{p_2-1}}, \frac{2^{q_2+1}}{(b-a)^{q_2-1}}\right\} \leq \big|u(x_0)\big|^\alpha\big|v(y_0)\big|^{\beta-\frac{p_2+q_2}{2}} \int_a^b g(x)\,dx. \tag{15}$$

Raising inequality (14) to a power $e_1 > 0$, inequality (15) to a power $e_2 > 0$, and multiplying the resulting inequalities, we obtain

$$\left[\min\left\{\frac{2^{p_1+1}}{(b-a)^{p_1-1}}, \frac{2^{q_1+1}}{(b-a)^{q_1-1}}\right\}\right]^{e_1}\left[\min\left\{\frac{2^{p_2+1}}{(b-a)^{p_2-1}}, \frac{2^{q_2+1}}{(b-a)^{q_2-1}}\right\}\right]^{e_2}$$
$$\leq |u(x_0)|^{(\alpha-\frac{p_1+q_1}{2})e_1+\alpha e_2}|v(y_0)|^{\beta e_1+(\beta-\frac{p_2+q_2}{2})e_2}\left(\int_a^b f(x)\,dx\right)^{e_1}\left(\int_a^b g(x)\,dx\right)^{e_2}.$$

Next, we take (e_1, e_2) any solution of the homogeneous linear system

$$\begin{cases} (\alpha - \frac{p_1+q_1}{2})e_1 + \alpha e_2 = 0, \\ \beta e_1 + (\beta - \frac{p_2+q_2}{2})e_2 = 0. \end{cases}$$

Using (1), we may take

$$\begin{cases} e_1 = \alpha, \\ e_2 = \frac{\beta(p_1+q_1)}{p_2+q_2}. \end{cases}$$

Therefore, we obtain

$$2^{\alpha+\frac{\beta(p_1+q_1)}{p_2+q_2}}\left[\min\left\{\frac{2^{p_1}}{(b-a)^{p_1-1}}, \frac{2^{q_1}}{(b-a)^{q_1-1}}\right\}\right]^{\alpha}\left[\min\left\{\frac{2^{p_2}}{(b-a)^{p_2-1}}, \frac{2^{q_2}}{(b-a)^{q_2-1}}\right\}\right]^{\frac{\beta(p_1+q_1)}{p_2+q_2}}$$
$$\leq \left(\int_a^b f(x)\,dx\right)^{\alpha}\left(\int_a^b g(x)\,dx\right)^{\frac{\beta(p_1+q_1)}{p_2+q_2}}.$$

Using again (1), we get

$$2\left[\min\left\{\frac{2^{p_1}}{(b-a)^{p_1-1}}, \frac{2^{q_1}}{(b-a)^{q_1-1}}\right\}\right]^{\frac{2\alpha}{p_1+q_1}}\left[\min\left\{\frac{2^{p_2}}{(b-a)^{p_2-1}}, \frac{2^{q_2}}{(b-a)^{q_2-1}}\right\}\right]^{\frac{2\beta}{p_2+q_2}}$$
$$\leq \left(\int_a^b f(x)\,dx\right)^{\frac{2\alpha}{p_1+q_1}}\left(\int_a^b g(x)\,dx\right)^{\frac{2\beta}{p_2+q_2}},$$

which proves Theorem 2.1. $\qquad\square$

As a consequence of Theorem 2.1, we deduce the following result for the case of a single equation.

Corollary 1 *Let us assume that there exists a nontrivial solution of*

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' - (|u'(x)|^{q-2}u'(x))' = f(x)|u(x)|^{\frac{p+q}{2}-2}u(x), \quad x \in (a,b), \\ u(a) = u(b) = 0, \end{cases}$$

where $p > 1, q > 1, f \geq 0$, and $f \in L^1(a,b)$. Then

$$\min\left\{\frac{2^p}{(b-a)^{p-1}}, \frac{2^q}{(b-a)^{q-1}}\right\} \leq \frac{1}{2}\int_a^b f(x)\,dx.$$

Proof An application of Theorem 2.1 with

$$p_1 = p_2 = p, \qquad q_1 = q_2 = q, \qquad \alpha = \frac{p+q}{2}, \qquad \beta = 0, \qquad v = u, \qquad g = f,$$

yields the desired result. □

Remark 1 Taking $f = 2h$ and $q = p$ in Corollary 1, we obtain Lyapunov-type inequality (3) for the one-dimensional p-Laplacian equation.

Remark 2 Taking $p_1 = q_1 = p$ and $p_2 = q_2 = q$ in Theorem 2.1, we obtain Lyapunov-type inequality (6).

3 Generalized eigenvalues

The concept of generalized eigenvalues was introduced by Protter [30] for a system of linear elliptic operators. The first work dealing with generalized eigenvalues for p-Laplacian systems is due to Nápoli and Pinasco [6]. Inspired by that work, we present in this section some applications to generalized eigenvalues related to problem (S)-(DBC).

Let us consider the generalized eigenvalue problem

$$(S)_{\lambda,\mu}: \begin{cases} -(|u'(x)|^{p_1-2}u'(x))' - (|u'(x)|^{q_1-2}u'(x))' = \lambda\alpha w(x)|u(x)|^{\alpha-2}|v(x)|^{\beta}u(x), \\ -(|v'(x)|^{p_2-2}v'(x))' - (|v'(x)|^{q_2-2}v'(x))' = \mu\beta w(x)|u(x)|^{\alpha}|v(x)|^{\beta-2}v(x), \end{cases}$$

on the interval (a,b), with Dirichlet boundary conditions (DBC). If problem $(S)_{\lambda,\mu}$-(DBC) admits a nontrivial solution $(u,v) \in C^2[a,b] \times C^2[a,b]$, we say that (λ,μ) is a generalized eigenvalue of $(S)_{\lambda,\mu}$-(DBC). The set of generalized eigenvalues is called generalized spectrum, and it is denoted by σ.

We assume that

$$\alpha \geq 2, \qquad \beta \geq 2, \qquad p_i \geq 2, \qquad q_i \geq 2, \quad i = 1, 2, \qquad w \geq 0, \qquad w \in L^1(a,b),$$

and (1) is satisfied.

The following result provides lower bounds of the generalized eigenvalues of $(S)_{\lambda,\mu}$-(DBC).

Theorem 3.1 *Let (λ,μ) be a generalized eigenvalue of $(S)_{\lambda,\mu}$-(DBC). Then*

$$\mu \geq h(\lambda), \tag{16}$$

where $h : (0,\infty) \to (0,\infty)$ is the function defined by

$$h(t) = \frac{1}{\beta}\left(\frac{C}{t^{\frac{2\alpha}{p_1+q_1}}\int_a^b w(x)\,dx}\right)^{\frac{p_2+q_2}{2\beta}}, \quad t > 0,$$

with

$$\alpha^{\frac{2\alpha}{p_1+q_1}} C = 2\left[\min\left\{\frac{2^{p_1}}{(b-a)^{p_1-1}}, \frac{2^{q_1}}{(b-a)^{q_1-1}}\right\}\right]^{\frac{2\alpha}{p_1+q_1}}$$

$$\times \left[\min\left\{\frac{2^{p_2}}{(b-a)^{p_2-1}}, \frac{2^{q_2}}{(b-a)^{q_2-1}}\right\}\right]^{\frac{2\beta}{p_2+q_2}}.$$

Proof Let (λ, μ) be a generalized eigenpair, and let u, v be the corresponding nontrivial solutions. By replacing in Lyapunov-type inequality (7) the functions

$$f(x) = \alpha \lambda w(x), \qquad g(x) = \beta \mu w(x),$$

and using condition (1), we obtain

$$2M \leq \alpha^{\frac{2\alpha}{p_1+q_1}} \lambda^{\frac{2\alpha}{p_1+q_1}} \beta^{\frac{2\beta}{p_2+q_2}} \mu^{\frac{2\beta}{p_2+q_2}} \int_a^b w(x)\, dx,$$

where

$$M = \left[\min\left\{ \frac{2^{p_1}}{(b-a)^{p_1-1}}, \frac{2^{q_1}}{(b-a)^{q_1-1}} \right\} \right]^{\frac{2\alpha}{p_1+q_1}} \left[\min\left\{ \frac{2^{p_2}}{(b-a)^{p_2-1}}, \frac{2^{q_2}}{(b-a)^{q_2-1}} \right\} \right]^{\frac{2\beta}{p_2+q_2}}.$$

Hence, we have

$$\mu^{\frac{2\beta}{p_2+q_2}} \geq \frac{C}{\lambda^{\frac{2\alpha}{p_1+q_1}} \beta^{\frac{2\beta}{p_2+q_2}} \int_a^b w(x)\, dx},$$

which yields

$$\mu \geq \frac{1}{\beta} \left(\frac{C}{\lambda^{\frac{2\alpha}{p_1+q_1}} \int_a^b w(x)\, dx} \right)^{\frac{p_2+q_2}{2\beta}},$$

and the proof is finished. □

As consequences of the previous obtained result, we deduce the following Protter's type results for the generalized spectrum.

Corollary 2 *There exists a constant $c_{a,b} > 0$ that depends on a and b such that no point of the generalized spectrum σ is contained in the ball $B(0, c_{a,b})$, where*

$$B(0, c_{a,b}) = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : \|x\|_\infty < c_{a,b} \right\},$$

and $\|\cdot\|_\infty$ is the Chebyshev norm in $\mathbb{R}^2$.

Proof Let $(\lambda, \mu) \in \sigma$. From (16), we obtain easily that

$$\lambda^{\frac{2\alpha}{p_1+q_1}} \mu^{\frac{2\beta}{p_2+q_2}} \geq \frac{C}{\beta^{\frac{2\beta}{p_2+q_2}} \int_a^b w(x)\, dx}. \tag{17}$$

On the other hand, using condition (1), we have

$$\lambda^{\frac{2\alpha}{p_1+q_1}} \mu^{\frac{2\beta}{p_2+q_2}} \leq \left\| (\lambda, \mu) \right\|_\infty^{\frac{2\alpha}{p_1+q_1} + \frac{2\beta}{p_2+q_2}} = \left\| (\lambda, \mu) \right\|_\infty.$$

Therefore, we obtain

$$\left\| (\lambda, \mu) \right\|_\infty \geq c_{a,b},$$

where

$$c_{a,b} = \frac{C}{\beta^{\frac{2\beta}{p_2+q_2}} \int_a^b w(x)\,dx}.$$

The proof is finished. □

Corollary 3 *Let* (λ, μ) *be fixed. There exists an interval* J *of sufficiently small measure such that, if* $I = [a, b] \subset J$, *then there are no nontrivial solutions of* $(S)_{\lambda,\mu}$-(DBC).

Proof Suppose that $(S)_{\lambda,\mu}$-(DBC) admits a nontrivial solution. Since $C \to +\infty$ as $b - a \to 0^+$, where C is defined in Theorem 3.1, there exists $\delta > 0$ such that

$$b - a < \delta \quad \implies \quad \frac{C}{\int_a^b w(x)\,dx} > \lambda^{\frac{2\alpha}{p_1+q_1}} \mu^{\frac{2\beta}{p_2+q_2}} \beta^{\frac{2\beta}{p_2+q_2}}.$$

Let $J = [a, a + \delta]$. Hence, if $I \subset J$, we have

$$\frac{C}{\beta^{\frac{2\beta}{p_2+q_2}} \int_a^b w(x)\,dx} > \lambda^{\frac{2\alpha}{p_1+q_1}} \mu^{\frac{2\beta}{p_2+q_2}},$$

which is a contradiction with (17). Therefore, if $I \subset J$, there are no nontrivial solutions of $(S)_{\lambda,\mu}$-(DBC). □

4 Conclusion

Lyapunov-type inequalities for a system of differential equations involving one-dimensional (p_i, q_i)-Laplacian operators $(i = 1, 2)$ are derived. It was shown that such inequalities are very useful to obtain geometric characterizations of the generalized spectrum associated to the considered problem.

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