Xue *Journal of Inequalities and Applications* 2014, **2014**:247
http://www.journalofinequalitiesandapplications.com/content/2014/1/247

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Rarefied sets at infinity associated with the Schrödinger operator

Gaixian Xue*

*Correspondence:
jingben84@163.com
School of Mathematics and Information Science, Henan University of Economics and Law, Zhengzhou, 450046, China

Abstract

This paper gives some criteria for a-rarefied sets at infinity associated with the Schrödinger operator in a cone. Our proofs are based on estimating Green a-potential with a positive measure by connecting with a kind of density of the modified measure. Meanwhile, the geometrical property of this a-rarefied sets at infinity is also considered. By giving an example, we show that the reverse of this property is not true.

Keywords: rarefied set; Schrödinger operator; Green a-potential

1 Introduction and results

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and the set of all positive real numbers, respectively. We denote by $\mathbf{R}^n$ ($n \geq 2$) the n-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P = (X, x_n)$, $X = (x_1, x_2, \ldots, x_{n-1})$. The Euclidean distance between two points P and Q in $\mathbf{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin O of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary and the closure of a set S in $\mathbf{R}^n$ are denoted by ∂S and $\overline{S}$, respectively.

We introduce a system of spherical coordinates (r, Θ), $\Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to Cartesian coordinates $(x_1, x_2, \ldots, x_{n-1}, x_n)$ by $x_n = r\cos\theta_1$.

Let D be an arbitrary domain in $\mathbf{R}^n$ and let $\mathscr{A}_a$ denote the class of non-negative radial potentials $a(P)$, *i.e.*, $0 \leq a(P) = a(r)$, $P = (r, \Theta) \in D$, such that $a \in L^b_{\mathrm{loc}}(D)$ with some $b > n/2$ if $n \geq 4$ and with $b = 2$ if $n = 2$ or $n = 3$.

If $a \in \mathscr{A}_a$, then the Schrödinger operator

$$Sch_a = -\Delta + a(P)I = 0,$$

where Δ is the Laplace operator and I is the identical operator, can be extended in the usual way from the space $C_0^\infty(D)$ to an essentially self-adjoint operator on $L^2(D)$ (see [1, Ch. 11]). We will denote it by Sch_a as well. This last one has a Green a-function $G^a_D(P, Q)$. Here $G^a_D(P, Q)$ is positive on D and its inner normal derivative $\partial G^a_D(P, Q)/\partial n_Q \geq 0$, where $\partial/\partial n_Q$ denotes the differentiation at Q along the inward normal into D.

We call a function $u \not\equiv -\infty$ that is upper semi-continuous in D a subfunction with respect to the Schrödinger operator Sch_a if its values belong to the interval $[-\infty, \infty)$ and at each point $P \in D$ with $0 < r < r(P)$ the generalized mean-value inequality (see [2])

$$u(P) \leq \int_{S(P,r)} u(Q) \frac{\partial G^a_{B(P,r)}(P, Q)}{\partial n_Q} \, d\sigma(Q)$$

is satisfied, where $G^a_{B(P,r)}(P,Q)$ is the Green a-function of Sch_a in $B(P,r)$ and $d\sigma(Q)$ is a surface measure on the sphere $S(P,r) = \partial B(P,r)$.

If $-u$ is a subfunction, then we call u a superfunction. If a function u is both subfunction and superfunction, it is, clearly, continuous and is called an a-harmonic function (with respect to the Schrödinger operator Sch_a).

The unit sphere and the upper half unit sphere in $\mathbf{R}^n$ are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}^{n-1}_+$, respectively. For simplicity, a point $(1,\Theta)$ on $\mathbf{S}^{n-1}$ and the set $\{\Theta; (1,\Theta) \in \Omega\}$ for a set Ω, $\Omega \subset \mathbf{S}^{n-1}$, are often identified with Θ and Ω, respectively. For two sets $\Xi \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, the set $\{(r,\Theta) \in \mathbf{R}^n; r \in \Xi, (1,\Theta) \in \Omega\}$ in $\mathbf{R}^n$ is simply denoted by $\Xi \times \Omega$. By $C_n(\Omega)$ we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ with the domain Ω on $\mathbf{S}^{n-1}$. We call it a cone. We denote the set $I \times \Omega$ with an interval on $\mathbf{R}$ by $C_n(\Omega; I)$.

We shall say that a set $H \subset C_n(\Omega)$ has a covering $\{r_j, R_j\}$ if there exists a sequence of balls $\{B_j\}$ with centers in $C_n(\Omega)$ such that $H \subset \bigcup_{j=0}^{\infty} B_j$, where r_j is the radius of B_j and R_j is the distance from the origin to the center of B_j.

From now on, we always assume $D = C_n(\Omega)$. For the sake of brevity, we shall write $G^a_\Omega(P,Q)$ instead of $G^a_{C_n(\Omega)}(P,Q)$. Throughout this paper, let c denote various positive constants, because we do not need to specify them. Moreover, ϵ appearing in the expression in the following sections will be a sufficiently small positive number.

Let Ω be a domain on $\mathbf{S}^{n-1}$ with smooth boundary. Consider the Dirichlet problem

$$(\Lambda_n + \lambda)\varphi = 0 \quad \text{on } \Omega,$$
$$\varphi = 0 \quad \text{on } \partial\Omega,$$

where Λ_n is the spherical part of the Laplace operator Δ_n

$$\Delta_n = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\Lambda_n}{r^2}.$$

We denote the least positive eigenvalue of this boundary value problem by λ and the normalized positive eigenfunction corresponding to λ by $\varphi(\Theta)$. In order to ensure the existence of λ and a smooth $\varphi(\Theta)$, we put a rather strong assumption on Ω: if $n \geq 3$, then Ω is a $C^{2,\alpha}$-domain $(0 < \alpha < 1)$ on $\mathbf{S}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (*e.g.*, see [3, pp.88-89] for the definition of $C^{2,\alpha}$-domain).

For any $(1,\Theta) \in \Omega$, we have (see [4, pp.7-8])

$$c^{-1} r\varphi(\Theta) \leq \delta(P) \leq cr\varphi(\Theta), \tag{1}$$

where $P = (r,\Theta) \in C_n(\Omega)$ and $\delta(P) = \operatorname{dist}(P, \partial C_n(\Omega))$.

Solutions of an ordinary differential equation

$$-Q''(r) - \frac{n-1}{r}Q'(r) + \left(\frac{\lambda}{r^2} + a(r)\right)Q(r) = 0, \quad 0 < r < \infty. \tag{2}$$

It is known (see, for example, [5]) that if the potential $a \in \mathscr{A}_a$, then equation (2) has a fundamental system of positive solutions $\{V, W\}$ such that V and W are increasing and decreasing, respectively.

We will also consider the class $\mathscr{B}_a$, consisting of the potentials $a \in \mathscr{A}_a$, such that there exists the finite limit $\lim_{r\to\infty} r^2 a(r) = k \in [0,\infty)$ and, moreover, $r^{-1}|r^2 a(r) - k| \in L(1,\infty)$. If $a \in \mathscr{B}_a$, then the (sub)superfunctions are continuous (see [6]).

In the rest of paper, we assume that $a \in \mathscr{B}_a$ and we shall suppress this assumption for simplicity.

Denote

$$\iota_k^{\pm} = \frac{2 - n \pm \sqrt{(n-2)^2 + 4(k+\lambda)}}{2},$$

then the solutions to equation (2) have the asymptotic (see [3])

$$c^{-1} r^{\iota_k^+} \le V(r) \le c r^{\iota_k^+}, \qquad c^{-1} r^{\iota_k^-} \le W(r) \le c r^{\iota_k^-}, \quad \text{as } r \to \infty. \tag{3}$$

Let ν be any positive measure on $C_n(\Omega)$ such that the Green a-potential

$$G_\Omega^a \nu(P) = \int_{C_n(\Omega)} G_\Omega^a(P,Q)\, d\nu(Q) \not\equiv +\infty$$

for any $P \in C_n(\Omega)$. Then the positive measure $m(\nu)$ on $\mathbf{R}^n$ is defined by

$$dm(\nu)(Q) = \begin{cases} W(t)\varphi(\Phi)\, d\nu(Q), & Q = (t,\Phi) \in C_n(\Omega;(1,+\infty)), \\ 0, & Q \in \mathbf{R}^n - C_n(\Omega;(1,+\infty)). \end{cases}$$

Remark 1 We remark that the total mass $m(\nu)$ is finite (see [2, Lemma 5]).

For each $P = (r,\Theta) \in \mathbf{R}^n - \{O\}$, the maximal function $M(P;\lambda,\beta)$ is defined by

$$M(P;\lambda,\beta) = \sup_{0<\rho<\frac{r}{2}} \frac{\lambda(B(P,\rho))}{\rho^\beta},$$

where $\beta \ge 0$ and λ is a positive measure on $\mathbf{R}^n$. The set

$$\{P = (r,\Theta) \in \mathbf{R}^n - \{O\}; M(P;\lambda,\beta) r^\beta > \epsilon\}$$

is denoted by $E(\epsilon;\lambda,\beta)$.

It is known that the Martin boundary of $C_n(\Omega)$ is the set $\partial C_n(\Omega) \cup \{\infty\}$, each of which is a minimal Martin boundary point. For $P \in C_n(\Omega)$ and $Q \in \partial C_n(\Omega) \cup \{\infty\}$, the Martin kernel can be defined by $M_\Omega^a(P,Q)$. If the reference point P is chosen suitably, then we have

$$M_\Omega^a(P,\infty) = V(r)\varphi(\Theta) \quad \text{and} \quad M_\Omega^a(P,O) = cW(r)\varphi(\Theta) \tag{4}$$

for any $P = (r,\Theta) \in C_n(\Omega)$.

In [7], Long *et al.* introduced the notations of a-thin (with respect to the Schrödinger operator Sch_a) at a point, a-polar set (with respect to the Schrödinger operator Sch_a) and a-rarefied sets at infinity (with respect to the Schrödinger operator Sch_a), which generalized earlier notations obtained by Brelot and Miyamoto (see [8, 9]). A set H in $\mathbf{R}^n$ is said to be a-thin at a point Q if there is a fine neighborhood E of Q which does not intersect $H\setminus\{Q\}$. Otherwise H is said to be not a-thin at Q on $C_n(\Omega)$. A set H in $\mathbf{R}^n$ is called a polar set if there is a superfunction u on some open set E such that $H \subset \{P \in E; u(P) = \infty\}$. A subset H of $C_n(\Omega)$ is said to be a-rarefied at infinity on $C_n(\Omega)$ if there exists a positive superfunction $v(P)$ on $C_n(\Omega)$ such that

$$\inf_{P\in C_n(\Omega)} \frac{v(P)}{M_\Omega^a(P,\infty)} \equiv 0$$

and

$$H \subset \{P = (r,\Theta) \in C_n(\Omega); v(P) \geq V(r)\}.$$

Let H be a bounded subset of $C_n(\Omega)$. Then $\hat{R}^H_{M^a_\Omega(\cdot,\infty)}$ is bounded on $C_n(\Omega)$ and the greatest a-harmonic minorant of $\hat{R}^H_{M^a_\Omega(\cdot,\infty)}$ is zero. We see from the Riesz decomposition theorem (see [10, Theorem 2]) that there exists a unique positive measure λ^a_H on $C_n(\Omega)$ such that (see [7, p.6])

$$\hat{R}^H_{M^a_\Omega(\cdot,\infty)}(P) = G^a_\Omega \lambda^a_H(P) \tag{5}$$

for any $P \in C_n(\Omega)$ and λ^a_H is concentrated on I_H, where

$$I_H = \{P \in C_n(\Omega); H \text{ is not } a\text{-thin at } P\}.$$

We denote the total mass $\lambda^a_H(C_n(\Omega))$ of λ^a_H by $\lambda^a_\Omega(H)$.

By using this positive measure λ^a_H (with respect to the Schrödinger operator Sch_a), we can further define another measure η^a_H on $C_n(\Omega)$ by

$$d\eta^a_H(P) = M^a_\Omega(P,\infty)\, d\lambda^a_H(P)$$

for any $P \in C_n(\Omega)$. It is easy to see that $\eta^a_H(C_n(\Omega)) < +\infty$.

Recently, Long *et al.* (see [7, Theorem 2.5]) gave a criterion for a subset H of $C_n(\Omega)$ to be a-rarefied set at infinity.

Theorem A *A subset H of $C_n(\Omega)$ is a-rarefied at infinity on $C_n(\Omega)$ if and only if*

$$\sum_{j=0}^{\infty} \lambda^a_\Omega(H_j) W(2^j) < \infty,$$

where $H_j = H \cap C_n(\Omega; [2^j, 2^{j+1}))$ and $j = 0, 1, 2, \ldots$.

In this paper, we shall obtain a series of new criteria for a-rarefied sets at infinity on $C_n(\Omega)$, which complement Theorem A. Our results are essentially based on Qiao and Deng, Ren and Zhao, Xue (see [2, 11–14]). In order to avoid complexity of our proofs, we shall assume $n \geq 3$. But our results in this paper are also true for $n = 2$.

First we shall state Theorem 1, which is the main result in this paper.

Theorem 1 *A subset H of $C_n(\Omega)$ is a-rarefied at infinity on $C_n(\Omega)$ if and only if there exists a positive measure ξ_H^a on $C_n(\Omega)$ such that*

$$G_\Omega^a \xi_H^a(P) \not\equiv +\infty \tag{6}$$

for any $P \in C_n(\Omega)$ and

$$H \subset \left\{P = (r, \Theta) \in C_n(\Omega); G_\Omega^a \xi_H^a(P) \geq V(r)\right\}. \tag{7}$$

Next we give the geometrical property of a-rarefied sets at infinity.

Theorem 2 *If a subset H of $C_n(\Omega)$ is a-rarefied at infinity on $C_n(\Omega)$, then H has a covering $\{r_j, R_j\}$ $(j = 0, 1, 2, \ldots)$ satisfying*

$$\sum_{j=0}^{\infty} \left(\frac{r_j}{R_j}\right) V\left(\frac{R_j}{r_j}\right) W\left(\frac{R_j}{r_j}\right) < \infty. \tag{8}$$

Finally, by an example we show that the reverse of Theorem 2 is not true.

Example Put

$$r_j = 3 \cdot 2^{j-1} \cdot j^{\frac{1}{2-n}} \quad \text{and} \quad R_j = 3 \cdot 2^{j-1} \quad (j = 1, 2, 3, \ldots).$$

A covering $\{r_j, R_j\}$ satisfies

$$\sum_{j=1}^{\infty} \left(\frac{r_j}{R_j}\right) V\left(\frac{R_j}{r_j}\right) W\left(\frac{R_j}{r_j}\right) \leq c \sum_{j=1}^{\infty} \left(\frac{r_j}{R_j}\right)^{n-1} = c \sum_{j=1}^{\infty} j^{\frac{n-1}{2-n}} < +\infty$$

from equation (3).

Let $C_n(\Omega')$ be a subset of $C_n(\Omega)$, *i.e.*, $\overline{\Omega}' \subset \Omega$. Suppose that this covering is located as follows: there is an integer j_0 such that $B_j \subset C_n(\Omega')$ and $R_j > 2r_j$ for $j \geq j_0$. Then the set $H = \bigcup_{j=j_0}^{\infty} B_j$ is not a-rarefied at infinity on $C_n(\Omega)$. This fact will be proved in Section 5.

2 Lemmas

Lemma 1 (see [1, Ch. 11] and [15, Lemma 4])

$$G_\Omega^a(P, Q) \leq c V(t) W(r) \varphi(\Theta) \varphi(\Phi)$$
$$\left(\textit{resp. } G_\Omega^a(P, Q) \leq c V(r) W(t) \varphi(\Theta) \varphi(\Phi)\right)$$

for any $P = (r, \Theta) \in C_n(\Omega)$ and any $Q = (t, \Phi) \in C_n(\Omega)$ satisfying $r \geq 2t$ (resp. $t \geq 2r$).

Lemma 2 (see [2, Lemma 5]) *Let ν be a positive measure on $C_n(\Omega)$ such that there is a sequence of points $P_i = (r_i, \Theta_i) \in C_n(\Omega)$, $r_i \to +\infty$ $(i \to +\infty)$ satisfying $G_\Omega^a \nu(P_i) < +\infty$ $(i = 1, 2, \ldots; Q \in C_n(\Omega))$. Then, for a positive number L,*

$$\int_{C_n(\Omega; (L, +\infty))} W(t) \varphi(\Phi) \, d\nu(Q) < +\infty$$

and

$$\lim_{R\to+\infty}\frac{W(R)}{V(R)}\int_{C_n(\Omega;(0,R))}V(t)\varphi(\Phi)\,d\nu(Q)=0.$$

Lemma 3 (see [2, Theorem 3]) *Let* ν *be any positive measure on* $C_n(\Omega)$ *such that* $G^a_\Omega\nu(P)\not\equiv +\infty$ *for any* $P\in C_n(\Omega)$. *Then, for a sufficiently large* L,

$$\bigl\{P=(r,\Theta)\in C_n\bigl(\Omega;(L,+\infty)\bigr);G^a_\Omega\nu(P)\geq V(r)\varphi(\Theta)\bigr\}\subset E\bigl(\epsilon;m(\nu),n-1\bigr).$$

Lemma 4 (see [2, Lemma 6]) *Let* λ *be any positive measure on* $\mathbf{R}^n$ *having finite total mass. Then* $E(\epsilon;\lambda,n-1)$ *has a covering* $\{r_j,R_j\}$ $(j=1,2,\ldots)$ *satisfying*

$$\sum_{j=1}^{\infty}\left(\frac{r_j}{R_j}\right)V\left(\frac{R_j}{r_j}\right)W\left(\frac{R_j}{r_j}\right)<\infty. \tag{9}$$

3 Proof of Theorem 1

Suppose that

$$H\subset\Pi\bigl(\xi^a_H\bigr)=\bigl\{P=(r,\Theta)\in C_n(\Omega);G^a_\Omega\xi^a_H(P)\geq V(r)\bigr\} \tag{10}$$

for a positive measure ξ^a_H on $C_n(\Omega)$ satisfying equation (6).

We write

$$G^a_\Omega\nu(P)=G^a_\Omega(1,j)(P)+G^a_\Omega(2,j)(P)+G^a_\Omega(3,j)(P),$$

where

$$G^a_\Omega(1,j)(P)=\int_{C_n(\Omega;(0,2^{j-1}))}G^a_\Omega(P,Q)\,d\nu(Q),$$

$$G^a_\Omega(2,j)(P)=\int_{C_n(\Omega;[2^{j-1},2^{j+2}))}G^a_\Omega(P,Q)\,d\nu(Q)$$

and

$$G^a_\Omega(3,j)(P)=\int_{C_n(\Omega;[2^{j+2},\infty))}G^a_\Omega(P,Q)\,d\nu(Q).$$

Now we shall show the existence of an integer N such that for any integer j $(\geq N)$, we have

$$\Pi\bigl(\xi^a_H\bigr)(j)\subset\bigl\{P=(r,\Theta)\in C_n\bigl(\Omega;\bigl[2^j,2^{j+1}\bigr)\bigr);2G^a_\Omega(2,j)(P)\geq V(r)\bigr\} \tag{11}$$

for any integer j $(\geq N)$.

For any $P=(r,\Theta)\in C_n(\Omega;[2^j,2^{j+1}))$, we have

$$G^a_\Omega(1,j)(P)\leq cW(r)\varphi(\Theta)\int_{C_n(\Omega;(0,2^{j-1}))}V(t)\varphi(\Phi)\,d\nu(Q)$$

and

$$G^a_\Omega(3,j)(P) \le cV(r)\varphi(\Theta)\int_{C_n(\Omega;[2^{j+2},\infty))} dm(\nu)(Q)$$

from Lemma 1.

By applying Lemma 2, we can take an integer N such that for any j ($\ge N$),

$$W\big(2^j\big)V^{-1}\big(2^j\big)\int_{C_n(\Omega;(0,2^{j-1}))} V(t)\varphi(\Phi)\,d\nu(Q) \le \frac{1}{4c}$$

and

$$\int_{C_n(\Omega;[2^{j+2},\infty))} dm(\nu)(Q) \le \frac{1}{4c}.$$

Thus we obtain

$$4G^a_\Omega(1,j)(P) \le V(r)\varphi(\Theta) \tag{12}$$

and

$$4G^a_\Omega(3,j)(P) \le V(r)\varphi(\Theta) \tag{13}$$

for any $P = (r,\Theta) \in C_n(\Omega;[2^j,2^{j+1}))$, where $j \ge N$.

Thus, if $P = (r,\Theta) \in \Pi(\nu)(j)$ ($j \ge N$), then we obtain

$$2G^a_\Omega(1,j)(P) \ge V(r)\varphi(\Theta)$$

from equations (12) and (13), which gives equation (11).

From equations (4), (7) and (11), we have

$$G^a_\Omega(2,j)(P) = \int_{C_n(\Omega)} G^a_\Omega(P,Q)\,d\tau^a_j(Q) \ge M^a_\Omega(P,\infty),$$

where $P \in I_j$ ($j \ge N$) and

$$d\tau^a_j(Q) = \begin{cases} 2^{1-j}\,d\xi^a_H(Q), & Q \in C_n(\Omega;[2^{j-1},2^{j+2})), \\ 0, & Q \in C_n(\Omega;(0,2^{j-1})) \cup C_n(\Omega;[2^{j+2},\infty)). \end{cases}$$

And then we obtain

$$\eta^a_{H_j}\big(C_n(\Omega)\big) \le \int_{C_n(\Omega)} V(t)\varphi(\Phi)\,d\tau^a_j(Q) = \int_{C_n(\Omega;[2^{j-1},2^{j+2}))} V(t)\varphi(\Phi)\,d\xi^a_H(Q)$$

for $j \ge N$. Then we have

$$\sum_{j=N}^{\infty}\lambda^a_\Omega(H_j)W\big(2^j\big) = \sum_{j=N}^{\infty}\eta^a_{H_j}\big(C_n(\Omega)\big)W\big(2^j\big) \le c\int_{C_n(\Omega;[2^{N-1},\infty))} dm\big(\xi^a_H\big),$$

in which the last integral is finite by Remark 1. And hence H is a-rarefied set at infinity from Theorem A.

Suppose that

$$\sum_{j=0}^{\infty} \lambda_{\Omega}^{a}(H_j) W\bigl(2^j\bigr) < \infty.$$

Consider a function $f_H^a(P)$ on $C_n(\Omega)$ defined by

$$f_H^a(P) = \sum_{j=-1}^{\infty} \hat{R}_{M_{\Omega}^{a}(\cdot,\infty)}^{H_j}(P)$$

for any $P \in C_n(\Omega)$, where $H_{-1} = H \cap C_n(\Omega;(0,1))$.

If we put $\mu_H^a(1)(P) = \sum_{j=-1}^{\infty} \lambda_{H_j}^{a}(P)$, then from equation (5) we have that

$$f_H^a(P) = \int_{C_n(\Omega)} G_{\Omega}^{a}(P,Q)\, d\mu_H^a(1)(Q)$$

for any $P \in C_n(\Omega)$.

Next we shall show that $f_H^a(P)$ is always finite on $C_n(\Omega)$. Take any point $P = (r,\Theta) \in C_n(\Omega)$ and a positive integer $j(P)$ satisfying $r \le 2^{j(P)+1}$. We write

$$f_H^a(P) = f_H^a(1)(P) + f_H^a(2)(P),$$

where

$$f_H^a(1)(P) = \sum_{j=-1}^{j(P)+1} \int_{C_n(\Omega)} G_{\Omega}^{a}(P,Q)\, d\lambda_{H_j}^{a}(Q) \quad \text{and}$$

$$f_H^a(2)(P) = \sum_{j=j(P)+2}^{\infty} \int_{C_n(\Omega)} G_{\Omega}^{a}(P,Q)\, d\lambda_{H_j}^{a}(Q).$$

Since $\lambda_{H_j}^{a}$ is concentrated on $I_{H_j} \subset \overline{H}_j \cap C_n(\Omega)$, we have that

$$\begin{aligned}\int_{C_n(\Omega)} G_{\Omega}^{a}(P,Q)\, d\lambda_{H_j}^{a}(Q) &\le cV(r)\varphi(\Theta) \int_{C_n(\Omega)} W(t)\varphi(\Phi)\, d\lambda_{H_j}^{a}(t,\Phi) \\ &\le cV(r)\varphi(\Theta) W\bigl(2^j\bigr) V^{-1}\bigl(2^j\bigr) \int_{C_n(\Omega)} V(t)\varphi(\Phi)\, d\lambda_{H_j}^{a}(t,\Phi)\end{aligned}$$

for $j \ge j(P)+2$. Hence we have

$$f_H^a(2)(P) \le cV(r)\varphi(\Theta) \sum_{j=j(P)+2}^{\infty} \eta_{H_j}^{a}\bigl(C_n(\Omega)\bigr) W\bigl(2^j\bigr) V^{-1}\bigl(2^j\bigr), \tag{14}$$

which, together with Theorem A, shows that $f_H^a(2)(P)$ is finite and hence $f_H^a(P)$ is also finite for any $P \in C_n(\Omega)$.

Since

$$\hat{R}_{M_{\Omega}^{a}(\cdot,\infty)}^{H_j}(P) = M_{\Omega}^{a}(P,\infty)$$

holds on I_{H_j} and $I_{H_j} \subset \overline{H}_j \cap C_n(\Omega)$, we see that for any $P = (r, \Theta) \in I_{H_j}$ $(j = -1, 0, 1, 2, 3, \ldots)$

$$f_H^a(P) \geq c\hat{R}^{H_j}_{M_\Omega^a(\cdot,\infty)}(P) \geq V(r)\varphi(\Theta). \tag{15}$$

And hence equation (15) also holds for any $P = (r, \Theta) \in H' = \bigcup_{j=-1}^{\infty} I_{H_j}$. Since H' is equal to H except a polar set H^0, we can take another positive superfunction $f_H^a(3)(P)$ on $C_n(\Omega)$ such that $f_H^a(3)(P) = G_\Omega^a \mu_H^a(2)(P)$ with a positive measure $\mu_H^a(2)(P)$ on $C_n(\Omega)$ and $f_H^a(3)(P)$ is identically $+\infty$ on H^0.

Finally, we can define a positive superfunction g on $C_n(\Omega)$ by $g(P) = f_H^a(P) + f_H^a(3)(P) = G_\Omega^a \xi_H^a(P)$ for any $P \in C_n(\Omega)$ with $\xi_H^a = \mu_H^a(1) + \mu_H^a(2)$. Also we see from equation (15) that equations (6) and (7) hold.

Thus we complete the proof of Theorem 1.

4 Proof of Theorem 2

From Theorem 1 and Lemma 3, we have a positive number L such that

$$H \cap C_n\big(\Omega; (L, +\infty)\big) \subset E\big(\epsilon; m\big(\xi_H^a\big), n-1\big).$$

Hence by Remark 1 and Lemma 4, $E(\epsilon; m(\xi_H^a), n-1)$ has a covering $\{r_j, R_j\}$ $(j = 1, 2, 3, \ldots)$ satisfying equation (9) and hence H has also a covering $\{r_j, R_j\}$ $(j = 0, 1, 2, 3, \ldots)$ with an additional finite B_0 covering $C_n(\Omega; (0, L])$, satisfying equation (8), which is the conclusion of Theorem 2.

5 Proof of an example

Since $\varphi(\Theta) \geq c$ for any $\Theta \in \Omega'$, we have $M_\Omega^a(P, \infty) \geq cV(R_j)$ for any $P \in \overline{B}_j$, where $j \geq j_0$. Hence we have

$$\hat{R}^{B_j}_{M_\Omega^a(\cdot,\infty)}(P) \geq cV(R_j) \tag{16}$$

for any $P \in \overline{B}_j$, where $j \geq j_0$.

Take a measure δ on $C_n(\Omega)$, $\operatorname{supp}\delta \subset \overline{B}_j$, $\delta(\overline{B}_j) = 1$ such that

$$\int_{C_n(\Omega)} |P - Q|^{2-n}\, d\delta(P) = \big\{\operatorname{Cap}(\overline{B}_j)\big\}^{-1} \tag{17}$$

for any $Q \in \overline{B}_j$, where Cap denotes the Newton capacity. Since

$$G_\Omega^a(P, Q) \leq |P - Q|^{2-n}$$

for any $P \in C_n(\Omega)$ and $Q \in C_n(\Omega)$ (see [16], the case $n = 2$ is implicitly contained in [17]),

$$\begin{aligned}
\big\{\operatorname{Cap}(\overline{B}_j)\big\}^{-1} \lambda_{B_j}^a\big(C_n(\Omega)\big) &= \int \left(\int |P - Q|^{2-n}\, d\delta(P) \right) d\lambda_{B_j}^a(Q) \\
&\geq \int \left(\int G_\Omega^a(P, Q)\, d\lambda_{B_j}^a(Q) \right) d\delta(P) \\
&= \int \hat{R}^{B_j}_{M_\Omega^a(\cdot,\infty)}\, d\delta(P) \\
&\geq cV(R_j)\delta(\overline{B}_j) = cV(R_j)
\end{aligned}$$

from equations (16) and (17). Hence we have

$$\lambda_{B_j}^a\big(C_n(\Omega)\big) \geq c\,\mathrm{Cap}(\overline{B}_j)V(R_j) \geq cr_j^{n-2}V(R_j). \tag{18}$$

If we observe $\lambda_{H_j}^a(C_n(\Omega)) = \lambda_{B_j}^a(C_n(\Omega))$, then we have by equation (3)

$$\sum_{j=j_0}^{\infty} W\big(2^j\big)\lambda_{H_j}^a\big(C_n(\Omega)\big) \geq c\sum_{j=j_0}^{\infty}\left(\frac{r_j}{R_j}\right)^{n-2} = c\sum_{j=j_0}^{\infty}\frac{1}{j} = +\infty,$$

from which it follows by Theorem A that H is not a-rarefied at infinity on $C_n(\Omega)$.

Competing interests
The author declares that they have no competing interests.

Acknowledgements
This work was supported by the National Natural Science Foundation of China under Grants Nos. 11301140 and U1304102.

Received: 26 March 2014 Accepted: 30 May 2014 Published: 18 July 2014

References
1. Escassut, A, Tutshke, W, Yang, CC: Some Topics on Value Distribution and Differentiability in Complex and *p*-Adic Analysis. Science Press, Beijing (2008)
2. Qiao, L, Deng, GT: Integral representations and growth properties for a class of superfunctions in a cone. Taiwan. J. Math. **15**, 2213-2233 (2011)
3. Gilbarg, D, Trudinger, NS: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (1977)
4. Courant, R, Hilbert, D: Methods of Mathematical Physics, vol. 1. Interscience, New York (2008)
5. Verzhbinskii, GM, Maz'ya, VG: Asymptotic behavior of solutions of elliptic equations of the second order close to a boundary. I. Sib. Mat. Zh. **12**, 874-899 (1971)
6. Simon, B: Schrödinger semigroups. Bull. Am. Math. Soc. **7**, 447-526 (1982)
7. Long, PH, Gao, ZQ, Deng, GT: Criteria of Wiener type for minimally thin sets and rarefied sets associated with the stationary Schrödinger operator in a cone. Abstr. Appl. Anal. **2012**, Article ID 453891 (2012)
8. Brelot, M: On Topologies and Boundaries in Potential Theory. Lecture Notes in Mathematics, vol. 175. Springer, Berlin (1971)
9. Miyamoto, I, Yoshida, H: Two criterions of Wiener type for minimally thin sets and rarefied sets in a cone. J. Math. Soc. Jpn. **54**, 487-512 (2002)
10. Qiao, L, Pan, GS: Generalization of the Phragmén-Lindelöf theorems for subfunctions. Int. J. Math. **24**(8), 1350062 (2013). doi:10.1142/S0129167X13500626
11. Qiao, L, Pan, GS: Integral representations of generalized harmonic functions. Taiwan. J. Math. **17**(5), 1503-1521 (2013)
12. Ren, YD: Solving integral representations problems for the stationary Schrödinger equation. Abstr. Appl. Anal. **2013**, Article ID 715252 (2013)
13. Zhao, T: Minimally thin sets associated with the stationary Schrödinger operator. J. Inequal. Appl. **2014**, 67 (2014)
14. Xue, GX: A remark on the *a*-minimally thin sets associated with the Schrödinger operator. Bound. Value Probl. **2014**, 133 (2014)
15. Azarin, VS: Generalization of a theorem of Hayman on subharmonic functions in an *m*-dimensional cone. Transl. Am. Math. Soc. **80**, 119-138 (1969)
16. Cranston, M: Conditional Brownian motion, Whitney squares and the conditional gauge theorem. In: Seminar on Stochastic Processes, 1988, pp. 109-119. Birkhäuser, Basel (1989)
17. Cranston, M, Fabes, E, Zhao, Z: Conditional gauge and potential theory for the Schrödinger operator. Trans. Am. Math. Soc. **307**, 415-425 (1964)

doi:10.1186/1029-242X-2014-247
Cite this article as: Xue: **Rarefied sets at infinity associated with the Schrödinger operator.** *Journal of Inequalities and Applications* 2014 **2014**:247.