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On an implicit convexity concept and some integral inequalities

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Abstract

We introduce a new concept of convexity that depends on a function $F:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times(0,1)\to\mathbb{R}$ satisfying certain axioms. The presented concept generalizes many kinds of convexity including ε-convex functions, α-convex functions, and h-convex functions. Moreover, some integral inequalities are provided via our notion of convexity.

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Keywords: F-convex; integral inequalities

1 Introduction

Convexity is an important concept in many branches of mathematics, pure and applied. In particular, many important integral inequalities are based on a convexity assumption of a certain function, such as Jensen's inequality, the Hermite-Hadamard inequality, the Hardy-Littlewood-Pólya majoration inequality, Petrović's inequality, Popoviciui's convex function inequality, and many others. For more details as regards inequalities via convex functions, we refer the reader to the monograph [1]. However, for many encountered problems, convexity is not satisfied. This leads to the necessity to extend this concept.

In the last 60 years, great attention has been focused on the generalization of the notion of convexity. Let us cite some references in this direction. In [2], Definetti introduced the class of quasi-convex functions. In [3], Mangasarian introduced the notion of pseudo-convex functions. Polyak [4] defined the concept of strongly convex functions. The class of ε-convex functions was introduced by Hyers and Ulam [5]. In [6], Varosanec introduced the notion of h-convexity that includes the class of s-convex functions (see Hudzik [7]). For other work in this direction, we refer the reader to [8–12] and the references therein.

In this paper, we present a new concept of convexity that depends on a certain function satisfying some axioms. This new notion generalizes different types of convexity, including ε-convex functions, α-convex functions, h-convex functions, and many others. Moreover, some integral inequalities are established via this new notion of convexity. As particular cases, we retrieve several existing inequalities from the literature.

2 An implicit convexity concept

We denote by $\mathcal{F}$ the family of mappings $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times (0,1) \to \mathbb{R}$ satisfying the following axioms:

(A1) If $u_i \in L^1(0,1)$, $i = 1,2,3$, then for every $\lambda \in (0,1)$, we have

$$\int_0^1 F\big(u_1(t), u_2(t), u_3(t), \lambda\big)\, dt = F\left(\int_0^1 u_1(t)\, dt, \int_0^1 u_2(t)\, dt, \int_0^1 u_3(t)\, dt, \lambda\right).$$

(A2) For every $u \in L^1(0,1)$, $w \in L^\infty(0,1)$, and $(z_1, z_2) \in \mathbb{R}^2$, we have

$$\int_0^1 F\big(w(t)u(t), w(t)z_1, w(t)z_2, t\big)\, dt = T_{F,w}\left(\int_0^1 w(t)u(t)\, dt, z_1, z_2\right),$$

where $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function that depends on (F, w), and it is nondecreasing with respect to the first variable.

(A3) For any $(w, u_1, u_2, u_3) \in \mathbb{R}^4$, $u_4 \in (0,1)$, we have

$$wF(u_1, u_2, u_3, u_4) = F(wu_1, wu_2, wu_3, u_4) + L_w,$$

where $L_w \in \mathbb{R}$ is a constant that depends only on w.

We introduce the new concept of convexity as follows.

Definition 2.1 Let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be a given function. We say that f is a convex function with respect to some $F \in \mathcal{F}$ (or F-convex function) iff

$$F\big(f\big(tx + (1-t)y\big), f(x), f(y), t\big) \leq 0, \quad (x,y,t) \in [a,b] \times [a,b] \times (0,1). \tag{2.1}$$

The following property follows immediately from (2.1).

Proposition 2.2 *Let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be an F-convex function, for some $F \in \mathcal{F}$. Then*

$$F\big(f(x), f(x), f(x), t\big) \leq 0, \quad (x,t) \in [a,b] \times (0,1).$$

Proof Taking $x = y$ in (2.1), the desired inequality follows. □

Now, we give some examples of F-convex functions.

Example 2.3 Let $\varepsilon \geq 0$, and let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be an ε-convex function, that is (see [5]),

$$f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y) + \varepsilon, \quad (x,y,t) \in [a,b] \times [a,b] \times [0,1].$$

Define the function $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}$ by

$$F(u_1, u_2, u_3, u_4) = u_1 - u_4 u_2 - (1-u_4)u_3 - \varepsilon, \quad (u_1, u_2, u_3, u_4) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1]. \tag{2.2}$$

Let $u_i \in L^1(0,1)$, $i = 1,2,3$, and let $\lambda \in [0,1]$. We have

$$\begin{aligned}\int_0^1 F\big(u_1(t), u_2(t), u_3(t), \lambda\big)\,dt &= \int_0^1 \big(u_1(t) - \lambda u_2(t) - (1-\lambda)u_3(t) - \varepsilon\big)\,dt \\ &= \int_0^1 u_1(t)\,dt - \lambda \int_0^1 u_2(t)\,dt - (1-\lambda)\int_0^1 u_3(t)\,dt - \varepsilon \\ &= F\left(\int_0^1 u_1(t)\,dt, \int_0^1 u_2(t)\,dt, \int_0^1 u_3(t)\,dt, \lambda\right).\end{aligned}$$

Therefore, the function F satisfies axiom (A1). Now, let $u \in L^1(0,1)$, $w \in L^\infty(0,1)$, and $(z_1, z_2) \in \mathbb{R}^2$. We have

$$\begin{aligned}&\int_0^1 F\big(w(t)u(t), w(t)z_1, w(t)z_2, t\big)\,dt \\ &\quad= \int_0^1 \big(w(t)u(t) - tw(t)z_1 - (1-t)w(t)z_2 - \varepsilon\big)\,dt \\ &\quad= \int_0^1 w(t)u(t)\,dt - z_1 \int_0^1 tw(t)\,dt - z_2 \int_0^1 (1-t)w(t)\,dt - \varepsilon \\ &\quad= T_{F,w}\left(\int_0^1 w(t)u(t)\,dt, z_1, z_2\right),\end{aligned}$$

where $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\begin{aligned}&T_{F,w}(u_1, u_2, u_3) = u_1 - \left(\int_0^1 tw(t)\,dt\right)u_2 - \left(\int_0^1 (1-t)w(t)\,dt\right)u_3 - \varepsilon, \\ &(u_1, u_2, u_3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}.\end{aligned} \tag{2.3}$$

Then the function F satisfies axiom (A2). Now, let $(w, u_1, u_2, u_3) \in \mathbb{R}^4$ and $u_4 \in (0,1)$. We have

$$\begin{aligned}wF(u_1, u_2, u_3, u_4) &= w\big(u_1 - u_4 u_2 - (1-u_4)u_3 - \varepsilon\big) \\ &= wu_1 - u_4(wu_2) - (1-u_4)(wu_3) - w\varepsilon \\ &= \big(wu_1 - u_4(wu_2) - (1-u_4)(wu_3) - \varepsilon\big) + (1-w)\varepsilon \\ &= F(wu_1, wu_2, wu_3, u_4) + (1-w)\varepsilon.\end{aligned}$$

Therefore, axiom (A3) is satisfied with

$$L_w = (1-w)\varepsilon. \tag{2.4}$$

Thus we proved that $F \in \mathcal{F}$. On the other hand, since f is ε-convex, for all $(x, y, t) \in [a,b] \times [a,b] \times (0,1)$, we have

$$F\big(f\big(tx + (1-t)y\big), f(x), f(y), t\big) = f\big(tx + (1-t)y\big) - tf(x) - (1-t)f(y) - \varepsilon \leq 0.$$

As a consequence, f is an F-convex function.

Remark 2.4 Taking $\varepsilon = 0$ in the above example, we observe that any convex function $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, is an F-convex function with respect to the function $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times (0,1) \to \mathbb{R}$ defined by

$$F(u_1,u_2,u_3,u_4) = u_1 - u_4u_2 - (1-u_4)u_3, \quad (u_1,u_2,u_3,u_4) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times (0,1).$$

Example 2.5 Let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be an α-convex function, $0 < \alpha \leq 1$, that is,

$$f\big(tx + (1-t)y\big) \leq t^\alpha f(x) + \big(1-t^\alpha\big)f(y), \quad (x,y,t) \in [a,b] \times [a,b] \times [0,1].$$

Define the function $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}$ by

$$F(u_1,u_2,u_3,u_4) = u_1 - u_4^\alpha u_2 - \big(1-u_4^\alpha\big)u_3, \quad (u_1,u_2,u_3,u_4) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1]. \tag{2.5}$$

Let $u_i \in L^1(0,1)$, $i = 1,2,3$, and let $\lambda \in [0,1]$. We have

$$\begin{aligned} \int_0^1 F\big(u_1(t),u_2(t),u_3(t),\lambda\big)\,dt &= \int_0^1 \big(u_1(t) - \lambda^\alpha u_2(t) - \big(1-\lambda^\alpha\big)u_3(t)\big)\,dt \\ &= \int_0^1 u_1(t)\,dt - \lambda^\alpha \int_0^1 u_2(t)\,dt - \big(1-\lambda^\alpha\big)\int_0^1 u_3(t)\,dt \\ &= F\bigg(\int_0^1 u_1(t)\,dt, \int_0^1 u_2(t)\,dt, \int_0^1 u_3(t)\,dt, \lambda\bigg). \end{aligned}$$

Therefore, the function F satisfies axiom (A1). Now, let $u \in L^1(0,1)$, $w \in L^\infty(0,1)$, and $(z_1,z_2) \in \mathbb{R}^2$. We have

$$\begin{aligned} &\int_0^1 F\big(w(t)u(t), w(t)z_1, w(t)z_2, t\big)\,dt \\ &\quad= \int_0^1 \big(w(t)u(t) - t^\alpha w(t)z_1 - \big(1-t^\alpha\big)w(t)z_2\big)\,dt \\ &\quad= \int_0^1 w(t)u(t)\,dt - z_1\int_0^1 t^\alpha w(t)\,dt - z_2\int_0^1 \big(1-t^\alpha\big)w(t)\,dt \\ &\quad= T_{F,w}\bigg(\int_0^1 w(t)u(t)\,dt, z_1, z_2\bigg), \end{aligned}$$

where $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\begin{aligned} &T_{F,w}(u_1,u_2,u_3) = u_1 - \bigg(\int_0^1 t^\alpha w(t)\,dt\bigg)u_2 - \bigg(\int_0^1 \big(1-t^\alpha\big)w(t)\,dt\bigg)u_3, \\ &(u_1,u_2,u_3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}. \end{aligned} \tag{2.6}$$

Then the function F satisfies axiom (A2). Now, let $(w,u_1,u_2,u_3) \in \mathbb{R}^4$ and $u_4 \in (0,1)$. We have

$$\begin{aligned} wF(u_1,u_2,u_3,u_4) &= w\big(u_1 - u_4^\alpha u_2 - \big(1-u_4^\alpha\big)u_3\big) = wu_1 - u_4^\alpha(wu_2) - \big(1-u_4^\alpha\big)(wu_3) \\ &= F(wu_1,wu_2,wu_3,u_4). \end{aligned}$$

Therefore, axiom (A3) is satisfied with

$$L_w = 0. \tag{2.7}$$

Thus we proved that $F \in \mathcal{F}$. On the other hand, since f is α-convex, for all $(x,y,t) \in [a,b] \times [a,b] \times (0,1)$, we have

$$\begin{aligned} F\big(f\big(tx+(1-t)y\big),f(x),f(y),t\big) &= f\big(tx+(1-t)y\big) - t^\alpha f(x) - \big(1-t^\alpha\big)f(y) \\ &\leq 0. \end{aligned}$$

As a consequence, f is an F-convex function.

Example 2.6 Let $h : J \to [0,\infty)$ be a given function which is not identical to 0, where J is an interval in $\mathbb{R}$ such that $(0,1) \subseteq J$. Let $f : [a,b] \to [0,\infty)$, $(a,b) \in \mathbb{R}^2$, $a < b$, be a h-convex function, that is (see [6]),

$$f\big(tx+(1-t)y\big) \leq h(t)f(x) + h(1-t)f(y), \quad (x,y,t) \in [a,b] \times [a,b] \times (0,1).$$

We suppose that $h \in L^1(0,1)$. Define the function $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times (0,1) \to \mathbb{R}$ by

$$\begin{aligned} &F(u_1,u_2,u_3,u_4) = u_1 - h(u_4)u_2 - h(1-u_4)u_3, \\ &\quad (u_1,u_2,u_3,u_4) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times (0,1). \end{aligned} \tag{2.8}$$

Let $u_i \in L^1(0,1)$, $i = 1,2,3$, and let $\lambda \in [0,1]$. We have

$$\begin{aligned} \int_0^1 F\big(u_1(t),u_2(t),u_3(t),\lambda\big)\,dt &= \int_0^1 \big(u_1(t) - h(\lambda)u_2(t) - h(1-\lambda)u_3(t)\big)\,dt \\ &= \int_0^1 u_1(t)\,dt - h(\lambda)\int_0^1 u_2(t)\,dt - h(1-\lambda)\int_0^1 u_3(t)\,dt \\ &= F\bigg(\int_0^1 u_1(t)\,dt, \int_0^1 u_2(t)\,dt, \int_0^1 u_3(t)\,dt, \lambda\bigg). \end{aligned}$$

Therefore, the function F satisfies axiom (A1). Now, let $u \in L^1(0,1)$, $w \in L^\infty(0,1)$, and $(z_1,z_2) \in \mathbb{R}^2$. We have

$$\begin{aligned} &\int_0^1 F\big(w(t)u(t),w(t)z_1,w(t)z_2,t\big)\,dt \\ &\quad = \int_0^1 \big(w(t)u(t) - h(t)w(t)z_1 - h(1-t)w(t)z_2\big)\,dt \\ &\quad = \int_0^1 w(t)u(t)\,dt - z_1\int_0^1 h(t)w(t)\,dt - z_2\int_0^1 h(1-t)w(t)\,dt \\ &\quad = T_{F,w}\bigg(\int_0^1 w(t)u(t)\,dt, z_1, z_2\bigg), \end{aligned}$$

where $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$T_{F,w}(u_1, u_2, u_3) = u_1 - \left(\int_0^1 h(t)w(t)\,dt\right)u_2 - \left(\int_0^1 h(1-t)w(t)\,dt\right)u_3,$$
$$(u_1, u_2, u_3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}. \tag{2.9}$$

Then the function F satisfies axiom (A2). Now, let $(w, u_1, u_2, u_3) \in \mathbb{R}^4$ and $u_4 \in (0,1)$. We have

$$\begin{aligned} wF(u_1, u_2, u_3, u_4) &= w\bigl(u_1 - h(u_4)u_2 - h(1-u_4)u_3\bigr) \\ &= wu_1 - h(u_4)(wu_2) - h(1-u_4)(wu_3) \\ &= F(wu_1, wu_2, wu_3, u_4). \end{aligned}$$

Therefore, axiom (A3) is satisfied with

$$L_w = 0. \tag{2.10}$$

Thus we proved that $F \in \mathcal{F}$. On the other hand, since f is h-convex, for all $(x, y, t) \in [a,b] \times [a,b] \times (0,1)$, we have

$$\begin{aligned} F\bigl(f\bigl(tx + (1-t)y\bigr), f(x), f(y), t\bigr) &= f\bigl(tx + (1-t)y\bigr) - h(t)f(x) - h(1-t)f(y) \\ &\le 0. \end{aligned}$$

As a consequence, f is an F-convex function.

3 Integral inequalities involving *F*-convex functions

Some integral inequalities via F-convex functions are presented in this section.

Theorem 3.1 *Let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be an F-convex function, for some $F \in \mathcal{F}$. Suppose that $f \in L^1(a,b)$. Then*

$$F\left(f\left(\frac{a+b}{2}\right), \frac{1}{b-a}\int_a^b f(x)\,dx, \frac{1}{b-a}\int_a^b f(x)\,dx, \frac{1}{2}\right) \le 0, \tag{3.1}$$

$$T_{F,1}\left(\frac{1}{b-a}\int_a^b f(x)\,dx, f(a), f(b)\right) \le 0. \tag{3.2}$$

Proof Since f is an F-convex function, for every $u, v \in [a,b]$, we have

$$F\left(f\left(\frac{1}{2}u + \left(1 - \frac{1}{2}\right)v\right), f(u), f(v), \frac{1}{2}\right) \le 0.$$

Taking

$$u = ta + (1-t)b, \qquad v = tb + (1-t)a, \quad t \in [0,1],$$

we obtain

$$F\left(f\left(\frac{a+b}{2}\right), f\bigl(ta+(1-t)b\bigr), f\bigl(tb+(1-t)a\bigr), \frac{1}{2}\right) \le 0.$$

Using axiom (A1), we get

$$F\left(\int_0^1 f\left(\frac{a+b}{2}\right)dt, \int_0^1 f\bigl(ta+(1-t)b\bigr)dt, \int_0^1 f\bigl(tb+(1-t)a\bigr)dt, \frac{1}{2}\right) \le 0.$$

On the other hand, we have

$$\int_0^1 f\bigl(ta+(1-t)b\bigr)dt = \int_0^1 f\bigl(tb+(1-t)a\bigr)dt = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Therefore,

$$F\left(\int_0^1 f\left(\frac{a+b}{2}\right)dt, \frac{1}{b-a}\int_a^b f(x)\,dx, \frac{1}{b-a}\int_a^b f(x)\,dx, \frac{1}{2}\right) \le 0,$$

which proves (3.1).

Again, since f is an F-convex function, for every $t \in (0,1)$, we have

$$F\bigl(f\bigl(ta+(1-t)b\bigr), f(a), f(b), t\bigr) \le 0.$$

Using axiom (A2) with $w \equiv 1$, and integrating over $(0,1)$ with respect to the variable t, we obtain

$$T_{F,1}\left(\int_0^1 f\bigl(ta+(1-t)b\bigr)dt, f(a), f(b)\right) \le 0,$$

that is,

$$T_{F,1}\left(\frac{1}{b-a}\int_a^b f(x)\,dx, f(a), f(b)\right) \le 0,$$

which proves (3.2). □

Remark 3.2 Note that in the proof of Theorem 3.1, we used only the axioms (A1) and (A2). So, Theorem 3.1 holds true for any function F satisfying (A1) and (A2).

The following lemma will be useful later (see [13]).

Lemma 3.3 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $(a,b) \in I^\circ \times I^\circ$, $a < b$. Then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{b-a}{2}\int_0^1 (1-2t)f'\bigl(ta+(1-t)b\bigr)dt.$$

We have the following result.

Theorem 3.4 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $(a,b) \in I^\circ \times I^\circ$, $a < b$. Suppose that*

(i) *$|f'|$ is F-convex on $[a,b]$, for some $F \in \mathcal{F}$.*
(ii) *The function $t \in (0,1) \mapsto L_{w(t)}$ belongs to $L^1(0,1)$, where $w(t) = |1-2t|$.*

Then

$$T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|, \left|f'(a)\right|, \left|f'(b)\right|\right) + \int_0^1 L_{w(t)}\,dt \le 0. \tag{3.3}$$

Proof Since $|f'|$ is F-convex, we have

$$F\left(\left|f'\left(ta+(1-t)b\right)\right|, \left|f'(a)\right|, \left|f'(b)\right|, t\right) \le 0, \quad t \in (0,1).$$

Multiplying this inequality by $w(t)$ and using axiom (A3), we get

$$F\left(w(t)\left|f'\left(ta+(1-t)b\right)\right|, w(t)\left|f'(a)\right|, w(t)\left|f'(b)\right|, t\right) + L_{w(t)} \le 0, \quad t \in (0,1).$$

Integration over $(0,1)$ with respect to the variable t and using axiom (A2), we obtain

$$T_{F,w}\left(\int_0^1 w(t)\left|f'\left(ta+(1-t)b\right)\right|dt, \left|f'(a)\right|, \left|f'(b)\right|\right) + \int_0^1 L_{w(t)}\,dt \le 0.$$

On the other hand, from Lemma 3.3, we have

$$\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \int_0^1 w(t)\left|f'\left(ta+(1-t)b\right)\right|dt.$$

Since $T_{F,w}$ is nondecreasing with respect to the first variable, we deduce that

$$T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|, \left|f'(a)\right|, \left|f'(b)\right|\right) + \int_0^1 L_{w(t)}\,dt \le 0,$$

which proves (3.3). □

Another similar result is given by the following theorem.

Theorem 3.5 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $(a,b) \in I^\circ \times I^\circ$, $a < b$, and let $p > 1$. Suppose that $|f'|^{p/(p-1)}$ is F-convex on $[a,b]$, for some $F \in \mathcal{F}$, and $|f'| \in L^{p/(p-1)}(a,b)$. Then*

$$T_{F,1}\left(A(p,f), \left|f'(a)\right|^{\frac{p}{p-1}}, \left|f'(b)\right|^{\frac{p}{p-1}}\right) \le 0, \tag{3.4}$$

where

$$A(p,f) = \left(\frac{2}{b-a}\right)^{\frac{p}{p-1}} (p+1)^{\frac{1}{p-1}} \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|^{\frac{p}{p-1}}.$$

Proof Since $|f'|^{\frac{p}{p-1}}$ is F-convex, we have

$$F\left(\left|f'\left(ta+(1-t)b\right)\right|^{\frac{p}{p-1}}, \left|f'(a)\right|^{\frac{p}{p-1}}, \left|f'(b)\right|^{\frac{p}{p-1}}, t\right) \le 0, \quad t \in (0,1).$$

Using axiom (A2) with $w \equiv 1$, and integrating over $(0,1)$ with respect to the variable t, we get

$$T_{F,1}\left(\int_0^1 \left|f'\bigl(ta+(1-t)b\bigr)\right|^{\frac{p}{p-1}}\,dt, \left|f'(a)\right|^{\frac{p}{p-1}}, \left|f'(b)\right|^{\frac{p}{p-1}}\right) \leq 0.$$

On the other hand, using Lemma 3.3 and Hölder's inequality, we obtain

$$\begin{aligned}
&\left(\frac{2}{b-a}\right)^{\frac{p}{p-1}}(p+1)^{\frac{1}{p-1}}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|^{\frac{p}{p-1}} \\
&\quad \leq \int_0^1 \left|f'\bigl(ta+(1-t)b\bigr)\right|^{\frac{p}{p-1}}\,dt,
\end{aligned}$$

that is,

$$A(p,f) \leq \int_0^1 \left|f'\bigl(ta+(1-t)b\bigr)\right|^{\frac{p}{p-1}}\,dt.$$

Since $T_{F,1}$ is nondecreasing with respect to the first variable, we obtain

$$T_{F,1}\bigl(A(p,f), \left|f'(a)\right|^{\frac{p}{p-1}}, \left|f'(b)\right|^{\frac{p}{p-1}}\bigr) \leq 0,$$

which proves (3.4). □

4 Particular cases

As consequences of the presented theorems, we obtain in this section some integral inequalities for different (and independent) kinds of convexity.

4.1 The case of ε-convexity

We have the following Hermite-Hadamard inequalities for ε-convex functions.

Corollary 4.1 *Let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be an ε-convex function, $\varepsilon \geq 0$. Suppose that $f \in L^1(a,b)$. Then*

$$f\left(\frac{a+b}{2}\right) - \varepsilon \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} + \varepsilon.$$

Proof From Example 2.3, we know that an ε-convex function is an F-convex. Using (2.2) and (2.3) with $w \equiv 1$, we have

$$F(u_1,u_2,u_3,u_4) = u_1 - u_4u_2 - (1-u_4)u_3 - \varepsilon, \quad (u_1,u_2,u_3,u_4) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times[0,1],$$

and

$$T_{F,1}(u_1,u_2,u_3) = u_1 - \frac{u_2+u_3}{2} - \varepsilon, \quad (u_1,u_2,u_3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}.$$

So, applying Theorem 3.1, we obtain the desired result. □

Taking $\varepsilon = 0$ in Corollary 4.1, we obtain the following standard Hermite-Hadamard inequalities for convex functions.

Corollary 4.2 *Let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be a convex function. Suppose that $f \in L^1(a,b)$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

Corollary 4.3 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $(a,b) \in I^\circ \times I^\circ$, $a < b$. Suppose that the function $|f'|$ is ε-convex on $[a,b]$, $\varepsilon \geq 0$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq (b-a)\left[\frac{|f'(a)|+|f'(b)|}{8} + \frac{\varepsilon}{4}\right].$$

Proof Using (2.4) with $w(t) = |1-2t|$, we obtain

$$\int_0^1 L_{w(t)}\,dt = \varepsilon\int_0^1 \bigl(1-w(t)\bigr)\,dt = \varepsilon\left(1 - \int_0^1 |1-2t|\,dt\right) = \frac{\varepsilon}{2}.$$

Using (2.3) with $w(t) = |1-2t|$, we obtain

$$T_{F,w}(u_1,u_2,u_3) = u_1 - \left(\int_0^1 t|1-2t|\,dt\right)u_2 - \left(\int_0^1 (1-t)|1-2t|\,dt\right)u_3 - \varepsilon,$$

for all $(u_1,u_2,u_3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$. On the other hand, simple computations yield

$$\int_0^1 t|1-2t|\,dt = \int_0^1 (1-t)|1-2t|\,dt = \frac{1}{4}.$$

Therefore, we have

$$T_{F,w}(u_1,u_2,u_3) = u_1 - \frac{u_2+u_3}{4} - \varepsilon, \quad (u_1,u_2,u_3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}.$$

Then

$$\begin{aligned} &T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|, \bigl|f'(a)\bigr|, \bigl|f'(b)\bigr|\right) + \int_0^1 L_{w(t)}\,dt \\ &\quad= \frac{2}{b-a}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| - \frac{|f'(a)|+|f'(b)|}{4} - \frac{\varepsilon}{2}. \end{aligned}$$

Now, by Theorem 3.4, we have

$$T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|, \bigl|f'(a)\bigr|, \bigl|f'(b)\bigr|\right) + \int_0^1 L_{w(t)}\,dt \leq 0,$$

that is,

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq (b-a)\left[\frac{|f'(a)|+|f'(b)|}{8} + \frac{\varepsilon}{4}\right],$$

which proves the desired inequality. □

Taking $\varepsilon = 0$ in Corollary 4.3, we obtain the following result (see [13]).

Corollary 4.4 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $(a,b) \in I^\circ \times I^\circ$, $a < b$. Suppose that the function $|f'|$ is convex on $[a,b]$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

Corollary 4.5 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $(a,b) \in I^\circ \times I^\circ$, $a < b$, and let $p > 1$. Suppose that $|f'|^{p/(p-1)}$ is ε-convex on $[a,b]$, $\varepsilon \geq 0$, and $|f'| \in L^{p/(p-1)}(a,b)$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}+\varepsilon\right)^{\frac{p-1}{p}}.$$

Proof Using (2.3) with $w \equiv 1$, we obtain

$$T_{F,1}(u_1,u_2,u_3) = u_1 - \frac{u_2+u_3}{2} - \varepsilon, \quad (u_1,u_2,u_3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}.$$

Then

$$\begin{aligned} &T_{F,1}\bigl(A(p,f), \bigl|f'(a)\bigr|^{\frac{p}{p-1}}, \bigl|f'(b)\bigr|^{\frac{p}{p-1}}\bigr) \\ &\quad = A(p,f) - \frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2} - \varepsilon, \end{aligned}$$

where

$$A(p,f) = \left(\frac{2}{b-a}\right)^{\frac{p}{p-1}}(p+1)^{\frac{1}{p-1}}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|^{\frac{p}{p-1}}.$$

By Theorem 3.5, we have

$$T_{F,1}\bigl(A(p,f), \bigl|f'(a)\bigr|^{\frac{p}{p-1}}, \bigl|f'(b)\bigr|^{\frac{p}{p-1}}\bigr) \leq 0,$$

that is,

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}+\varepsilon\right)^{\frac{p-1}{p}},$$

which proves the desired inequality. □

Taking $\varepsilon = 0$ in Corollary 4.5, we obtain the following result (see [13]).

Corollary 4.6 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $(a,b) \in I^\circ \times I^\circ$, $a < b$, and let $p > 1$. Suppose that $|f'|^{p/(p-1)}$ is convex on $[a,b]$, and $|f'| \in L^{p/(p-1)}(a,b)$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right)^{\frac{p-1}{p}}.$$

4.2 The case of α-convexity

We have the following Hermite-Hadamard inequalities for α-convex functions.

Corollary 4.7 *Let $f:[a,b]\to\mathbb{R}$, $(a,b)\in\mathbb{R}^2$, $a<b$, be an α-convex function, $\alpha\in(0,1]$. Suppose that $f\in L^1(a,b)$. Then*

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+\alpha f(b)}{\alpha+1}.$$

Proof From Example 2.5, we know that an α-convex function is F-convex. Using (2.5) and (2.6) with $w\equiv 1$, we have

$$F(u_1,u_2,u_3,u_4)=u_1-u_4^\alpha u_2-\left(1-u_4^\alpha\right)u_3,\quad (u_1,u_2,u_3,u_4)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times[0,1],$$

and

$$T_{F,1}(u_1,u_2,u_3)=u_1-\frac{u_2}{\alpha+1}-\frac{\alpha}{\alpha+1}u_3,\quad (u_1,u_2,u_3)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}.$$

So, applying Theorem 3.1, we obtain the desired result. □

Remark 4.8 Taking $\alpha=1$ in Corollary 4.7, we obtain the standard Hermite-Hadamard inequalities for convex functions (see Corollary 4.2).

Corollary 4.9 *Let $f:I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $(a,b)\in I^\circ\times I^\circ$, $a<b$. Suppose that the function $|f'|$ is α-convex on $[a,b]$, $\alpha\in(0,1]$. Then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\quad\le\frac{(b-a)}{2(\alpha+1)(\alpha+2)}\left(\left(2^{-\alpha}+\alpha\right)\left|f'(a)\right|+\frac{[\alpha(\alpha+1)+2(1-2^{-\alpha})]}{2}\left|f'(b)\right|\right).\end{aligned}$$

Proof Using (2.7), we have

$$\int_0^1 L_{w(t)}\,dt=0.$$

Using (2.6) with $w(t)=|1-2t|$, we obtain

$$T_{F,w}(u_1,u_2,u_3)=u_1-\left(\int_0^1 t^\alpha|1-2t|\,dt\right)u_2-\left(\int_0^1\left(1-t^\alpha\right)|1-2t|\,dt\right)u_3,$$

for all $(u_1,u_2,u_3)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}$. Simple computations yield

$$\int_0^1 t^\alpha|1-2t|\,dt=\frac{1}{(\alpha+1)(\alpha+2)}\left(2^{-\alpha}+\alpha\right)$$

and

$$\int_0^1\left(1-t^\alpha\right)|1-2t|\,dt=\frac{\alpha(\alpha+1)+2(1-2^{-\alpha})}{2(\alpha+1)(\alpha+2)}.$$

Therefore,

$$T_{F,w}(u_1,u_2,u_3) = u_1 - \frac{(2^{-\alpha}+\alpha)u_2}{(\alpha+1)(\alpha+2)} - \frac{[\alpha(\alpha+1)+2(1-2^{-\alpha})]u_3}{2(\alpha+1)(\alpha+2)},$$

for all $(u_1,u_2,u_3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$. Now, we have

$$\begin{aligned}
&T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|, \left|f'(a)\right|, \left|f'(b)\right|\right) + \int_0^1 L_{w(t)}\,dt \\
&\quad = T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|, \left|f'(a)\right|, \left|f'(b)\right|\right) \\
&\quad = \frac{2}{b-a}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\qquad - \frac{(2^{-\alpha}+\alpha)|f'(a)|}{(\alpha+1)(\alpha+2)} - \frac{[\alpha(\alpha+1)+2(1-2^{-\alpha})]|f'(b)|}{2(\alpha+1)(\alpha+2)}.
\end{aligned}$$

So, applying Theorem 3.4, we obtain the desired result. □

Remark 4.10 Taking $\alpha = 1$ in Corollary 4.9, we obtain the result given by Corollary 4.4.

Corollary 4.11 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $(a,b) \in I^\circ \times I^\circ$, $a < b$, and let $p > 1$. Suppose that $|f'|^{p/(p-1)}$ is α-convex on $[a,b]$, $\alpha \in (0,1]$, and $|f'| \in L^{p/(p-1)}(a,b)$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^{\frac{p}{p-1}} + \alpha|f'(b)|^{\frac{p}{p-1}}}{\alpha+1}\right)^{\frac{p-1}{p}}.$$

Proof Using (2.6) with $w \equiv 1$, we obtain

$$T_{F,1}(u_1,u_2,u_3) = u_1 - \frac{u_2}{\alpha+1} - \frac{\alpha}{\alpha+1}u_3, \quad (u_1,u_2,u_3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}.$$

Then

$$\begin{aligned}
&T_{F,1}\left(A(p,f), \left|f'(a)\right|^{\frac{p}{p-1}}, \left|f'(b)\right|^{\frac{p}{p-1}}\right) \\
&\quad = A(p,f) - \frac{|f'(a)|^{\frac{p}{p-1}} + \alpha|f'(b)|^{\frac{p}{p-1}}}{\alpha+1},
\end{aligned}$$

where

$$A(p,f) = \left(\frac{2}{b-a}\right)^{\frac{p}{p-1}}(p+1)^{\frac{1}{p-1}}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|^{\frac{p}{p-1}}.$$

By Theorem 3.5, we have

$$T_{F,1}\left(A(p,f), \left|f'(a)\right|^{\frac{p}{p-1}}, \left|f'(b)\right|^{\frac{p}{p-1}}\right) \le 0,$$

that is,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^{\frac{p}{p-1}}+\alpha|f'(b)|^{\frac{p}{p-1}}}{\alpha+1}\right)^{\frac{p-1}{p}},$$

which is the desired inequality. □

Remark 4.12 Taking $\alpha = 1$ in Corollary 4.11, we obtain the result given by Corollary 4.6.

4.3 The case of *h*-convex functions

Let $h : J \to [0,\infty)$ be a given function which is not identical to 0, where J is an interval in $\mathbb{R}$ such that $(0,1) \subseteq J$. We suppose that $h \in L^1(0,1)$ and $h(\frac{1}{2}) \neq 0$.

We have the following Hermite-Hadamard inequalities for h-convex functions (obtained by Sarikaya *et al.* [14]).

Corollary 4.13 *Let $f : [a,b] \to [0,\infty)$, $(a,b) \in \mathbb{R}^2$, $a < b$, be a h-convex function. Suppose that $f \in L^1(a,b)$. Then*

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \left(\int_0^1 h(t)\,dt\right)\big(f(a)+f(b)\big).$$

Proof From Example 2.6, we know that a h-convex function is F-convex. Using (2.8) and (2.9) with $w \equiv 1$, we have

$$F(u_1,u_2,u_3,u_4) = u_1 - h(u_4)u_2 - h(1-u_4)u_3, \quad (u_1,u_2,u_3,u_4) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times(0,1).$$

and

$$T_{F,1}(u_1,u_2,u_3) = u_1 - \left(\int_0^1 h(t)\,dt\right)(u_2+u_3), \quad (u_1,u_2,u_3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}.$$

So, applying Theorem 3.1, we obtain the desired result. □

Corollary 4.14 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $(a,b) \in I^\circ \times I^\circ$, $a < b$. Suppose that the function $|f'|$ is h-convex on $[a,b]$. Then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq (b-a)\left(\int_0^1 h(t)|1-2t|\,dt\right)\left(\frac{|f'(a)|+|f'(b)|}{2}\right).$$

Proof Using (2.10), we have

$$\int_0^1 L_{w(t)}\,dt = 0.$$

Using (2.9) with $w(t) = |1-2t|$, we obtain

$$T_{F,w}(u_1,u_2,u_3) = u_1 - \left(\int_0^1 h(t)|1-2t|\,dt\right)(u_2+u_3), \quad (u_1,u_2,u_3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}.$$

Now, we have

$$\begin{aligned}
&T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|, \left|f'(a)\right|, \left|f'(b)\right|\right) + \int_0^1 L_{w(t)}\,dt \\
&= T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|, \left|f'(a)\right|, \left|f'(b)\right|\right) \\
&= \frac{2}{b-a}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad - \left(\int_0^1 h(t)|1-2t|\,dt\right)\left(\left|f'(a)\right| + \left|f'(b)\right|\right).
\end{aligned}$$

So, applying Theorem 3.4, we obtain the desired result. □

Corollary 4.15 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $(a,b) \in I^\circ \times I^\circ$, $a < b$, and let $p > 1$. Suppose that $|f'|^{p/(p-1)}$ is h-convex on $[a,b]$, and $|f'| \in L^{p/(p-1)}(a,b)$. Then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad \leq \frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1 h(t)\,dt\right)^{\frac{p-1}{p}}\left(\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}.
\end{aligned}$$

Proof Using (2.9) with $w \equiv 1$, we obtain

$$T_{F,1}(u_1,u_2,u_3) = u_1 - \left(\int_0^1 h(t)\,dt\right)(u_2+u_3), \quad (u_1,u_2,u_3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}.$$

Then

$$T_{F,1}\left(A(p,f), \left|f'(a)\right|^{\frac{p}{p-1}}, \left|f'(b)\right|^{\frac{p}{p-1}}\right) = A(p,f) - \left(\int_0^1 h(t)\,dt\right)\left(\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}}\right),$$

where

$$A(p,f) = \left(\frac{2}{b-a}\right)^{\frac{p}{p-1}}(p+1)^{\frac{1}{p-1}}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|^{\frac{p}{p-1}}.$$

By Theorem 3.5, we have

$$T_{F,1}\left(A(p,f), \left|f'(a)\right|^{\frac{p}{p-1}}, \left|f'(b)\right|^{\frac{p}{p-1}}\right) \leq 0,$$

that is,

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad \leq \frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1 h(t)\,dt\right)^{\frac{p-1}{p}}\left(\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}},
\end{aligned}$$

which is the desired inequality. □

Competing interests
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