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A note on solution lower semicontinuity for parametric generalized vector equilibrium problems

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Abstract

By using a density technique, sufficient conditions for lower semicontinuity of strong solutions to a parametric generalized vector equilibrium problem are established, where the monotonicity is not necessary. The obtained results are different from the corresponding ones in the literature (Gong and Yao in *J. Optim. Theory Appl.* 138:197-205, 2008; Gong in *J. Optim. Theory Appl.* 139:35-46, 2008; Chen and Li in *Pac. J. Optim.* 6:141-151, 2010; Li and Fang in *J. Optim. Theory Appl.* 147:507-515, 2010; Gong and Yao in *J. Optim. Theory Appl.* 138:189-196, 2008). Some examples are given to illustrate the results.

MSC: 49K40; 90C29; 90C31**Keywords:** stability; strong solution; lower semicontinuity; parametric vector equilibrium problem; density

1 Introduction

It is well known that the vector equilibrium problem (VEP, in short) is a very general mathematical model, which embraces the formats of several disciplines, as those for Nash equilibria, those from Game Theory, those from (Vector) Optimization and (Vector) Variational Inequalities, and so on (see [1-4]).

The stability analysis of solution maps for parametric vector equilibrium problems (PVEP, in short) is an important topic in optimization theory and applications. There are some papers discussing the upper and/or lower semicontinuity of solution maps. Cheng and Zhu [5] obtained a result on the lower semicontinuity of the solution set map to a PVEP in finite-dimensional spaces by using a scalarization method. Huang *et al.* [6] used local existence results of the models considered and additional assumptions to establish the lower semicontinuity of solution mappings for parametric implicit vector equilibrium problems. Recently, by virtue of a density result and scalarization technique, Gong and Yao [7] have first discussed the lower semicontinuity of efficient solutions to parametric vector equilibrium problems, which are called generalized systems in their paper. By using the idea of Cheng and Zhu [5], Gong [8] has discussed the continuity of the solution maps to a weak PVEP in Hausdorff topological vector spaces. Kimura and Yao [9] discussed the semicontinuity of solution maps for parametric vector quasi-equilibrium problems by virtue of the closedness or openness assumptions for some certain sets. Xu and Li [10] proved the lower semicontinuity for PVEP by using a new proof method, which is different

from the one used in Gong and Yao [7]. Chen and Li [11] studied the continuity of solution sets for parametric generalized systems without the uniform compactness assumption, which improves the corresponding results in [8]. We observe that the semicontinuity of solution maps for the PVEP has been discussed under the assumption of C -strict (strong) monotonicity, which implies that the f -solution set of the PVEP is a singleton for a linear continuous functional f (see [5, 7, 8, 10, 11]). However, it is well known that the f -solution set of (weak) PVEPs should be general, but not a singleton. Moreover, to the best of our knowledge, there are few results of semicontinuity have been established for strong solution maps of PVEP in the literature. So, in this paper, by using a density skill, we aim at studying the lower semicontinuity of the strong solution map for a class of parametric generalized vector equilibrium problems (PGVEPs), when the f -solution set is a general set by removing the assumption of C -strict monotonicity.

The rest of the paper is organized as follows. In Section 2, we introduce a class of parametric generalized vector equilibrium problem, and recall some concepts and their properties. In Section 3, by the density and scalarization technique, we discuss the lower semicontinuity of strong solution mappings to the PGVEP, and compare our main results with the corresponding ones in the recent literature [7, 8, 10, 11]. We also give some examples to illustrate our results.

2 Preliminaries

Throughout this paper, unless specified otherwise, let X, Y be normed spaces and Z be Banach space. Let Y^* be the topological dual space of Y , and C be a closed convex pointed cone in Y with nonempty topological interior $\text{int } C$.

Let

$$C^* := \{f \in Y^* : f(y) \geq 0, \forall y \in C\}$$

be the dual cone of C . Denote the quasi-interior of C^* by C^\sharp , i.e.,

$$C^\sharp := \{f \in Y^* : f(y) > 0, \forall y \in C \setminus \{0\}\}.$$

Let A be a nonempty subset of X and $F : A \times A \rightarrow Y$ be a vector-valued mapping. We consider the following generalized vector equilibrium problem:

$$\text{Find } x \in A \text{ such that } F(x, y) \notin -K, \quad \forall y \in A,$$

where $K \cup \{0\}$ is a convex cone in Y .

When the subset A and the mapping F are perturbed by a parameter $\mu \in \Lambda$, in which Λ is a nonempty subset of Z , we consider the following parametric generalized vector equilibrium problem (PGVEP):

$$\text{Find } x \in A(\mu) \text{ such that } F(x, y, \mu) \notin -K, \quad \forall y \in A(\mu),$$

where $A : \Lambda \rightarrow 2^X \setminus \{\emptyset\}$ is a set-valued mapping, $F : B \times B \times \Lambda \subset X \times X \times Z \rightarrow Y$ is a vector-valued mapping with $A(\Lambda) = \bigcup_{\mu \in \Lambda} A(\mu) \subset B$.

Definition 2.1 [12] A vector $x \in A(\mu)$ is called a weak solution to the (PGVEP), iff

$$F(x, y, \mu) \notin -\text{int } C, \quad \forall y \in A(\mu).$$

The set of the weak solutions to the (PGVEP) is denoted by $V_w(\mu)$.

Definition 2.2 A vector $x \in A(\mu)$ is called a strong solution to the (PGVEP), iff

$$F(x, y, \mu) \notin -C \setminus \{0\}, \quad \forall y \in A(\mu).$$

The set of the strong solutions to the (PGVEP) is denoted by $V_s(\mu)$.

Definition 2.3 [7] Let $f \in C^* \setminus \{0\}$. A vector $x \in A(\mu)$ is called an f -solution to the (PGVEP), iff

$$f(F(x, y, \mu)) \geq 0, \quad \forall y \in A(\mu).$$

The set of the f -solution to the (PGVEP) is denoted by $V_f(\mu)$.

Definition 2.4 Let $F : X \times X \times \Lambda \rightarrow Y$ be a vector-valued mapping.

- (i) $F(\cdot, \cdot, \cdot)$ is called C -monotone on $A(\mu) \times A(\mu) \times \Lambda$, iff for any given $\mu \in \Lambda$, for each $x, y \in A(\mu)$, $F(x, y, \mu) + F(y, x, \mu) \in -C$.
- (ii) $F(\cdot, \cdot, \cdot)$ is called C -strictly monotone (i.e., C -strongly monotone in [13]) on $A(\mu) \times A(\mu) \times \Lambda$, iff F is a C -monotone on $A(\mu) \times A(\mu) \times \Lambda$, and for any given $\mu \in \Lambda$, for each $x, y \in A(\mu)$ with $x \neq y$, $F(x, y, \mu) + F(y, x, \mu) \in -\text{int } C$.
- (iii) $F(x, \cdot, \mu)$ is called C -convex if, for each $x_1, x_2 \in A(\mu)$ and $t \in [0, 1]$,
 $tF(x, x_1, \mu) + (1 - t)F(x, x_2, \mu) \in F(x, tx_1 + (1 - t)x_2, \mu) + C$.
- (iv) $F(x, \cdot, \mu)$ is called C -like-convex on $A(\mu)$, iff for any $x_1, x_2 \in A(\mu)$ and any $t \in [0, 1]$, there exists $x_3 \in A(\mu)$ such that $tF(x, x_1, \mu) + (1 - t)F(x, x_2, \mu) \in F(x, x_3, \mu) + C$.
- (v) A set $D \subset Y$ is called a C -convex set, iff $D + C$ is a convex set in Y .

Throughout this paper, we always assume $V_w(\mu) \neq \emptyset$ and $V_s(\mu) \neq \emptyset$ for all $\mu \in \Lambda$. This paper aims at investigating the semicontinuity of the strong solution mappings to (PGVEP).

Lemma 2.1 [14] Let $F(x, A(\mu), \mu)$ be a C -convex set for each $\mu \in \Lambda$ and $x \in A(\mu)$. If $\text{int } C \neq \emptyset$, then $V_w(\mu) = \bigcup_{f \in C^* \setminus \{0\}} V_f(\mu)$.

The notation $B(\bar{\lambda}, \delta)$ denotes the open ball with center $\bar{\lambda} \in \Lambda$ and radius $\delta > 0$. Let $F : \Lambda \rightarrow 2^X$ be a set-valued mapping, and let there be given $\bar{\lambda} \in \Lambda$.

Definition 2.5 [15]

- (i) F is called lower semicontinuous (l.s.c., for short) at $\bar{\lambda}$, iff for any open set V satisfying $V \cap F(\bar{\lambda}) \neq \emptyset$, there exists $\delta > 0$, such that for every $\lambda \in B(\bar{\lambda}, \delta)$, $V \cap F(\lambda) \neq \emptyset$.
- (ii) F is called upper semicontinuous (u.s.c., for short) at $\bar{\lambda}$, iff for any open set V satisfying $F(\bar{\lambda}) \subset V$, there exists $\delta > 0$, such that for every $\lambda \in B(\bar{\lambda}, \delta)$, $F(\lambda) \subset V$.

We say F is l.s.c. (resp. u.s.c.) on Λ , iff it is l.s.c. (resp. u.s.c.) at each $\lambda \in \Lambda$. F is said to be continuous on Λ , iff it is both l.s.c. and u.s.c. on Λ .

Proposition 2.1 [15]

- (i) F is l.s.c. at $\bar{\lambda}$ if and only if for any sequence $\{\lambda_n\} \subset \Lambda$ with $\lambda_n \rightarrow \bar{\lambda}$ and any $\bar{x} \in F(\bar{\lambda})$, there exists $x_n \in F(\lambda_n)$, such that $x_n \rightarrow \bar{x}$.
- (ii) If F has compact values (i.e., $F(\lambda)$ is a compact set for each $\lambda \in \Lambda$), then F is u.s.c. at $\bar{\lambda}$ if and only if for any sequences $\{\lambda_n\} \subset \Lambda$ with $\lambda_n \rightarrow \bar{\lambda}$ and $\{x_n\}$ with $x_n \in F(\lambda_n)$, there exist $\bar{x} \in F(\bar{\lambda})$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $x_{n_k} \rightarrow \bar{x}$.

The following lemma plays an important role in the proof of the lower semicontinuity of the solution map $V_s(\mu)$.

Lemma 2.2 [16, Theorem 2, p.114] *The union $\Gamma = \bigcup_{i \in I} \Gamma_i$ of a family of l.s.c. set-valued mappings Γ_i from a topological space X into a topological space Y is also an l.s.c. set-valued mapping from X into Y , where I is an index set.*

Let $Q: X \rightarrow 2^Y$ be a set-valued mapping between two topological spaces. The lower limit of Q is defined as

$$\text{Liminf}_{x \rightarrow x_0} Q(x) = \{y \in Y : \forall x_\alpha \rightarrow x_0, \exists y_\alpha \in Q(x_\alpha), y_\alpha \rightarrow y\}.$$

Proposition 2.2 [17]

- (i) $\text{Liminf}_{x \rightarrow x_0} Q(x)$ is a closed set.
- (ii) Q is l.s.c. at $x_0 \in \text{dom } Q := \{x | Q(x) \neq \emptyset\}$ if and only if $Q(x_0) \subset \text{Liminf}_{x \rightarrow x_0} Q(x)$.

3 Lower semicontinuity of strong solution sets to (PGVEP)

In this section, we discuss the lower semicontinuity of the strong solutions for (PVEP). Firstly, we obtain two important lemmas relevant to the (PVEP) as follows.

Lemma 3.1 *Let $f \in C^* \setminus \{0\}$. Suppose the following conditions are satisfied:*

- (i) $A(\cdot)$ is continuous with compact convex values on Λ .
- (ii) For each $\mu \in \Lambda$, $F(\cdot, \cdot, \cdot)$ is continuous on $B \times B \times \mu$.
- (iii) For each $\mu \in \Lambda$, $x \in A(\mu) \setminus V_f(\mu)$, there exist $y \in V_f(\mu)$ and $r > 0$, such that

$$F(x, y, \mu) + F(y, x, \mu) + B(0, d^r(x, y)) \subset -C.$$

Then $V_f(\cdot)$ is l.s.c. on Λ .

Proof Suppose to the contrary that there exists $\mu_0 \in \Lambda$, such that $V_f(\cdot)$ is not l.s.c. at μ_0 . Then there exist a sequence $\{\mu_n\}$ with $\mu_n \rightarrow \mu_0$ and $x_0 \in V_f(\mu_0)$, such that for any $x_n \in V_f(\mu_n)$, $x_n \not\rightarrow x_0$.

Since $A(\cdot)$ is l.s.c. at μ_0 , there exists $\bar{x}_n \in A(\mu_n)$, such that $\bar{x}_n \rightarrow x_0$. Obviously, $\bar{x}_n \in A(\mu_n) \setminus V_f(\mu_n)$. By (iii), there exists $y_n \in V_f(\mu_n)$ such that

$$F(\bar{x}_n, y_n, \mu_n) + F(y_n, \bar{x}_n, \mu_n) + B(0, d^r(\bar{x}_n, y_n)) \subset -C. \tag{3.1}$$

For $y_n \in A(\mu_n)$, because $A(\cdot)$ is u.s.c. at μ_0 with compact values, there exist $y_0 \in A(\mu_0)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$, such that $y_{n_k} \rightarrow y_0$. In particular, for (3.1), we have

$$F(\bar{x}_{n_k}, y_{n_k}, \mu_{n_k}) + F(y_{n_k}, \bar{x}_{n_k}, \mu_{n_k}) + B(0, d'(\bar{x}_{n_k}, y_{n_k})) \subset -C. \tag{3.2}$$

Taking the limit as $n_k \rightarrow +\infty$, we have

$$F(x_0, y_0, \mu_0) + F(y_0, x_0, \mu_0) + B(0, d'(x_0, y_0)) \subset -C. \tag{3.3}$$

Noting that $x_0 \in V_f(\mu_0)$ and $y_0 \in A(\mu_0)$, we have

$$f(F(x_0, y_0, \mu_0)) \geq 0. \tag{3.4}$$

Moreover, since $y_{n_k} \in V_f(\mu_{n_k})$ and $\bar{x}_{n_k} \in A(\mu_{n_k})$, it follows from the continuity of f and F that

$$f(F(y_0, x_0, \mu_0)) \geq 0. \tag{3.5}$$

By (3.4), (3.5), and the linearity of f , we have

$$f(F(x_0, y_0, \mu_0) + F(y_0, x_0, \mu_0)) \geq 0. \tag{3.6}$$

Assume that $x_0 \neq y_0$; by (3.3), we obtain $F(x_0, y_0, \mu_0) + F(y_0, x_0, \mu_0) \subset -\text{int } C$. Thus, we have

$$f(F(x_0, y_0, \mu_0) + F(y_0, x_0, \mu_0)) < 0,$$

which contradicts (3.6). Therefore $x_0 = y_0$, which leads to a contradiction. Hence, for each $f \in C^* \setminus \{0\}$, $S_f(\cdot)$ is l.s.c. on Λ . □

Lemma 3.2 *Let $f \in C^* \setminus \{0\}$. Suppose the following conditions are satisfied:*

- (i) A is nonempty compact set.
- (ii) $F(\cdot, \cdot)$ is continuous on $B \times B$.
- (iii) For each $x \in A \setminus V_f$ there exist $y \in V_f$ and $r > 0$, such that

$$F(x, y) + F(y, x) + B(0, d'(x, y)) \subset -C.$$

Let us define the set-valued mapping $H : C^* \setminus \{0\} \rightarrow 2^A$ by

$$H(f) = V_f, \quad f \in C^* \setminus \{0\};$$

then we see that $H(\cdot)$ is l.s.c. on $C^* \setminus \{0\}$.

Proof By using Lemma 3.1 and following a similar way to the proof to Lemma 3.1 of [18], we can obtain the conclusion. □

Now, we discuss the lower semicontinuity of strong solution mappings to (PGVEP).

Theorem 3.1 Let $f \in C^* \setminus \{0\}$. Suppose the following conditions are satisfied:

- (i) $A(\cdot)$ is continuous with compact convex values on Λ .
- (ii) For each $\mu \in \Lambda$ and $x \in A(\mu)$, $F(x, \cdot, \mu)$ is C -like-convex on $A(\mu)$.
- (iii) For each $\mu \in \Lambda$, $F(\cdot, \cdot, \cdot)$ is continuous on $B \times B \times \mu$.
- (iv) For each $\mu \in \Lambda$, $x \in A(\mu) \setminus V_f(\mu)$, there exist $y \in V_f(\mu)$ and $r > 0$, such that

$$F(x, y, \mu) + F(y, x, \mu) + B(0, d^r(x, y)) \subset -C.$$

- (v) $F(A(\mu), A(\mu), \mu)$ are bounded subsets of Y for each $\mu \in \Lambda$.
- (vi) $C^\# \neq \emptyset$ and $\text{int } C \neq \emptyset$.

Then $V_s(\cdot)$ is lower semicontinuous on Λ .

Proof We prove the result in the following three steps.

Step 1. We first show that

$$\bigcup_{f \in C^\#} V_f(\mu) \subset V_s(\mu) \subset \text{cl} \left(\bigcup_{f \in C^\#} V_f(\mu) \right). \tag{3.7}$$

Since $V_f(\mu) \neq \emptyset$, for each $f \in C^* \setminus \{0\}$. Then, by definition, we have

$$\bigcup_{f \in C^\#} V_f(\mu) \subset V_s(\mu) \subset V_w(\mu). \tag{3.8}$$

Since for any $x \in A(\mu)$, $F(x, \cdot, \mu)$ is C -like-convex, then $F(x, A(\mu), \mu) + C$ is a convex set. From Lemma 2.1, we have

$$V_w(\mu) = \bigcup_{f \in C^* \setminus \{0\}} V_f(\mu). \tag{3.9}$$

By (3.7) and (3.8), we get

$$\bigcup_{f \in C^\#} V_f(\mu) \subset V_s(\mu) \subset \bigcup_{f \in C^* \setminus \{0\}} V_f(\mu). \tag{3.10}$$

To show that

$$\bigcup_{f \in C^* \setminus \{0\}} V_f(\mu) \subset \text{cl} \left(\bigcup_{f \in C^\#} V_f(\mu) \right), \tag{3.11}$$

we first prove

$$\bigcup_{f \in C^* \setminus \{0\}} V_f \subset \text{cl} \left(\bigcup_{f \in C^\#} V_f \right). \tag{3.12}$$

Let us define the set-valued mapping $H : C^* \setminus \{0\} \rightarrow 2^A$ by

$$H(f) = V_f, \quad f \in C^* \setminus \{0\}.$$

By Lemma 3.2, we know that $H(\cdot)$ is lower semicontinuous on $C^* \setminus \{0\}$.

Let $x_0 \in \bigcup_{f \in C^* \setminus \{0\}} V_f$. Then there exists $f_0 \in C^* \setminus \{0\}$ such that

$$x_0 \in V_{f_0} = H(f_0).$$

Since $C^\sharp \neq \emptyset$, let $g \in C^\sharp$ and set

$$f_n = f_0 + (1/n)g.$$

Then $f_n \in C^\sharp$. We show that $\{f_n\}$ converges to f_0 with respect to the topology $\beta(Y^*, Y)$.

For any neighborhood U of 0 with respect to $\beta(Y^*, Y)$, there exist bounded subsets $B_i \subset Y$ ($i = 1, 2, \dots, m$) and $\epsilon > 0$ such that

$$\bigcap_{i=1}^m \left\{ f \in Y^* : \sup_{y \in B_i} |f(y)| < \epsilon \right\} \subset U.$$

Since B_i is bounded and $g \in Y^*$, $|g(B_i)|$ is bounded for $i = 1, \dots, m$. Thus, there exists N such that

$$\sup_{y \in B_i} |(1/n)g(y)| < \epsilon, \quad i = 1, \dots, m, n \geq N.$$

Hence $(1/n)g \in U$, that is, $f_n - f_0 \in U$. This means that $\{f_n\}$ converges to f_0 with respect to $\beta(Y^*, Y)$.

Since $H(f)$ is l.s.c. at f_0 , for sequence $\{f_n\} \subset C^* \setminus \{0\}$, $f_n \rightarrow f_0$ and $x_0 \in H(f_0)$, there exists $x_n \in H(f_n) = V_{f_n} \subset \bigcup_{f \in C^\sharp} V_f$, such that $x_n \rightarrow x_0$. This means that

$$x_0 \in cl\left(\bigcup_{f \in C^\sharp} V_f\right).$$

By the arbitrariness of $x_0 \in \bigcup_{f \in C^* \setminus \{0\}} V_f$, we have

$$\bigcup_{f \in C^* \setminus \{0\}} V_f \subset cl\left(\bigcup_{f \in C^\sharp} V_f\right).$$

Then we can obtain the result that for each $\mu \in \Lambda$ (3.11) is valid, and the validity of (3.7) follows readily from (3.10) and (3.11).

Step 2. For each $\mu \in \Lambda$, let $\tilde{S}(\mu) := \bigcup_{f \in C^\sharp} V_f(\mu)$. By a similar argument to the proof of Lemma 3.1, we find for each $f \in C^\sharp$, that $V_f(\cdot)$ is l.s.c. on Λ . It follows from Lemma 2.2 that $\tilde{S}(\cdot)$ is l.s.c. on Λ .

Step 3. Now we show that $V_s(\cdot)$ is lower semicontinuous on Λ . From Step 1, we have

$$\tilde{S}(\mu) \subset V_s(\mu) \subset cl(\tilde{S}(\mu)), \quad \forall \mu \in \Lambda.$$

Let $\mu_0 \in \Lambda$ be any fixed point. Because of the lower semicontinuity of $\tilde{S}(\cdot)$ at μ_0 and the closedness of the lower limit of \tilde{S} (see Proposition 2.2), together with the above inclusion relation, we have

$$V_s(\mu_0) \subset cl(\tilde{S}(\mu_0)) \subset \liminf_{\mu_\alpha \rightarrow \mu_0} \tilde{S}(\mu_\alpha) \subset \liminf_{\mu_\alpha \rightarrow \mu_0} V_s(\mu_\alpha).$$

Hence, $V_s(\cdot)$ is l.s.c. at μ_0 . By the arbitrariness of μ_0 , $V_s(\cdot)$ is l.s.c. on Λ . This completes the proof. \square

Remark 3.2 In Theorem 3.1, by using the density technique, we obtain a sufficient condition for the lower semicontinuity of strong solutions to (PGVEP). Our approach is different from the corresponding ones in [7] (see Theorem 2.1 of [7]). Furthermore, the condition of C -strong (strict) monotonicity is not required, and the C -convexity of F is generalized to the C -like-convexity. Thus, Theorem 3.1 improves and extends the corresponding results in the literature [5, 7, 8, 10, 11]. The following example is given to illustrate the case.

Example 3.1 Let $X = R$, $Y = R^2$, $C = R_+^2$, $\Lambda = [1, 2]$, $A(\mu) = [-1, 1]$, and $F(x, y, \mu) = (1 - 2\mu^2 + y, 2\mu x)$. For any $f \in C^* \setminus \{0\}$, it follows from a direct computation that $V_f(\mu) = [0, 1]$. It is clear that $A(\cdot)$ is a continuous set-valued mapping with nonempty compact convex values and for each $\mu \in \Lambda$, $F(\cdot, \cdot, \cdot)$ is continuous on $A(\mu) \times A(\mu) \times \mu$. Therefore, conditions (i)-(ii) of Theorem 3.1 are satisfied. For any given $\mu \in \Lambda$ and for any given $x \in A(\mu)$, we have, for any $y_1, y_2 \in A(\mu)$, $t \in [0, 1]$,

$$\begin{aligned} & F(x, ty_1 + (1-t)y_2, \mu) \\ &= (1 - 2\mu^2 + ty_1 + (1-t)y_2, 2\mu x) \\ &= t(1 - 2\mu^2 + y_1, 2\mu x) + (1-t)(1 - 2\mu^2 + y_2, 2\mu x) \\ &\in tF(x, y_1, \mu) + (1-t)F(x, y_2, \mu) - C. \end{aligned}$$

Thus, for any given $\mu \in \Lambda$ and for any given $x \in A(\mu)$, $F(x, \cdot, \mu)$ is R_+^2 -like-convex on $A(\mu)$, that is, condition (iii) of Theorem 3.1 is satisfied.

For any $x \in A(\mu) \setminus V_f(\mu)$, there exists $y = 0 \in V_f(\mu)$, such that

$$\begin{aligned} & F(x, y, \mu) + F(y, x, \mu) + B(0, d^r(x, y)) \\ &= (2 - 4\mu^2 + x, 2\mu x) + B(0, d^r(x, 0)) \subset -C. \end{aligned}$$

Thus, the condition (iv) of Theorem 3.1 is satisfied. It is clear that $F(A(\mu), A(\mu), \mu)$ are bounded subsets of Y for each $\mu \in \Lambda$, $(R_+^2)^\# \neq \emptyset$, and $\text{int} R_+^2 \neq \emptyset$. Consequently, by Theorem 3.1, $V_s(\cdot)$ is lower semicontinuous on Λ .

However, the condition of C -strong monotonicity does not hold. Indeed, for any $x \in A(\mu) \setminus V_f(\mu) = [-1, 0)$, there exists $y = -x \in V_f(\mu) = [0, 1]$, such that

$$F(x, y, \mu) + F(y, x, \mu) = (-4\mu^2 + 2, 0) \notin -\text{int} C.$$

Obviously, $F(\cdot, \cdot, \mu)$ is not C -strongly monotone on $A(\mu) \times A(\mu)$. Then Theorem 2.1 in [7] and Theorem 3.2 in [11] are not applicable, and the corresponding results in references [8, Theorem 4.2], [10, Theorem 3.1] are also not applicable.

Finally, we give an example to illustrate that the assumption (iv) of Theorem 3.1 is essential.

Example 3.2 Let $X = R, Y = R^2, C = R_+^2$. Let $\Lambda = [4, 5]$ and

$$A(\mu) = [\mu - 4, 1], \quad \mu \in \Lambda.$$

It is clear that A is a continuous set-valued mapping from Λ to X with nonempty compact convex values. Define the mapping $F : A(\mu) \times A(\mu) \times \Lambda \rightarrow R^2$ by

$$F(x, y, \mu) = \left((30 - \mu^2) \frac{\mu}{5} x(x - y), 0 \right), \quad \forall x, y \in A(\mu), \mu \in \Lambda.$$

Obviously, we know that conditions (ii), (iii), (v), and (vi) are satisfied.

We have, for any $f \in C^* \setminus \{0\}, 0 \in V_f(\mu)$, and

$$V_s(\mu) = V_f(\mu) = \{0, 1\}, \quad \text{if } \mu = 4 \quad \text{and} \quad V_s(\mu) = \{1\}, \quad \text{if } \mu \in (4, 5].$$

In fact, there exists $x = \frac{1}{2} \in A(\mu) \setminus V_f(\mu)$, for $y = 0 \in V_f(\mu)$, we have

$$\begin{aligned} & F(x, y, \mu) + F(y, x, \mu) + B(0, d^r(x, y)) \\ &= \left(\frac{14}{5}, 0 \right) + B\left(0, d^r\left(\frac{1}{2}, 0 \right) \right) \notin -C. \end{aligned}$$

Using a similar method, there exists $x = \frac{1}{2} \in A(\mu) \setminus V_f(\mu)$, for $y = 1 \in V_f(\mu)$, we have $F(x, y, \mu) + F(y, x, \mu) + B(0, d^r(x, y)) \notin -C$. Thus, condition (iv) of Theorem 3.1 is not satisfied.

Now, we show that $V_s(\mu)$ is not lower semicontinuous at $\mu_0 = 4$. There exists $0 \in V_s(4)$ and there exists a neighborhood $(-\frac{1}{2}, \frac{1}{2})$ of 0, for any neighborhood $U(4)$ of 4, there exists $4 < \tilde{\mu} < 5$ such that $\tilde{\mu} \in U(4)$ and

$$V_s(\tilde{\mu}) = 1 \notin \left(-\frac{1}{2}, \frac{1}{2} \right).$$

Thus,

$$V_s(\tilde{\mu}) \cap \left(-\frac{1}{2}, \frac{1}{2} \right) = \emptyset.$$

By virtue of Definition 2.5, we know that $V_s(\mu)$ is not lower semicontinuous at $\mu_0 = 4$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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