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Nonlinear fractional differential equations in nonreflexive Banach spaces and fractional calculus

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Abstract
The aim of this paper is to correct some ambiguities and inaccuracies in Agarwal *et al.* (Commun. Nonlinear Sci. Numer. Simul. 20(1):59-73, 2015; Adv. Differ. Equ. 2013:302, 2013, doi:10.1186/1687-1847-2013-302) and to present new ideas and approaches for fractional calculus and fractional differential equations in nonreflexive Banach spaces.

1 Introduction

One of the sections, Section 5, of our paper [1] contains a number of ambiguities (and inaccuracies) which we correct here. The notion of pseudo-solution in [1] is not adequately defined and assumption (h2) in Theorem 5.1 is strong. Also, there is some ambiguity regarding the use of the space $C(T,E)$ of all continuous functions from T into E with its weak topology $\sigma(C(T,E),C(T,E)^*)$ and the space $C_w(T,E)$ of all weakly continuous functions from T into E_w endowed with the topology of weak uniform convergence. Parts of Corollaries 5.1-5.6 are no longer valid in their current form. Similar comments also apply to [2]. In [3] the authors developed fractional calculus for vector-valued functions using the weak Riemann integral and they established the existence of weak solutions for a class of fractional differential equations with fractional weak derivatives. In this paper we present new ideas in fractional calculus and we present a new approach to establishing existence to some fractional differential equations in nonreflexive Banach spaces. References [4–6], and [7] were helpful in presenting these new ideas.

2 Preliminaries

In the following we outline some aspects of fractional calculus in a nonreflexive Banach space. This subject has been treated extensively in [1, 3]. Let E be a Banach space with norm $\|\cdot\|$ and let E^* be the topological dual of E. If $x^* \in E^*$, then its value on an element $x \in E$ will be denoted by $\langle x^*, x\rangle$. The space E endowed with the weak topology $\sigma(E,E^*)$ will be denoted by E_w. Consider an interval $T = [0,b]$ of $\mathbb{R}$, the set of real numbers, endowed with the Lebesgue σ-algebra $\mathcal{L}(T)$ and the Lebesgue measure λ. A function $x(\cdot): T \to E$ is said to be *strongly measurable* on T if there exists a sequence of simple functions $x_n(\cdot): T \to E$ such that $\lim_{n\to\infty} x_n(t) = x(t)$ for a.e. $t \in T$. Also, a function $x(\cdot): T \to E$ is said to be *weakly measurable* (or *scalarly measurable*) on T if, for every $x^* \in E^*$, the real valued function $t \mapsto \langle x^*, x(t)\rangle$ is Lebesgue measurable on T.

We denote by $L^p(T)$ the space of all real measurable functions $f : T \to \mathbb{R}$, whose absolute value raised to the pth power has finite integral, or equivalently, that

$$\|f\|_p := \left(\int_T |f(t)|^p \, dt\right)^{\frac{1}{p}} < \infty,$$

where $1 \leq p < \infty$. Moreover, by $L^\infty(T)$ we denote the space of all measurable and essential bounded real functions defined on T. Let $C(T,E)$ denote the space of all strong continuous functions $y(\cdot) : T \to E$, endowed with the supremum norm $\|y(\cdot)\|_c = \sup_{t\in T} \|y(t)\|$. Also, we consider the space $C(T,E)$ with its weak topology $\sigma(C(T,E), C(T,E)^*)$. It is well known that (see [8, 9])

$$C(T,E)^* = M(T,E^*),$$

where $M(T,E^*)$ is the space of all bounded regular vector measures from $\mathcal{B}(T)$ into E^* which are of bounded variation. Here, $\mathcal{B}(T)$ denotes the σ-algebra of Borel measurable subsets of T. Therefore, a sequence $\{y_n(\cdot)\}_{n\geq 1}$ converges weakly to $y(\cdot)$ in $C(T,E)$ if and only if

$$\langle m(\cdot), y_n(\cdot) - y(\cdot)\rangle \to 0 \quad \text{as } n \to \infty, \tag{1}$$

for all $m(\cdot) \in M(T,E^*)$. In [10], Lemma 9, it is shown that a sequence $\{y_n(\cdot)\}_{n\geq 1}$ converges weakly to $y(\cdot)$ in $C(T_0,E)$ if and only if $y_n(t)$ tends weakly to $y(t)$ for each $t \in T$.

Let $C_w(T,E)$ denote the space of all weakly continuous functions from T into E_w endowed with the topology of weak uniform convergence. A set $N \in \mathcal{L}(T)$ is called a null set if $\lambda(N) = 0$.

A function $x(\cdot) : T \to E$ is said to be *pseudo-differentiable* on T to a function $y(\cdot) : T \to E$ if, for every $x^* \in E^*$, there exists a null set $N(x^*) \in \mathcal{L}(T)$ such that the real function $t \mapsto \langle x^*, x(t)\rangle$ is differentiable on $T \setminus N(x^*)$ and

$$\frac{d}{dt}\langle x^*, x(t)\rangle = \langle x^*, y(t)\rangle, \quad t \in T \setminus N(x^*). \tag{2}$$

The function $y(\cdot)$ is called a *pseudo-derivative* of $x(\cdot)$ and it will be denoted by $x_p'(\cdot)$ or by $\frac{d_p}{dt}x(\cdot)$. A pseudo-derivative $x_p'(\cdot)$ of a pseudo-differentiable function $x(\cdot) : T \to E$ is weakly measurable on T (see [11]).

We recall that a function $x(\cdot) : T \to E$ is said to be *weakly differentiable on* T if there exists a function $x_w'(\cdot) : T \to E$ such that

$$\lim_{h\to 0}\left\langle x^*, \frac{x(t_0 + h) - x(t_0)}{h}\right\rangle = \langle x^*, x_w'(t_0)\rangle,$$

for every $x^* \in E^*$. If it exists, $x_w'(\cdot)$ is uniquely determined and it is called the *weak derivative* of $x(\cdot)$ on T. Obviously, if $x(\cdot) : T \to E$ is a weakly differentiable function on T, then the real function $t \mapsto \langle x^*, x(t)\rangle$ is differentiable on T. Moreover, in this case we have

$$\frac{d}{dt}\langle x^*, x(t)\rangle = \langle x^*, x_w'(t)\rangle, \quad t \in T,$$

for every $x^* \in E^*$. It is easy to see that, if $x(\cdot) : T \to E$ is a function a.e. weakly differentiable on T, then $x(\cdot)$ is pseudo-differentiable on T and $x'_p(\cdot) = x'_w(\cdot)$ a.e. on T.

The concept of a Bochner integral and a Pettis integral are well known [12–14].

We recall that a weakly measurable function $x(\cdot) : T \to E$ is said to be *Pettis integrable* on T if

(a) $x(\cdot)$ is *scalarly integrable*; that is, for every $x^* \in E^*$, the real function $t \mapsto \langle x^*, x(t)\rangle$ is Lebesgue integrable on T;

(b) for every set $A \in \mathcal{L}(T)$, there exists an element $x_A \in E$ such that

$$\langle x^*, x_A \rangle = \int_A \langle x^*, x(s)\rangle\, ds, \tag{3}$$

for every $x^* \in E^*$. The element $x_A \in E$ is called the *Pettis integral* on A and it will be denoted by $\int_A x(s)\, ds$.

It is easy to show that a Bochner integrable function $x(\cdot) : T \to E$ is Pettis integrable and both integrals of $x(\cdot)$ are equal on each Lebesgue measurable subset A of T ([14], Proposition 2.3.1). The best result for a descriptive definition of the Pettis integral is that given by Pettis in [15].

Proposition 1 *Let $x(\cdot) : T \to E$ be a weakly measurable function.*

(a) *If $x(\cdot)$ is Pettis integrable on T, then the indefinite Pettis integral*

$$y(t) := \int_0^t x(s)\, ds, \quad t \in T$$

is AC on T and $x(\cdot)$ is a pseudo-derivative of $y(\cdot)$.

(b) *If $y(\cdot) : T \to E$ is an AC function on T and it has a pseudo-derivative $x(\cdot)$ on T, then $x(\cdot)$ is Pettis integrable on T and*

$$y(t) = y(0) + \int_0^t x(s)\, ds, \quad t \in T.$$

It is well known that the Pettis integrals of two strongly measurable functions $x(\cdot) : T \to E$ and $y(\cdot) : T \to E$ coincide over every Lebesgue measurable set in T if and only if $x(\cdot) = y(\cdot)$ a.e. on T ([15], Theorem 5.2). Since a pseudo-derivative of a pseudo-differentiable function $x(\cdot) : T \to E$ is not unique (see [11]) and two pseudo-derivatives of $x(\cdot)$ need not be a.e. equal, the concept of weakly equivalence plays an important role in the following.

Two weak measurable functions $x(\cdot) : T \to E$ and $y(\cdot) : T \to E$ are said to be *weakly equivalent* on T if, for every $x^* \in E^*$, we have $\langle x^*, x(t)\rangle = \langle x^*, y(t)\rangle$ for a.e. $t \in T$. In the following, if two weak measurable functions $x(\cdot) : T \to E$ and $y(\cdot) : T \to E$ are weakly equivalent on T, then we will write $x(\cdot) \approx y(\cdot)$ or $x(t) \approx y(t)$, $t \in T$.

Proposition 2 ([15]) *A weakly measurable function $x(\cdot) : T \to E$ is Pettis integrable on T and $\langle x^*, x(\cdot)\rangle \in L^\infty(T)$, for every $x^* \in E^*$, if and only if the function $t \mapsto \varphi(t)x(t)$ is Pettis integrable on T, for every $\varphi(\cdot) \in L^1(T)$.*

Let us denote by $P^\infty(T, E)$ the space of all weakly measurable and Pettis integrable functions $x(\cdot) : T \to E$ with the property that $\langle x^*, x(\cdot)\rangle \in L^\infty(T)$, for every $x^* \in E^*$. Since for each

$t \in T$ the real valued function $s \mapsto (t-s)^{\alpha-1}$ is Lebesgue integrable on $[0,t]$, the *fractional Pettis integral*

$$I^\alpha x(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\,ds, \quad t \in T,$$

exists, for every function $x(\cdot) \in P^\infty(T,E)$, as a function from T into E (see [16]). Moreover, we have

$$\langle x^*, I^\alpha x(t)\rangle = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \langle x^*, x(s)\rangle\,ds, \quad t \in T,$$

for every $x^* \in E^*$, and the real function $t \mapsto \langle x^*, I^\alpha x(t)\rangle$ is continuous (in fact, bounded and uniformly continuous on T if $T = \mathbb{R}$) on T, for every $x^* \in E^*$ ([17], Proposition 1.3.2).

In the following, consider $\alpha \in (0,1)$ and for a given function $x(\cdot) \in P^\infty(T,E)$ we also denote by $x_{1-\alpha}(t)$ the fractional Pettis integral

$$I^{1-\alpha} x(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)\,ds, \quad t \in T.$$

Lemma 1 ([1], Lemma 3.1) *If $x(\cdot), y(\cdot) \in P^\infty(T,E)$ are weakly equivalent on T, then $I^\alpha x(t) = I^\alpha y(t)$ on T.*

Lemma 2 ([1, 16]) *The fractional Pettis integral is a linear operator from $P^\infty(T,E)$ into $P^\infty(T,E)$. Moreover, if $x(\cdot) \in P^\infty(T,E)$, then for $\alpha, \beta > 0$ we have*

(a) $I^\alpha I^\beta x(t) = I^{\alpha+\beta} x(t)$, $t \in T$;
(b) $\lim_{\alpha\to 1} I^\alpha x(t) = I^1 x(t) = x(t) - x(0)$ *weakly uniformly on* T;
(c) $\lim_{\alpha\to 0} I^\alpha x(t) = x(t)$ *weakly on* T.

If $y(\cdot): T \to E$ is a pseudo-differentiable function on T with a pseudo-derivative $x(\cdot) \in P^\infty(T,E)$, then the fractional Pettis integral $I^{1-\alpha} x(t)$ exists on T. The fractional Pettis integral $I^{1-\alpha} x(\cdot)$ is called a *fractional pseudo-derivative* of $y(\cdot)$ on T and it will be denoted by $D_p^\alpha y(\cdot)$; that is,

$$D_p^\alpha y(t) = I^{1-\alpha} x(t), \quad t \in T. \tag{4}$$

Remark 1 If $x(\cdot), \widetilde{x}(\cdot) \in P^\infty(T,E)$ are two pseudo-derivatives of $y(\cdot): T \to E$, then $x(\cdot) \approx \widetilde{x}(\cdot)$ on T. Thus, Lemma 1 implies that $I^{1-\alpha} x(t) = I^{1-\alpha}\widetilde{x}(t)$ on T, and so $D_p^\alpha y(\cdot)$ does not depend on the choice of a pseudo-derivatives of the function $y(\cdot)$. Therefore, we can write (4) as

$$D_p^\alpha y(t) = I^{1-\alpha} y_p'(t), \quad t \in T, \tag{5}$$

where $y_p'(\cdot)$ is a given pseudo-derivatives of $y(\cdot)$.

We recall that a function $x(\cdot): T \to E$ is said to be *weakly absolutely continuous* (*wAC*, for short) on T if, for every $x^* \in E^*$, the real valued function $t \mapsto \langle x^*, x(t)\rangle$ is absolutely continuous on T.

Lemma 3 ([1]) *If $y(\cdot) \in P^\infty(T,E)$ is a pseudo-differentiable function on T with a pseudo-derivative $x(\cdot) \in P^\infty(T,E)$, then the function*

$$y_{1-\alpha}(t) := \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} y(s)\,ds, \quad t \in T,$$

is wAC and it has a pseudo-derivative $\frac{d_p}{dt} y_{1-\alpha}(\cdot) \in P^\infty(T,E)$ such that

$$\frac{d_p}{dt} y_{1-\alpha}(t) \approx \frac{t^{-\alpha}}{\Gamma(1-\alpha)} y(0) + I^{1-\alpha} x(t) \quad \textit{on } T. \tag{6}$$

Remark 2 Relation (6) can be written as

$$D_p^\alpha y(t) \approx \frac{d_p}{dt} y_{1-\alpha}(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} y(0) \quad \text{on } T. \tag{7}$$

Note (7) suggests us that we can extend the definition of the fractional pseudo-derivative for functions $y(\cdot) \in P^\infty(T,E)$ for which the function $t \mapsto y_{1-\alpha}(t)$ is pseudo-differentiable on T. If $\frac{d_p}{dt} y_{1-\alpha}(t)$ exists on T, then $\frac{d_p}{dt} y_{1-\alpha}(t)$ will be called the *Riemann-Liouville fractional pseudo-derivative* of $y(\cdot)$ and it will be denoted by $\mathcal{D}_p^\alpha y(\cdot)$; that is, $\mathcal{D}_p^\alpha y(\cdot) = \frac{d_p}{dt} y_{1-\alpha}(\cdot)$. Usually, $D_p^\alpha y(\cdot)$ is called the *Caputo fractional pseudo-derivative* of $y(\cdot)$. Relation (6) can be written as

$$D_p^\alpha y(t) \approx \mathcal{D}_p^\alpha y(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} y(0) \quad \text{on } T. \tag{8}$$

Therefore, the Caputo fractional pseudo-derivative $D_p^\alpha y(\cdot)$ exists together with the Riemann-Liouville fractional pseudo-derivative $\mathcal{D}_p^\alpha y(\cdot)$ and they satisfy (8). It is easy to see that if $y(0) = 0$, then

$$D_p^\alpha y(t) \approx \mathcal{D}_p^\alpha y(t) \quad \text{on } T. \tag{9}$$

Remark 3 Let $y(\cdot) : T \to E$ be a pseudo-differentiable function with a pseudo-derivative $y_p'(\cdot) \in P^\infty(T,E)$. Then from Lemma 3 we find that the function

$$y_\alpha(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds, \quad t \in T,$$

is *wAC* and has a pseudo-derivative $\frac{d_p}{dt} y_\alpha(t)$ such that

$$D_p^{1-\alpha} y(t) \approx \frac{d_p}{dt} y_\alpha(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} y(0) \quad \text{on } T.$$

Lemma 4 *Let $\alpha, \beta \in (0,1)$.*

(a) *If $y(\cdot) \in P^\infty(T,E)$, then*

$$D_p^\alpha I^\alpha y(t) = y(t), \quad t \in T.$$

(b) *If* $y(\cdot) \in P^\infty(T,E)$ *and* $y_{1-\alpha}(\cdot)$ *is pseudo-differentiable with a pseudo-derivative* $\frac{d_p}{dt} y_{1-\alpha}(\cdot) \in P^\infty(T,E)$, *then*

$$I^\alpha D_p^\alpha y(t) = y(t) - y(0), \quad t \in T.$$

Proof (a) Indeed, since $y(\cdot) \in P^\infty(T,E)$, then $t \mapsto \langle x^*, y(t)\rangle$ is essentially bounded on T, for every $x^* \in E^*$. Hence we have

$$\bigl|\bigl\langle x^*, I^\alpha y(t)\bigr\rangle\bigr| = \bigl|I^\alpha\bigl\langle x^*, y(t)\bigr\rangle\bigr| \le M\bigl(x^*\bigr)\frac{t^\alpha}{\Gamma(1+\alpha)}, \quad t \in T,$$

where $M(x^*) = \operatorname{ess\,sup}_{t\in T} |\langle x^*, y(t)\rangle| < \infty$, $x^* \in E^*$. Since the real function $t \mapsto \langle x^*, I^\alpha y(t)\rangle$ is continuous on T, it follows that $\langle x^*, I^\alpha y(0)\rangle = 0$, for every $x^* \in E^*$, and thus $I^\alpha y(0) = 0$. Then by Remark 2 we have $D_p^\beta I^\alpha y(t) = \mathcal{D}_p^\beta I^\alpha y(t)$, and so by Lemma 2 and Proposition 1 we have

$$D_p^\alpha I^\alpha y(t) = \mathcal{D}_p^\alpha I^\alpha y(t) = \frac{d_p}{dt} I^{1-\alpha} I^\alpha y(t) = \frac{d_p}{dt} I^1 y(t) = \frac{d_p}{dt}\int_0^t y(s)\,ds = y(t), \quad t \in T.$$

(b) By Lemma 2 and Proposition 1 we have

$$I^\alpha D_p^\alpha y(t) = I^\alpha I^{1-\alpha} y'_p(t) = I^1 y'_p(t) = \int_0^t y'_p(s)\,ds = y(t) - y(0), \quad t \in T. \qquad \square$$

Lemma 5 *Let* $y(\cdot): T \to E$ *be a pseudo-differentiable function on* T *with* $y'_p(\cdot) \in P^\infty(T,E)$ *and* $0 < \alpha \le \beta < 1$. *Then we have*

(a)

$$I^\alpha D_p^\beta y(t) = D_p^{\beta-\alpha} y(t) \quad \text{on } T. \tag{10}$$

(b) *If* $y(0) = 0$, *then*

$$D_p^\beta I^\alpha y(t) = D_p^{\beta-\alpha} y(t) \quad \text{on } T \tag{11}$$

and

$$I^\beta D_p^\alpha y(t) = I^{\beta-\alpha} y(t) \quad \text{on } T. \tag{12}$$

Proof If $y(\cdot): T \to E$ is a pseudo-differentiable function on T, then by Lemma 2 we have

$$I^\alpha D_p^\beta y(t) = I^\alpha I^{1-\beta} y'_p(t) = I^{1-(\beta-\alpha)} y'_p(t) = D_p^{\beta-\alpha} y(t), \quad t \in T.$$

If $y(0) = 0$, then by Remark 3 and (10) we have

$$D_p^\beta I^\alpha y(t) = I^{1-\beta}\frac{d_p}{dt} y_\alpha(t) = I^{1-\beta} D_p^{1-\alpha} y(t) = D_p^{\beta-\alpha} y(t), \quad t \in T.$$

Also, since $y(0) = 0$, then by Lemma 2 and Proposition 1 we have

$$I^\beta D_p^\alpha y(t) = I^\beta I^{1-\alpha} y'_p(t) = I^{\beta-\alpha+1} y'_p(t) = I^{\beta-\alpha} I^1 y'_p(t) = I^{\beta-\alpha} y(t), \quad t \in T. \qquad \square$$

3 Differential equations with fractional pseudo-derivatives

The existence of weak solutions or pseudo-solutions for ordinary differential equations in Banach spaces were investigated in many papers (see [18–31]). In reflexive Banach spaces, the existence of weak solutions or pseudo-solutions for fractional differential equations were studied in [32–36]. In this section we establish an existence result for the following fractional differential equation:

$$\begin{cases} D_p^\alpha y(t) = f(t,y(t)), \\ y(0) = y_0, \end{cases} \tag{13}$$

where $D_p^\alpha y(\cdot)$ is a fractional pseudo-derivative of the function $y(\cdot): T \to E$ and $f(\cdot,\cdot): T \times E \to E$ is a given function. Along with the Cauchy problem (13) consider the following integral equation:

$$y(t) = y_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,y(s)\bigr)\,ds, \quad t \in T, \tag{14}$$

where the integral is in the sense of Pettis.

A continuous function $y(\cdot): T \to E$ is said to be a *solution* of (13) if $y(\cdot)$ has a pseudo-derivative belonging to $P^\infty(T,E)$, $D_p^\alpha y(t) \approx f(t,y(t))$ for $t \in T$ and $y(0) = y_0$.

To prove a result on the existence of solutions for (13) we need some preliminary results.

Lemma 6 *Let $f(\cdot,\cdot): T \times E \to E$ be a function such that $f(\cdot,y(\cdot)) \in P^\infty(T,E)$, for every continuous function $y(\cdot): T \to E$. Then a continuous function $y(\cdot): T \to E$ is a solution of* (13) *if and only if it satisfies the integral equation* (14).

Proof Indeed, if a continuous function $y(\cdot): T \to E$ is a solution of (13), then $y(\cdot)$ has a pseudo-derivative belonging to $P^\infty(T,E)$, $D_p^\alpha y(t) \approx f(t,y(t))$ for $t \in T$ and $y(0) = y_0$. Then we have $I^\alpha D_p^\alpha y(t) = I^\alpha f(t,y(t))$ on T, and thus from Lemma 4(b) it follows that $y(t) - y(0) = I^\alpha f(t,y(t))$ on T; that is, $y(\cdot)$ satisfies the integral equation (14). Conversely, suppose that a continuous function $y(\cdot): T \to E$ satisfies the integral equation (14). Then the function $z(\cdot) := f(\cdot,y(\cdot)) \in P^\infty(T,E)$ satisfies the Abel equation

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} z(s)\,ds = v(t), \quad t \in T,$$

where $v(t) := y(t) - y_0$, $t \in T$. Then from [1], Theorem 3.1, and Lemma 3 it follows that $v_{1-\alpha}(\cdot)$ has a pseudo-derivative on T and

$$z(t) \approx \frac{d_p}{dt} v_{1-\alpha}(t) = \frac{d_p}{dt} y_{1-\alpha}(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} y(0) \quad \text{for } t \in T.$$

Then by Remark 2 we have $z(t) \approx D_p^\alpha y(t)$ for $t \in T$; that is, $D_p^\alpha y(t) \approx f(t,y(t))$ on T. □

In this section we shall discuss the existence of solutions of fractional differential equations in nonreflexive Banach spaces. We recall that a function $f(\cdot): E \to E$ is said to be *sequentially continuous* from E_w *into* E_w (or *weakly-weakly sequentially continuous*) if, for every weakly convergent sequence $\{x_n\}_{n\geq 1} \subset E$, the sequence $\{f(x_n)\}_{n\geq 1}$ is weakly convergent in E.

By a *Gripenberg function* we mean a function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $g(\cdot)$ is continuous, nonincreasing with $g(0) = 0$, and $u \equiv 0$ is the only continuous solution of

$$u(t) \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g\bigl(u(s)\bigr)\,ds, \qquad u(0) = 0. \tag{15}$$

The problem of uniqueness of the null solution of (15) was studied by Gripenberg in [37].

Let us denote by $P_{wk}(E)$ the set of all weakly compact subset of E. The *weak measure of noncompactness* [38] is the set function $\beta : P_{wk}(E) \to [0,\infty)$ defined by

$$\beta(A) = \inf\bigl\{r > 0;\ \text{there exist } K \in P_{wk}(E) \text{ such } A \subset K + rB_1\bigr\},$$

where B_1 is the closed unit ball in E. The properties of the weak noncompactness measure are analogous to the properties of the measure of noncompactness, namely (see [38]):

(1) $A \subseteq B$ implies that $\beta(A) \le \beta(B)$;
(2) $\beta(A) = \beta(cl_w A)$, where $cl_w A$ denotes the weak closure of A;
(3) $\beta(A) = 0$ if and only if $cl_w A$ is weakly compact;
(4) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$;
(5) $\beta(A) = \beta(\operatorname{conv}(A))$;
(6) $\beta(A + B) \le \beta(A) + \beta(B)$;
(7) $\beta(x + A) = \beta(A)$, for all $x \in E$;
(8) $\beta(\lambda A) = |\lambda|\beta(A)$, for all $\lambda \in \mathbb{R}$;
(9) $\beta(\bigcup_{0\le r\le r_0} rA) = r_0\beta(A)$;
(10) $\beta(A) \le 2\operatorname{diam}(A)$.

Lemma 7 ([39]) *Let $H \subset C(T,E)$ be bounded and equicontinuous. Then*
(i) *the function $t \to \beta(H(t))$ is continuous on T,*
(ii) $\beta_c(H) = \sup_{t\in T} \beta(H(t))$,
where $\beta_c(\cdot)$ denote the weak noncompactness measure on $C(T,E)$ and $H(t) = \{u(t), u \in H\}$, $t \in T$.

Lemma 8 ([21]) *Let E be a metrizable locally convex topological vector space and let K be a closed convex subset of E, and let Q be a weakly sequentially continuous map of K into itself. If for some $y \in K$ the implication*

$$\overline{V} = \overline{\operatorname{conv}}\bigl(Q(V) \cup \{y\}\bigr) \quad \Rightarrow \quad V \text{ is relatively weakly compact}$$

holds, for every subset V of K, then Q has a fixed point.

Theorem 1 *Assume $f(\cdot,\cdot) : T \times E \to E$ is a function such that*:
(H1) *$f(t,\cdot)$ is weakly-weakly sequentially continuous, for every $t \in T$;*
(H2) *$f(\cdot, y(\cdot)) \in P^\infty(T,E)$, for every continuous function $y(\cdot) : T \to E$;*
(H3) *$\|f(t,y)\| \le M$, for all $(t,y) \in T \times E$;*
(H4) *for every bounded set $A \subseteq E$ we have*

$$\beta\bigl(f(T \times A)\bigr) \le g\bigl(\beta(A)\bigr),$$

where $g(\cdot)$ *is a Gripenberg function. Then* (13) *admits a solution* $y(\cdot)$ *on an interval* $T_0 = [0,a]$ *with* $a = \min\{b, (\frac{\Gamma(\alpha+1)}{M})^{1/\alpha}\}$.

Proof In our proof we shall use some ideas from [5] and [6]. We define the nonlinear operator $Q(\cdot) : C(T_0,E) \to C(T_0,E)$ by

$$(Qy)(t) = y_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,y(s)\bigr)\,ds, \quad t \in T_0.$$

If $y(\cdot) \in C(T_0,E)$, then by (H2) we have $f(\cdot,y(\cdot)) \in P^\infty(T,E)$ and so the operator Q makes sense. To show that Q is well defined, let $t_1, t_2 \in T_0$ with $t_2 > t_1$. Without loss of generality, assume that $(Qy)(t_2) - (Qy)(t_1) \neq 0$. Then by the Hahn-Banach theorem, there exists a $y^* \in E^*$ with $\|y^*\| = 1$ and $\|(Qy)(t_2) - (Qy)(t_1)\| = |\langle y^*, (Qy)(t_2) - (Qy)(t_1)\rangle|$. Then

$$\begin{aligned}
&\bigl\|(Qy)(t_2) - (Qy)(t_1)\bigr\| \\
&\quad = \bigl|\bigl\langle y^*, (Qy)(t_2) - (Qy)(t_1)\bigr\rangle\bigr| \\
&\quad = \biggl|\int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl\langle y^*, f\bigl(s,y(s)\bigr)\bigr\rangle\,ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl\langle y^*, f\bigl(s,y(s)\bigr)\bigr\rangle\,ds\biggr| \\
&\quad \le \int_0^{t_1}\biggl(\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\biggr)\bigl|\bigl\langle y^*, f\bigl(s,y(s)\bigr)\bigr\rangle\bigr|\,ds \\
&\qquad + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|\bigl\langle y^*, f\bigl(s,y(s)\bigr)\bigr\rangle\bigr|\,ds \\
&\quad \le \frac{M}{\Gamma(1+\alpha)}\bigl[t_1^\alpha - t_2^\alpha + 2(t_2-t_1)^\alpha\bigr] \le \frac{2M}{\Gamma(1+\alpha)}(t_2-t_1)^\alpha,
\end{aligned} \tag{16}$$

so Q maps $C(T_0,E)$ into itself. Let K be the convex, closed, and equicontinuous set defined by

$$\begin{aligned}
K = \biggl\{&y(\cdot) \in C(T_0,E); \bigl\|y(\cdot)\bigr\|_c \le \|y_0\| + 1, \bigl\|y(t_2) - y(t_1)\bigr\| \\
&\le \frac{2M}{\Gamma(1+\alpha)}|t_2-t_1|^\alpha, \text{ for all } t_1, t_2 \in T_0\biggr\}.
\end{aligned}$$

We will show that Q maps K into itself and Q restricted to the set K is weakly-weakly sequentially continuous. To show that $Q : K \to K$, let $y(\cdot) \in K$ and $t \in T_0$. Again, without loss of generality, assume that $(Qy)(t) \neq 0$. By the Hahn-Banach theorem, there exists a $y^* \in E^*$ with $\|y^*\| = 1$ and $\|(Qy)(t)\| = |\langle y^*, (Qy)(t)\rangle|$. Then by (H3), we have

$$\begin{aligned}
\bigl\|(Qy)(t)\bigr\| &\le \|y_0\| + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|\bigl\langle y^*, f\bigl(s,y(s)\bigr)\bigr\rangle\bigr|\,ds \\
&\le \|y_0\| + \frac{Ma^\alpha}{\Gamma(\alpha+1)} \le \|y_0\| + 1,
\end{aligned}$$

and using (16) it follows that Q maps K into K. Next, we show that Q is weakly-weakly sequentially continuous. First, we recall that the weak convergence in $K \subset C(T_0,E)$ is exactly the weak pointwise convergence. Let $\{y_n(\cdot)\}_{n\ge1}$ be a sequence in K such that $y_n(\cdot)$ converges weakly to $y(\cdot)$ in K. Then $y_n(t)$ converges weakly to $y(t)$ in E for each $t \in T_0$.

Since K is a closed convex set, by Mazur's lemma we have $y(\cdot) \in K$. Further, by (H1) it follows that $f(t, y_n(t))$ converges weakly to $f(t, y(t))$ for each $t \in T_0$. Then the Lebesgue dominated convergence theorem for the Pettis integral (see [40]) yields $I^\alpha y_n(t)$ converging weakly to $I^\alpha y(t)$ in E for each $t \in T_0$. Since K is an equicontinuous subset of $C(T_0, E)$ it follows that $Q(\cdot)$ is weakly-weakly sequentially continuous.

Suppose that $V \subset K$ is such that $V = \overline{\text{co}}(Q(V) \cup \{y(\cdot)\})$ for some $y(\cdot) \in K$. We will show that V is relatively weakly compact in $C(T_0, E)$. Let

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, V(s)\bigr)\, ds = \left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, y(s)\bigr)\, ds; y(\cdot) \in V \right\}$$

and $(QV)(t) = y_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, V(s))\, ds$. Let $t \in T_0$ and $\varepsilon > 0$. If we choose $\eta > 0$ such that $\eta < (\frac{\varepsilon\Gamma(\alpha+1)}{M})^{1/\alpha}$ and $\int_{t-\eta}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))\, ds \neq 0$ then, by the Hahn-Banach theorem, there exists a $y^* \in E^*$ with $\|y^*\| = 1$ and

$$\left\| \int_{t-\eta}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, y(s)\bigr)\, ds \right\| = \left| \left\langle y^*, \int_{t-\eta}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, y(s)\bigr)\, ds \right\rangle \right|.$$

It follows that

$$\left\| \int_{t-\eta}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, y(s)\bigr)\, ds \right\| \le \int_{t-\eta}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl| \bigl\langle y^*, f\bigl(s, y(s)\bigr) \bigr\rangle \bigr|\, ds \le \varepsilon,$$

and thus using property (10) of the noncompactness measure we infer

$$\beta\left(\int_{t-\eta}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, V(s)\bigr)\, ds \right) \le 2\varepsilon. \tag{17}$$

Since by Lemma 7 the function $s \to v(s) := \beta(V(s))$ is continuous on $[0, t-\eta]$ it follows that $s \to (t-s)^{\alpha-1} g(v(s))$ is continuous on $[0, t-\eta]$. Hence, there exists $\delta > 0$ such that

$$\bigl\| (t-\tau)^{\alpha-1} g\bigl(v(\tau)\bigr) - (t-s)^{\alpha-1} g\bigl(v(s)\bigr) \bigr\| < \frac{\varepsilon}{2}$$

and

$$\bigl\| g\bigl(v(\xi)\bigr) - g\bigl(v(\tau)\bigr) \bigr\| < \frac{\varepsilon}{2\eta^{\alpha-1}},$$

for all $\tau, s, \xi \in [0, t-\eta]$ with $|\tau - s| < \delta$ and $|\tau - \xi| < \delta$. It follows that

$$\begin{aligned} &\bigl| (t-\tau)^{\alpha-1} g\bigl(v(\xi)\bigr) - (t-s)^{\alpha-1} g\bigl(v(s)\bigr) \bigr| \\ &\quad \le \bigl| (t-\tau)^{\alpha-1} g\bigl(v(\tau)\bigr) - (t-s)^{\alpha-1} g\bigl(v(s)\bigr) \bigr| + (t-\tau)^{\alpha-1} \bigl| g\bigl(v(\xi)\bigr) - g\bigl(v(\tau)\bigr) \bigr| < \varepsilon, \end{aligned}$$

that is,

$$\bigl| (t-\tau)^{\alpha-1} g\bigl(v(\xi)\bigr) - (t-s)^{\alpha-1} g\bigl(v(s)\bigr) \bigr| < \varepsilon, \tag{18}$$

for all $\tau, s, \xi \in [0, t-\xi]$ with $|\tau - s| < \delta$ and $|\tau - \xi| < \delta$. Consider a partition of the interval $[0, t-\eta]$ into n parts $0 = t_0 < t_1 < \cdots < t_n = t - \eta$ such that $t_i - t_{i-1} < \delta$, $i = 1, 2, \ldots, n$.

From Lemma 7 it follows that for each $i \in \{1,2,\dots,n\}$ there exists $s_i \in [t_{i-1},t_i]$ such that $\beta(V([t_{i-1},t_i])) = \nu(s_i)$, $i = 1,2,\dots,n$. Then we have (see [41], Theorem 2.2)

$$\begin{aligned}
&\int_0^{t-n} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,V(s)\bigr)\,ds \\
&\quad\subset \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (t-s)^{\alpha-1} f\bigl(s,V(s)\bigr)\,ds \\
&\quad\subset \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1}) \overline{\operatorname{conv}}\bigl\{(t-s)^{\alpha-1} f\bigl(s,y(s)\bigr); s \in [t_{i-1},t_i], y \in V\bigr\},
\end{aligned}$$

and so

$$\begin{aligned}
&\beta\biggl(\int_0^{t-\eta} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,V(s)\bigr)\,ds\biggr) \\
&\quad\le \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1}) \beta\bigl(\overline{\operatorname{conv}}\bigl\{(t-s)^{\alpha-1} f\bigl(s,y(s)\bigr); s \in [t_{i-1},t_i], y \in V\bigr\}\bigr) \\
&\quad= \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1}) \beta\bigl(\bigl\{(t-s)^{\alpha-1} f\bigl(s,y(s)\bigr); s \in [t_{i-1},t_i], y \in V\bigr\}\bigr) \\
&\quad\le \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1})(t-t_i)^{\alpha-1} \beta\bigl(f\bigl(T_0 \times V[t_{i-1},t_i]\bigr)\bigr) \\
&\quad\le \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1})(t-t_i)^{\alpha-1} g\bigl(\beta\bigl(V[t_{i-1},t_i]\bigr)\bigr) \\
&\quad= \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1})(t-t_i)^{\alpha-1} g\bigl(\nu(s_i)\bigr).
\end{aligned}$$

Using (18) we have

$$\bigl|(t-t_i)^{\alpha-1} g\bigl(\nu(s_i)\bigr) - (t-s)^{\alpha-1} g\bigl(\nu(s)\bigr)\bigr| < \varepsilon\Gamma(\alpha+1).$$

This implies that

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1})(t-t_i)^{\alpha-1} g\bigl(\nu(s_i)\bigr) \le \int_0^{t-\eta} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(\nu(s)\bigr)\,ds + \varepsilon\bigl(t^\alpha - \eta^\alpha\bigr).$$

Thus we obtain

$$\beta\biggl(\int_0^{t-\eta} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,V(s)\bigr)\,ds\biggr) \le \int_0^{t-\eta} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(\nu(s)\bigr)\,ds + \varepsilon\bigl(t^\alpha - \eta^\alpha\bigr). \tag{19}$$

Now because

$$(QV)(t) \subset \int_0^{t-\eta} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,V(s)\bigr)\,ds + \int_{t-\eta}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,V(s)\bigr)\,ds,$$

then from (17) and (19) we have

$$\begin{aligned}\beta\big((QV)(t)\big) &\le \int_0^{t-\eta} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\big(v(s)\big)\,ds + \varepsilon\big(t^\alpha - \eta^\alpha\big) + 2\varepsilon \\ &\le \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\big(v(s)\big)\,ds + \varepsilon\big(t^\alpha - \eta^\alpha + 2\big).\end{aligned}$$

As the last inequality is true, for every $\varepsilon > 0$, we infer

$$\beta\big((QV)(t)\big) \le \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\big(v(s)\big)\,ds.$$

Because $V = \overline{\text{co}}(Q(V) \cup \{y(\cdot)\})$ then

$$\beta\big(V(t)\big) = \beta\big(\overline{\text{co}}\big(Q(V) \cup \big\{y(\cdot)\big\}\big)\big) \le \beta\big((QV)(t)\big)$$

and thus

$$v(t) \le \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\big(v(s)\big)\,ds \quad \text{for } t \in T_0.$$

Since $g(\cdot)$ is a Gripenberg function, it follows that $v(t) = 0$ for $t \in T_0$. Since V as a subset of K is equicontinuous, by Lemma 7 we infer

$$\beta_c\big(V(T_0)\big) = \sup_{t\in T_0} \beta\big(V(t)\big) = 0.$$

Thus, by Arzelá-Ascoli's theorem we find that V is weakly relatively compact in $C(T_0, E)$. Using Lemma 8 there exists a fixed point of the operator Q which is a solution of (13). □

If E is reflexive and $f(\cdot,\cdot) : T \times E \to E$ is bounded, then (H4) is automatically satisfied since a subset of a reflexive Banach space is weakly compact iff it is closed in the weak topology and bounded in the norm topology.

If for $\alpha = 1$ we put $D_p^1 y(\cdot) = y_p'(\cdot)$, then from Theorem 1 we obtain the following result (see [18, 23]).

Corollary 1 *If $f(\cdot,\cdot) : T \times E \to E$ is a function that satisfies the conditions* (H1)-(H4) *in Theorem* 1, *then the differential equation*

$$\begin{cases} y_p'(t) = f(t, y(t)), \\ y(0) = y_0, \end{cases} \tag{20}$$

has a solution on $[0, a]$ *with* $a = \min\{b, 1/M\}$.

4 Multi-term fractional differential equation

The case of multi-term fractional differential equations in reflexive Banach spaces was recently considered in [42–45]. Consider the following multi-term fractional differential equation:

$$\left(D^{\alpha_m} - \sum_{i=1}^{m-1} a_i D^{\alpha_i}\right) y(t) = f\big(t, y(t)\big) \quad \text{for } t \in [0,1], \qquad y(0) = 0, \tag{21}$$

where $D^{\alpha_i} y(\cdot)$, $i = 1, 2, \ldots, m$, are fractional pseudo-derivatives of order $\alpha_i \in (0,1)$ of a pseudo-differentiable function $y(\cdot) : [0,1] \to E$, $f(t,\cdot) : [0,1] \times E \to E$ is weakly-weakly sequentially continuous, for every $t \in [0,1]$, and $f(\cdot, y(\cdot))$ is Pettis integrable, for every continuous function $y(\cdot) : [0,1] \to E$, E is a nonreflexive Banach space, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < 1$ and $a_1, a_2, \ldots, a_{m-1}$ are real numbers such that $a := \sum_{i=1}^{m-1} \frac{|a_i|}{\Gamma(\alpha_m - \alpha_i + 1)} < 1$.

Along with the Cauchy problem (21) consider the following integral equation:

$$y(t) = \sum_{i=1}^{m-1} a_i I^{\alpha_m - \alpha_i} y(t) + I^{\alpha_m} f\big(t, y(t)\big), \tag{22}$$

$t \in T$, where the integral is in the sense of Pettis and $T = [0,1]$.

A continuous function $y(\cdot) : T \to E$ is said to be a solution of (21) if

(i) $y(\cdot)$ has Caputo fractional pseudo-derivatives of orders $\alpha_i \in (0,1)$, $i = 1, 2, \ldots, m$,
(ii) $(D^{\alpha_m} - \sum_{i=1}^{m-1} a_i D^{\alpha_i}) y(t) \approx f(t, y(t))$, for all $t \in T$,
(iii) $y(0) = 0$.

Lemma 9 *Assume that* $f(\cdot,\cdot) : T \times E \to E$ *satisfy the assumptions* (H2) *and* (H3) *in Theorem* 1. *Then every continuous function* $y(\cdot) : T \to E$ *which satisfies the integral equation* (22) *is a solution of* (21).

Proof Suppose that a continuous function $y(\cdot) : T \to E$ satisfies the integral equation (14). Then $z(\cdot) := f(\cdot, y(\cdot)) \in P^\infty(T, E)$ satisfies the Abel equation

$$\int_0^t \frac{(t-s)^{\alpha_m - 1}}{\Gamma(\alpha_m)} z(s)\, ds = v(t), \quad t \in T,$$

where $v(t) := y(t) - \sum_{i=1}^{m-1} a_i I^{\alpha_m - \alpha_i} y(t)$, $t \in T$. From [1], Theorem 3.1, it follows that $v_{1-\alpha_m}(\cdot)$ has a pseudo-derivative on T and

$$z(t) \approx \frac{d_p}{dt} v_{1-\alpha_m}(t) \quad \text{for } t \in T,$$

Since $y(\cdot)$ is continuous on T and $f(\cdot, y(\cdot)) \in P^\infty(T, E)$ satisfies (H3), we have

$$\lim_{t \to 0^+} I^\alpha y(t) = \lim_{t \to 0^+} I^\alpha f\big(t, y(t)\big) = 0$$

for $\alpha \in (0,1)$ and thus, taking the limit as $t \to 0^+$ on both equalities in (22), we obtain $y(0) = 0$ and consequently $v(0) = 0$. Since $v(0) = 0$, by Remark 2 we have

$$z(t) \approx \frac{d_p}{dt} v_{1-\alpha_m}(t) = \mathcal{D}_p^{\alpha_m} v(t) = D_p^{\alpha_m} v(t), \quad t \in T.$$

Since by Lemma 5(b) we have

$$D_p^{\alpha_m} v(t) = D_p^{\alpha_m} y(t) - \sum_{i=1}^{m-1} a_i D_p^{\alpha_m} I^{\alpha_m-\alpha_i} y(t) = D_p^{\alpha_m} y(t) - \sum_{i=1}^{m-1} a_i D_p^{\alpha_i} y(t),$$

we obtain

$$\left(D_p^{\alpha_m} - \sum_{i=1}^{m-1} a_i D_p^{\alpha_i}\right) y(t) \approx f\big(t, y(t)\big), \quad t \in T.$$

Hence the continuous function $y(\cdot)$ satisfy the conditions (i)-(iii) from definition and thus $y(\cdot)$ is a solution of (21). □

Lemma 10 ([24], Theorem 2.2) *Let K be a nonempty, bounded, convex, closed set in a Banach space E. Assume $Q : K \to K$ is weakly sequentially continuous and β-contractive (that is, there exists $0 \leq k_0 < 1$ such that $\beta(Q(A)) \leq k_0 \beta(A)$, for all bounded sets $A \subset E$). Then Q has a fixed point.*

Remark 4 Since the function $\sigma \mapsto \Gamma(\sigma)$ is convex and $\Gamma(\sigma) \geq \Gamma(3/2) \approx 0.8856032$ for $\sigma \in (1,2)$, for every $r \in (0, \Gamma(3/2))$ we have $\Gamma(\alpha_m + 1) > r$.

Next we establish an existence result for the multi-term fractional integral equation (22) in nonreflexive Banach spaces.

Theorem 2 *Suppose that $f(\cdot,\cdot) : T \times E \to E$ satisfies the conditions* (H1)-(H3) *in Theorem* 1 *and there exists $L > 0$ such that, for every bounded set $A \subseteq E$, we have*

$$\beta\big(f(T \times A)\big) \leq L\beta(A).$$

If $r \in (0,1)$ is such that $\Gamma(\alpha_m + 1) > r$, then (22) *admits a solution $y(\cdot)$ on an interval $T_0 = [0, a_0]$ with*

$$a_0 < \min\left\{\frac{r}{r+L}, \left[\frac{(1-a)\Gamma(\alpha_m+1)}{M}\right]^{1/\alpha_m}\right\}.$$

Proof We define the nonlinear operator $Q(\cdot) : C(T_0, E) \to C(T_0, E)$ by

$$(Qy)(t) = \sum_{i=1}^{m-1} a_i I^{\alpha_m-\alpha_i} y(t) + I^{\alpha_m} f\big(t, y(t)\big),$$

for all $t \in T_0$. We remark that a solution of integral equation (22) is a fixed point of the operator Q. If $y(\cdot) \in C(T_0, E)$, then by (H2) we have $f(\cdot, y(\cdot)) \in P^\infty(T_0, E)$ and so the operator Q makes sense. To show that Q is well defined, let $t, s \in T_0$ with $t > s$. Without loss of generality, assume that $(Qy)(t) - (Qy)(s) \neq 0$. Then by the Hahn-Banach theorem, there exists a $y^* \in E^*$ with $\|y^*\| = 1$ and $\|(Qy)(t) - (Qy)(s)\| = |\langle y^*, (Qy)(t) - (Qy)(s)\rangle|$. Then

$$\begin{aligned}
&\big\|(Qy)(t) - (Qy)(s)\big\| \\
&\quad = \big|\langle y^*, (Qy)(t) - (Qy)(s)\rangle\big| \\
&\quad \le \sum_{i=1}^{m-1} \frac{|a_i|}{\Gamma(\alpha_m - \alpha_i)} \biggl[\int_0^s \bigl[(s-\tau)^{\alpha_m-\alpha_i-1} - (t-\tau)^{\alpha_m-\alpha_i-1}\bigr] \big|\langle y^*, y(\tau)\rangle\big|\, d\tau \\
&\qquad + \int_s^t (t-\tau)^{\alpha_m-\alpha_i-1} \big|\langle y^*, y(\tau)\rangle\big|\, d\tau\biggr] \\
&\qquad + \frac{1}{\Gamma(\alpha_m)} \biggl[\int_0^s \bigl[(s-\tau)^{\alpha_m-1} - (t-\tau)^{\alpha_m-1}\bigr] \big|\langle y^*, f(\tau, y(\tau))\rangle\big|\, d\tau \\
&\qquad + \int_s^t (t-\tau)^{\alpha_m-1} \big|\langle y^*, f(\tau, y(\tau))\rangle\big|\, d\tau\biggr] \\
&\quad \le 2\Biggl[\sum_{i=1}^{m-1} \frac{|a_i|}{\Gamma(\alpha_m - \alpha_i + 1)} \|y\|_c + \frac{M}{\Gamma(\alpha_m + 1)}\Biggr] (t-s)^{\alpha_m},
\end{aligned} \tag{23}$$

so Q maps $C(T_0, E)$ into itself. Let $\delta \geq 1$ and let K be the convex, closed, bounded and equicontinuous set defined by

$$\begin{aligned}
K = \Biggl\{ &y(\cdot) \in C(T_0, E); \big\|y(\cdot)\big\|_c \le \delta, \big\|y(t) - y(s)\big\| \\
&\le 2\Biggl[\sum_{i=1}^{m-1} \frac{\delta|a_i|}{\Gamma(\alpha_m - \alpha_i + 1)} + \frac{M}{\Gamma(\alpha_m + 1)}\Biggr] |t-s|^{\alpha_m}, \text{ for all } t, s \in T_0 \Biggr\}.
\end{aligned}$$

Without loss of generality, assume that $(Qy)(t) \neq 0$. By the Hahn-Banach theorem, there exists a $y^* \in E^*$ with $\|y^*\| = 1$ and $\|(Qy)(t)\| = |\langle y^*, (Qy)(t)\rangle|$. Then by (H3), we have

$$\begin{aligned}
\big\|(Qy)(t)\big\| &= \big|\langle y^*, (Qy)(t)\rangle\big| \\
&\le \sum_{i=1}^{m-1} |a_i| \int_0^t \frac{(t-s)^{\alpha_m-\alpha_i-1}}{\Gamma(\alpha_m - \alpha_i)} \big|\langle y^*, y(\tau)\rangle\big|\, ds + \int_0^t \frac{(t-s)^{\alpha_m-1}}{\Gamma(\alpha_m)} \big|\langle y^*, f(\tau, y(\tau))\rangle\big|\, ds \\
&\le \sum_{i=1}^{m-1} \frac{\delta|a_i|}{\Gamma(\alpha_m - \alpha_i + 1)} + \frac{M a_0^{\alpha_m}}{\Gamma(\alpha_m + 1)} \le \delta a + (1-a)\delta = \delta
\end{aligned}$$

and using (23) it follows that Q maps K into K. Following the same reasoning as in the proof of Theorem 1 it is easy to show that Q is weakly-weakly sequentially continuous from K to K. Next, we will prove that Q has at least one fixed point $y_0(\cdot) \in K$. Let $V \subset K$ be such that $\beta_c(V) > 0$. Next, to simplify the writing of some relations, we will use the following notations:

$$\begin{aligned}
A(t) &:= \int_0^{t-\eta} \frac{(t-s)^{\alpha_m-1}}{\Gamma(\alpha_m)} f\bigl(s, y(s)\bigr)\, ds, \\
B(t) &:= \sum_{i=1}^{m-1} a_i \int_0^{t-\eta} \frac{(t-s)^{\alpha_m-\alpha_i-1}}{\Gamma(\alpha_m - \alpha_i)} y(s)\, ds,
\end{aligned}$$

$$C(t) := \int_{t-\eta}^{t} \frac{(t-s)^{\alpha_m-1}}{\Gamma(\alpha_m)} f\bigl(s, y(s)\bigr)\,ds,$$
$$D(t) := \sum_{i=1}^{m-1} a_i \int_{t-\eta}^{t} \frac{(t-s)^{\alpha_m-\alpha_i-1}}{\Gamma(\alpha_m-\alpha_i)} y(s)\,ds,$$

for $t \in T_0$. Then it is easy to see that $|\langle y^*, C(t)\rangle| \le \frac{M\eta^{\alpha_m}}{\Gamma(\alpha_m+1)}$ and $|\langle y^*, D(t)\rangle| \le \sum_{i=1}^{m-1} \frac{r|a_i|\eta^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m-\alpha_i+1)}$, for all $y^* \in E^*$ with $\|y^*\| = 1$. Let $t \in T_0$ and $\varepsilon > 0$. If we choose $\eta > 0$ such that $\eta < (\frac{\varepsilon\Gamma(\alpha_m+1)}{r[M+\Gamma(\alpha_m+1)]})^{1/\alpha_m}$ and $C(t) + D(t) \neq 0$, then by the Hahn-Banach theorem, there exists a $y^* \in E^*$ with $\|y^*\| = 1$ and

$$\begin{aligned}\bigl\|C(t) + D(t)\bigr\| &= \bigl|\langle y^*, C(t) + D(t)\rangle\bigr| \\ &\le \sum_{i=1}^{m-1} \frac{r|a_i|\eta^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m-\alpha_i+1)} + \frac{M\eta^{\alpha_m}}{\Gamma(\alpha_m+1)} \\ &\le r\eta^{\alpha_m} + \frac{M\eta^{\alpha_m}}{\Gamma(\alpha_m+1)} \le r\frac{M+\Gamma(\alpha_m+1)}{\Gamma(\alpha_m+1)}\eta^{\alpha_m} < \varepsilon,\end{aligned}$$

and thus using property (10) of the measure of noncompactness we infer

$$\beta\bigl((CV)(t) + (DV)(t)\bigr) \le 2\varepsilon.$$

As in the proof of Theorem 1, with $g(t) = L$, $t \in T_0$, we obtain

$$\beta\bigl((AV)(t)\bigr) \le L\int_0^{t-\eta} \frac{(t-s)^{\alpha_m-1}}{\Gamma(\alpha_m)} \beta\bigl(V(s)\bigr)\,ds + \varepsilon\bigl(t^\alpha - \eta^\alpha\bigr).$$

Also, with $y(\cdot)$ instead of $f(\cdot, y(\cdot))$, we have

$$\begin{aligned}&\beta\left(a_i \int_0^{t-\eta} \frac{(t-s)^{\alpha_m-\alpha_i-1}}{\Gamma(\alpha_m-\alpha_i)} V(s)\,ds\right) \\ &\quad\le |a_i| \int_0^{t-\eta} \frac{(t-s)^{\alpha_m-\alpha_i-1}}{\Gamma(\alpha_m-\alpha_i)} \beta\bigl(V(s)\bigr)\,ds + \frac{\varepsilon}{m-1}\bigl(t^\alpha - \eta^\alpha\bigr),\end{aligned}$$

and so

$$\beta\bigl((BV)(t)\bigr) \le \sum_{i=1}^{m-1} |a_i| \int_0^{t-\eta} \frac{(t-s)^{\alpha_m-\alpha_i-1}}{\Gamma(\alpha_m-\alpha_i)} \beta\bigl(V(s)\bigr)\,ds + \varepsilon\bigl(t^\alpha - \eta^\alpha\bigr).$$

Next, since $(QV)(t) = (AV)(t) + (CV)(t) + (BV)(t) + (DV)(t)$, $t \in T_0$, then from the last inequalities and using properties of the noncompactness measure we infer

$$\begin{aligned}\beta\bigl((QV)(t)\bigr) &\le \beta\bigl((AV)(t)\bigr) + \beta\bigl((BV)(t)\bigr) + \beta\bigl((CV)(t) + (DV)(t)\bigr) \\ &\le L\int_0^{t} \frac{(t-s)^{\alpha_m-1}}{\Gamma(\alpha_m)} \beta\bigl(V(s)\bigr)\,ds + \sum_{i=1}^{m-1} |a_i| \int_0^{t} \frac{(t-s)^{\alpha_m-\alpha_i-1}}{\Gamma(\alpha_m-\alpha_i)} \beta\bigl(V(s)\bigr)\,ds \\ &\quad + 3\varepsilon\bigl(t^\alpha - \eta^\alpha\bigr) + 2\varepsilon.\end{aligned}$$

As the last inequality is true, for every $\varepsilon > 0$, it follows that

$$\beta\big((QV)(t)\big) \le LI^{\alpha_m}\beta\big(V(t)\big) + \sum_{i=1}^{m-1}|a_i|I^{\alpha_m-\alpha_i}\beta\big(V(t)\big), \quad t \in T_0.$$

Since $\beta(V(t)) \le \beta_c(V)$, $t \in T_0$, we have

$$\beta\big((QV)(t)\big) \le \left(\frac{Lt^{\alpha_m}}{\Gamma(\alpha_m+1)} + t^{\alpha_m-\alpha_i}\right)\beta_c(V) \le \left(\frac{a_0L}{r} + a_0\right)\beta_c(V) \le k_0\beta_c(V),$$

where $k_0 = a_0(1 + \frac{L}{r}) < 1$. It follows that $\beta_c(QV) < k_0\beta_c(V)$, for every set $V \subset K$ with $\beta_c(V) > 0$; that is, $Q : K \to K$ is a β_c-contractive operator. Since K is a nonempty, closed, convex, bounded subset in $C(T_0, E)$, and $Q : K \to K$ is weakly sequentially continuous and β_c-contractive, by Lemma 10 it follows that the operator Q has a fixed point $y_0(\cdot) \in K$. □

Using Lemma 9 we obtain the following result.

Corollary 2 *If the assumptions of Theorem* 2 *are satisfied*, *then the problem* (21) *has at least one solution*.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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