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Uniform stability of fractional neutral systems: a Lyapunov-Krasovskii functional approach

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Abstract

In this paper, we study the stability of nonlinear fractional neutral systems equipped with the Caputo derivative. We extend the Lyapunov-Krasovskii approach for the nonlinear fractional neutral systems. Conditions of uniform stability are obtained for the nonlinear fractional neutral systems.

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1 Introduction

In recent decades, fractional calculus and fractional differential equations have attracted great attention. It has been proved that fractional calculus and fractional differential equations are valuable tools in the modeling of many phenomena in various fields of engineering, physics and economics. For details and examples, see [1–5] and the references therein.

Stability analysis is always one of the most important issues in the theory of differential equations and their applications for both deterministic and stochastic cases. The analysis on stability of fractional differential equations is more complex than that of classical differential equations, since fractional derivatives are nonlocal and have weakly singular kernels. Recently, stability of fractional differential equations has attracted increasing interest. The earliest study on stability of fractional differential equations started in [6], the author studied the case of linear fractional differential equations with Caputo derivative and the same fractional order α, where $0 < \alpha \leq 1$. The stability problem comes down to the eigenvalue problem of system matrix. Since then, many researchers have done further studies on the stability of linear fractional differential systems [7–11]. For the nonlinear fractional differential systems, the stability analysis is much more difficult and a few results are available in [12–18]. For more details about the stability results and the methods available to analyze the stability of fractional differential equations, the reader may refer to the recent survey papers [19, 20] and the references therein.

As we all know, Lyapunov's second method provides a way to analyze the stability of a system without explicitly solving the differential equations. It is necessary to extend Lyapunov's second method to fractional systems. In [12, 13], the fractional Lyapunov's second method was proposed, and the authors extended the exponential stability of integer order differential system to the Mittag-Leffler stability of fractional differential system. In [14], by using Bihari's and Bellman-Gronwall's inequality, an extension of Lyapunov's

second method for fractional-order systems was proposed. In [15–17], Baleanu *et al.* extended Lyapunov's method to fractional functional differential systems and developed the Lyapunov-Krasovskii stability theorem, Lyapunov-Razumikhin stability theorem and Mittag-Leffler stability theorem for fractional functional differential systems. As far as we know, there are few papers with respect to the stability of fractional neutral systems. In this paper, we consider the stability of a class of nonlinear fractional neutral functional differential equations with the Caputo derivative. Motivated by Li *et al.* [12, 13], Baleanu *et al.* [15] and Cruz and Hale [21], we aim in this paper to extend the Lyapunov-Krasovskii method for the nonlinear fractional neutral systems.

The rest of the paper is organized as follows. In Section 2, we give some notations and recall some concepts and preparation results. In Section 3, we extend the Lyapunov-Krasovskii approach for the nonlinear fractional neutral systems, results of uniform stability for the nonlinear fractional neutral systems are presented. Conclusions are presented in Section 4.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts needed here. Throughout this paper, let $\mathbb{R}^n$ be a real n-dimensional linear vector space with the norm $|\cdot|$, let $\mathcal{C} = C([-r, 0], \mathbb{R}^n)$ be the space of continuous functions taking $[-r, 0]$ into $\mathbb{R}^n$ with $\|\phi\|$, $\phi \in \mathcal{C}$ defined by $\|\phi\| = \sup_{s \in [-r,0]} |\phi(s)|$, $r > 0$ be a real constant. If $\sigma \in \mathbb{R}$, $A > 0$ and $x \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$, then for any $t \in [\sigma, \sigma + A]$, we let $x_t \in \mathcal{C}$ be defined by $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$.

Let us recall the following known definitions. For more details, we refer the reader to [1, 2, 4, 5].

Definition 2.1 The fractional order integral of a function $f : [t_0, \infty) \to \mathbb{R}$ of order $\alpha \in \mathbb{R}^+ = [0, +\infty)$ is defined by

$$I_{t_0}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t - s)^{\alpha-1} f(s) \, ds,$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 For a function f given on the interval $[t_0, \infty)$, the α order Riemann-Liouville fractional derivative of f is defined by

$$D_{t_0}^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_{t_0}^t (t - s)^{n-\alpha-1} f(s) \, ds, \quad n - 1 < \alpha < n, n \in \mathbb{N},$$

where $\mathbb{N} = \{1, 2, 3, \dots\}$.

Definition 2.3 For a function f given on the interval $[t_0, \infty)$, the α order Caputo fractional derivative of f is defined by

$$^cD_{t_0}^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_{t_0}^t (t - s)^{n-\alpha-1} f^{(n)}(s) \, ds, \quad n - 1 < \alpha < n,$$

$$^cD_{t_0}^\alpha f(t) = f^{(n)}(t), \quad \alpha = n, n \in \mathbb{N}.$$

Some properties of the aforementioned operators are recalled below [1].

Property 2.1 *The following results are especially interesting*:

(i) *For $\nu > -1$, we have $D_{t_0}^\alpha (t - t_0)^\nu = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)}(t - t_0)^{\nu-\alpha}$.*

(ii) *When $n - 1 < \alpha < n$, $n \in \mathbb{N}$, we have*

$$^cD_{t_0}^\alpha f(t) = D_{t_0}^\alpha \left[f(t) - \sum_{i=0}^{n-1} \frac{f^{(i)}(t_0)(t - t_0)^i}{i!} \right].$$

(iii) *For $\alpha \in (0,1)$, $T \geq t_0$ and $f \in C([t_0, T], \mathbb{R}^n)$, we have $^cD_{t_0}^\alpha I_{t_0}^\alpha f(t) = f(t)$,*
$I_{t_0}^\alpha {}^cD_{t_0}^\alpha f(t) = f(t) - f(t_0)$.

Remark 2.1 From Property 2.1, if $^cD_{t_0}^\alpha f(t) \geq 0$, $\alpha \in (0,1)$, then for $t \geq t_0$, we have

(i) $f(t) \geq f(t_0)$.

(ii) In general, it is not true that $f(t)$ is nondecreasing in t.

In [21], Cruz and Hale studied a class of functional difference operators which are very useful in stability theory and the asymptotic behavior of solutions of functional differential equations of neutral type. In monograph [22], Hale *et al.* presented the following definitions and results of the difference operators.

For Banach spaces X and Y, $\mathcal{L}(X, Y)$ is the Banach space of bounded linear mappings from X to Y with the operator topology. If $L \in \mathcal{L}(\mathcal{C}, \mathbb{R}^n)$, then the Riesz representation theorem implies that there is an $n \times n$ matrix function μ on $[-r, 0]$ of bounded variation such that

$$L\phi = \int_{-r}^{0} \left[d\mu(\theta) \right] \phi(\theta).$$

Definition 2.4 Let Ω be an open subset of a metric space. We say $L : \Omega \to \mathcal{L}(\mathcal{C}, \mathbb{R}^n)$ has smoothness on the measure if, for any $B \in \mathbb{R}$, there is a scalar function $\gamma(\lambda, t)$ continuous for $\lambda \in \Omega$, $t \in \mathbb{R}$, $\gamma(\lambda, 0) = 0$, such that if $L(\lambda)\phi = \int_{-r}^{0}[d\mu(\lambda, \theta)]\phi(\theta)$, $\lambda \in \Omega$, $s > 0$, then

$$\lim_{h \to 0^+} \left| \int_{B+h}^{B+s} + \int_{B-s}^{B-h} \left[d\mu(\lambda, \theta) \right] \phi(\theta) \right| \leq \gamma(\lambda, s) \|\phi\|.$$

If $B \in \mathbb{R}$ and the matrix $A(\lambda; B, L) = \mu(\lambda, B^+) - \mu(\lambda, B^-)$ is nonsingular at $\lambda = \lambda_0$, we say $L(\lambda)$ is atomic at B at λ_0. If $A(\lambda; B, L)$ is nonsingular on a set $K \subseteq \Lambda$, we say $L(\lambda)$ is atomic at B on K.

Definition 2.5 Suppose that $\Omega \subseteq \mathbb{R} \times \mathcal{C}$ is open with elements (t, ϕ). A function $\mathcal{D} : \Omega \to \mathbb{R}^n$ (not necessarily linear) is said to be atomic at B on Ω if D is continuous together with its first and second Fréchet derivatives with respect to ϕ; and $\mathcal{D}_\phi$, the derivative with respect to ϕ, is atomic at B on Ω.

Remark 2.2 If $\mathcal{D}(t, \phi)$ is linear in ϕ and continuous in $(t, \phi) \in \mathbb{R} \times \mathcal{C}$,

$$\mathcal{D}(t, \phi) = \int_{-r}^{0} \left[d_\theta \mu(t, \theta) \right] \phi(\theta),$$

then $A(t, \phi, B) = A(t, B)$ is independent of ϕ and

$$A(t, B) = \mu(t, B^+) - \mu(t, B^-).$$

Thus, $\mathcal{D}(t,\phi)$ is atomic at B on $\mathbb{R} \times \mathcal{C}$ for all $t \in \mathbb{R}$. In particular, if $B \neq 0$, $B \in [-r,0]$, $\mathcal{D}(t,\phi) = \phi(0) + F(t)\phi(B)$, then $A(t,B) = F(t)$ and $\mathcal{D}(t,\phi)$ is atomic at B on $\mathbb{R} \times \mathcal{C}$ if $\det F(t) \not\equiv 0$ for all $t \in \mathbb{R}$.

In the sequel, we consider the following nonlinear fractional neutral system:

$$^{c}D_{t_0}^{\alpha}\big[\mathcal{D}(t,x_t)\big] = f(t,x_t), \quad t > t_0, \tag{2.1}$$

with the initial condition

$$x_{t_0} = \varphi, \tag{2.2}$$

where $0 < \alpha < 1$, $t_0 \in \mathbb{R}$ is a constant, $\mathcal{D}, f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ are given continuous functions, nonlinear difference operator $\mathcal{D}$ atomic at zero. For more details about the operator $\mathcal{D}$, the reader may refer to [21, 22, 24] and the references therein. In the sequel, we always assume that, for any given $t_0 \in \mathbb{R}$ and a given function $\varphi \in \mathcal{C}$, there exists a unique continuous solution of (2.1), denoted by $x(t) = x(t,t_0,\varphi)$, such that it satisfies (2.1) for all $t > t_0$ and (2.2). To deal with stability, as usual, we assume that $\mathcal{D}(t,0) = f(t,0) = 0$ so that (2.1) has the zero solution.

Definition 2.6 [23] The zero solution $x = 0$ of (2.1) is stable if for any $t_0 \in \mathbb{R}$ and any $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon)$ such that any solution $x(t) = x(t,t_0,\varphi)$ of (2.1) with initial value φ at t_0, $\|\varphi\| < \delta$ satisfies $|x(t)| < \varepsilon$ for $t \geq t_0$. It is asymptotically stable if it is stable and for any $t_0 \in \mathbb{R}$ and any $\varepsilon > 0$, there exists $\delta_0 = \delta(t_0, \varepsilon)$, $T(t_0, \varepsilon) > 0$ such that $\|\varphi\| < \delta_0$ implies $|x(t)| \leq \varepsilon$ for $t \geq t_0 + T(t_0, \varepsilon)$, i.e., $\lim_{t \to +\infty} x(t) = 0$. It is uniformly stable if it is stable and $\delta = \delta(\varepsilon) > 0$ can be chosen independently of t_0. It is uniformly asymptotically stable if it is uniformly stable and there exists $\delta_0 > 0$ for any $\eta > 0$, there exists $T = T(\eta) > 0$ such that $\|\varphi\| < \delta_0$ implies $|x(t)| < \eta$ for $t > t_0 + T$. It is globally (uniformly) asymptotically stable if it is (uniformly) asymptotically stable and δ_0 can be an arbitrary large, finite number.

For a nonlinear operator $\mathcal{D}$, in [24], Zhang gave the following definition.

Definition 2.7 $\mathcal{D}(t,\phi)$ is said to be uniformly stable if there exist positive constants a, b, c such that for any $h : \mathbb{R} \to \mathbb{R}^n$ and any $\sigma \in \mathbb{R}$, $\varphi \in \mathbb{C}$, with $\mathcal{D}(\sigma,\varphi) = h(\sigma)$, the solution $x(t,\sigma,\varphi)$ of

$$\mathcal{D}(t,x_t) = h(t), \quad t \geq \sigma,$$
$$x_\sigma = \varphi. \tag{2.3}$$

satisfies the following estimate:

$$\|x_t\| \leq be^{-a(t-\sigma)}\|\varphi\| + c \sup_{u \in [\sigma,t]} \big|h(u)\big|, \quad t \geq \sigma. \tag{2.4}$$

The following lemma plays a major role in our analysis.

Lemma 2.1 *Let* $\mathcal{D}(t,\phi)$ *be uniformly stable,* $x_t = x_t(\sigma,\varphi)$ *be the solution of equation* (2.3). *Suppose that* $\alpha(t)$ *is any continuous and nondecreasing function with* $\alpha(0) = 0$, *and* $\alpha(t) > 0$

for $t > 0$. Then, for small $s > 0$, there is a continuous and strictly increasing function $\beta(s) \geq bs + c\alpha(s)$, $\beta(0) = 0$, $\beta(t) > 0$ for $t > 0$ such that

(i) *for each small $\delta > 0$, $\|\varphi\| < \delta$, $|\mathcal{D}(t, x_t)| < \alpha(\delta)$, $t \geq \sigma$, then*

$$\|x_t\| \leq \beta(\delta), \quad t \geq \sigma. \tag{2.5}$$

(ii) *for each small $\delta > 0$, $\mu > 0$ and a nonnegative constant L, there exists $T(\mu, \delta, L) > 0$ such that $\|\varphi\| < \delta$ and $|\mathcal{D}(t, x_t)| \leq \alpha(\mu)$, $t \geq \sigma$ imply*

$$\|x_t\| \leq \beta(\mu) + L, \quad t \geq \sigma + T(\mu, \delta, L). \tag{2.6}$$

Proof (i) From Definition 2.5, for sufficiently small $\delta > 0$ and $t \geq \sigma$, we have

$$\|x_t\| \leq be^{-a(t-\sigma)}\|\varphi\| + c \sup_{u \in [\sigma, t]} |h(u)|$$
$$\leq b\delta + c\alpha(\delta).$$

Trivially, we can choose a continuous and increasing function $\beta(\delta) \geq b\delta + c\alpha(\delta)$, $\beta(0) = 0$, $\beta(t) > 0$ for $t > 0$, so that (2.5) holds.

(ii) For $L \geq 0$ and sufficiently small $\mu > 0$, if $\|\varphi\| < \delta$ and $|\mathcal{D}(t, x_t)| \leq \alpha(\mu)$, $t \geq \sigma$, then by Definition 2.5 it suffices to show that

$$be^{-a(t-\sigma)}\delta + c\alpha(\mu) \leq b\mu + c\alpha(\mu) + L,$$

which implies

$$\|x_t\| \leq \beta(\mu) + L, \quad t \geq \sigma + T(\mu, \delta, L).$$

Therefore, if we take $T(\mu, \delta, L) = \max\{0, \frac{1}{a} \ln \frac{b\delta}{b\mu + L}\}$, then (2.6) holds. $\qquad\square$

3 Main results

In this section, we consider the stability of nonlinear fractional neutral system (2.1). Here, we always assume that fractional neutral system (2.1) with initial condition (2.2) has a unique continuous solution $x(t)$ which depends continuously upon t_0, φ. By Property 2.1(iii), we can obtain that initial value problem (2.1)-(2.2) is equivalent to the integral equation

$$\mathcal{D}(t, x_t) = \mathcal{D}(t_0, \varphi) + I_{t_0}^{\alpha} f(t, x_t), \quad t \geq t_0, x_{t_0} = \varphi. \tag{3.1}$$

If $V : \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ is continuously differentiable, we define the Caputo fractional derivative ${}^{c}D_{t_0}^{\alpha} V(t, \phi)$ along the solution $x_t = x_t(t_0, \varphi)$ of (2.1)-(2.2) as

$${}^{c}D_{t_0}^{\alpha} V(t, x_t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-s)^{-\alpha} V'(s, x_s) \, ds.$$

Now, we give the following Lyapunov-Krasovskii methods for nonlinear fractional neutral systems as counterpart to Lyapunov-Krasovskii methods for classical neutral systems proposed in [21].

Theorem 3.1 *Suppose that $\mathcal{D}(t,\phi)$ is uniformly stable, f takes closed bounded sets into bounded sets, and suppose that $u(s)$, $v(s)$ are strictly increasing functions with $u(0) = v(0) = 0$, and $w(s)$ is a continuous, nonnegative, nondecreasing function. If there exists a continuously differentiable functional $V : \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ such that*

$$u\big(\big|\mathcal{D}(t,\phi)\big|\big) \le V(t,\phi) \le v\big(\|\phi\|\big),$$
$$^cD_{t_0}^{\gamma} V(t,x_t) \le -w\big(\big|\mathcal{D}(t,x_t)\big|\big), \tag{3.2}$$

where $\gamma \in (0,1]$. Then the zero solution of (2.1) is uniformly stable. If, in addition, $w(s) > 0$ for $s > 0$, then it is uniformly asymptotically stable.

Proof It is possible to choose a continuous function $\alpha(t)$ so that $\alpha(0) = 0$ and $\alpha(t) > u^{-1}(v(t)) \ge 0$ for small $t > 0$. Then $v(t) < u(\alpha(t))$ for small $t > 0$. For the above-chosen $\alpha(t)$, by Lemma 2.1, we can find a corresponding $\beta(t)$ with the desired properties. Now, for any $\varepsilon > 0$, we can find a sufficiently small δ such that $\beta(\delta) < \varepsilon$. Hence, for any initial time t_0 and any initial condition $x_{t_0} = \varphi$ with $\|\varphi\| < \delta$, (3.2) implies

$$^cD_{t_0}^{\gamma} V\big(t,x_t(t_0,\varphi)\big) \le 0, \quad t \ge t_0,$$

by Property 2.1, we have

$$V\big(t,x_t(t_0,\varphi)\big) \le V(t_0,\varphi), \quad t \ge t_0.$$

By (3.2), this implies that

$$u\big(\big|\mathcal{D}\big(t,x_t(t_0,\varphi)\big)\big|\big) \le V\big(t,x_t(t_0,\varphi)\big) \le V(t_0,\varphi) \le v(\delta) < u\big(\alpha(\delta)\big), \quad t \ge t_0,$$

which implies that $|\mathcal{D}(t,x_t(t_0,\varphi))| < \alpha(\delta)$ for $t \ge t_0$. Since $\mathcal{D}(t,\phi)$ is uniformly stable, Lemma 2.1 implies

$$\big\|x_t(t_0,\varphi)\big\| < \varepsilon, \quad t \ge t_0.$$

Therefore, the zero solution is uniformly stable.

To prove uniform asymptotic stability, let $\varepsilon_0 := \varepsilon = 1$, choose $\delta_0 := \delta(1) > 0$ corresponding to uniform stability. Then, for any $t_0 \in \mathbb{R}$, $\|\varphi\| < \delta_0$ implies

$$\big\|x_t(t_0,\varphi)\big\| < \varepsilon_0, \qquad \big|\mathcal{D}\big(t,x_t(t_0,\varphi)\big)\big| < \alpha(\delta_0), \quad t \ge t_0. \tag{3.3}$$

Next, for any $\eta > 0$, we wish to show that there is $T(\delta_0,\eta)$ such that any solution $x(t,t_0,\varphi)$ of (2.1) with $\|\varphi\| < \delta_0$ satisfies

$$\big\|x_t(t_0,\varphi)\big\| < \eta, \quad t \ge t_0 + T(\delta_0,\eta). \tag{3.4}$$

Suppose that (3.4) is not true, then there is a solution $x = x(t,t_0,\varphi)$ of (2.1) with $\|\varphi\| < \delta_0$ and $\|x_t(t_0,\varphi)\| \ge \delta = \delta(\eta)$ ($\delta < 4b\alpha(\delta_0)$) for $t \ge t_0$. Then, for any $s \ge t_0$, from (2.4) we have

$$\delta \le \|x_t\| \le be^{-a(t-s)}\|x_s\| + c \sup_{u \in [s,t]} \big|\mathcal{D}(u,x_u)\big|. \tag{3.5}$$

Choose $N = N(\delta_0, \eta)$ so that $N > \max\{o, \frac{1}{a} \ln \frac{2b}{\delta}\}$. Let $t_1 = t_0 + N$, from (3.5) we have

$$\delta \leq \|x_{t_1}\| \leq be^{-aN}\|x_{t_0}\| + c \sup_{u \in [t_0, t_1]} |\mathcal{D}(u, x_u)|$$

$$\leq \frac{\delta}{2} + b \sup_{u \in [t_0, t_1]} |\mathcal{D}(u, x_u)|.$$

Then we have

$$b \sup_{u \in [t_0, t_1]} |\mathcal{D}(u, x_u)| \geq \frac{\delta}{2}.$$

Therefore, there exists $\zeta_1 \in [t_0, t_1]$ such that

$$|\mathcal{D}(\zeta_1, x_{\zeta_1})| \geq \frac{\delta}{2b}. \tag{3.6}$$

Now consider a sequence of intervals I_k with $I_k = [t_{k-1}, t_k]$, $t_k = t_0 + kN$, $k = 1, 2, \ldots$. By (3.5) and (3.6), there must be some ζ_k in each I_k such that

$$|\mathcal{D}(\zeta_k, x_{\zeta_k})| \geq \frac{\delta}{2b}.$$

Then there exists a sequence $\{\zeta_k\}_{k=1}^{\infty}$, $\zeta_k \to \infty$, as $k \to \infty$ such that

$$|\mathcal{D}(\zeta_k, x_{\zeta_k})| \geq \frac{\delta}{2b}, \quad k \geq 1.$$

Without loss of generality, we may assume that $\zeta_{k+1} - \zeta_k \geq r$ for $k \geq 0$. It follows from equation (3.1) that

$$\mathcal{D}(t, x_t(t_0, \varphi)) = \mathcal{D}(t_0, \varphi) + I_{t_0}^{\alpha} f(t, x_t(t_0, \varphi)), \quad t \geq t_0,$$

and

$$\mathcal{D}(\zeta_k, x_{\zeta_k}(t_0, \varphi)) = \mathcal{D}(t_0, \varphi) + I_{t_0}^{\alpha} f(\zeta_k, x_{\zeta_k}(t_0, \varphi)).$$

Then we have

$$\mathcal{D}(t, x_t(t_0, \varphi)) = \mathcal{D}(\zeta_k, x_{\zeta_k}(t_0, \varphi)) + I_{t_0}^{\alpha} f(t, x_t(t_0, \varphi)) - I_{t_0}^{\alpha} f(\zeta_k, x_{\zeta_k}(t_0, \varphi)).$$

Since f takes bounded sets into bounded sets, there is a constant L_0 such that $|f(t, x_t(t_0, \varphi))| \leq L_0$ for $t \geq t_0$, $\|\varphi\| \leq \delta_0$. Then

$$|\mathcal{D}(t, x_t(t_0, \varphi))| \geq \frac{\delta}{2b} - \frac{L_0(t - \zeta_k)^{\alpha}}{\Gamma(1 + \alpha)}, \quad t \geq \zeta_k.$$

Hence, for any $k \geq 1$, we have

$$|\mathcal{D}(x_t(t_0, \varphi))| \geq \frac{\delta}{4b}, \quad t \in [\zeta_k, \zeta_k + \theta],$$

where $\theta = \min\{r, [\frac{\delta \Gamma(1+\alpha)}{4bL_0}]^{\frac{1}{\alpha}}\}$.

Let

$$\lambda = \min_{\frac{\delta}{4b} \leq s \leq \alpha(\delta_0)} w(s) > 0.$$

Then there exists some positive integer k_0 such that

$$V(t_0, \varphi) < \frac{\lambda}{\Gamma(1 + \gamma)} (\zeta_{k_0} - t_0)^\gamma. \tag{3.7}$$

By (3.2), we have

$${}^c D_{t_0}^\gamma V\big(\zeta_{k_0}, x_{\zeta_{k_0}}(t_0, \varphi)\big) \leq -w\big(\big|\mathcal{D}\big(\zeta_{k_0}, x_{\zeta_{k_0}}(t_0, \varphi)\big)\big|\big) \leq -\lambda.$$

By (3.2), (3.7) and Property 2.1, we have

$$0 < u\left(\frac{\delta}{2b}\right) \leq V\big(\zeta_{k_0}, x_{\zeta_{k_0}}(t_0, \varphi)\big) \leq V(t_0, \varphi) - \frac{\lambda}{\Gamma(1 + \gamma)} (\zeta_{k_0} - t_0)^\gamma < 0, \tag{3.8}$$

which is a contradiction. Then there must be some $t^* \in [t_0, t_0 + k_0 N]$ such that $\|x_{t^*}(t_0, \varphi)\| < \delta$. Therefore, for $\|\varphi\| < \delta_0$, we have

$$\big|x(t, t_0, \varphi)\big| < \eta, \quad t \geq t_0 + k_0 N.$$

This proves the uniform asymptotic stability of the zero solution of (2.1). □

Remark 3.1 From the proof of inequality (3.8), we can know that the analysis on stability of fractional differential equations is more complex than that of classical differential equations, since fractional derivatives are nonlocal and have weakly singular kernels.

Remark 3.2 If $\mathcal{D}(t, \phi)$ is linear in ϕ and $\alpha = \gamma = 1$, Theorem 3.1 is just the same as Theorem 4.1 in [21].

Remark 3.3 If $\mathcal{D}(t, \phi) = \phi(0)$, the conclusions of Theorem 3.1 are just the same as the corresponding conclusions of Theorem in [15].

Theorem 3.2 *Suppose that the assumptions in Theorem* 3.1 *are satisfied except replacing* ${}^c D_{t_0}^\gamma$ *by* $D_{t_0}^\gamma$, *then one has the same result for uniform stability and uniform asymptotic stability.*

Proof By using Property 2.1, we have

$${}^c D_{t_0}^\gamma V(t, x_t) = D_{t_0}^\gamma V(t, x_t) - V(t_0, \varphi) \frac{(t - t_0)^{-\gamma}}{\Gamma(1 - \gamma)}.$$

Since $V(t_0, \varphi) \geq 0$, then ${}^c D_{t_0}^\gamma V(t, x_t) \leq D_{t_0}^\gamma V(t, x_t)$. Then we can obtain the same result for uniform stability and uniform asymptotic stability. □

4 Conclusions

In this paper, we have studied the stability of nonlinear fractional order neutral systems. We introduce the Lyapunov-Krasovskii approach for the nonlinear fractional neutral sys-

tems, which enrich the knowledge of both the system theory and the fractional calculus. We partly extend the application of Caputo fractional systems by using Lyapunov-Krasovskii approach. By using Caputo and Riemann-Liouville derivatives and Lyapunov-Krasovskii technique, uniform stability criteria are obtained for the nonlinear fractional neutral systems. The obtained conclusions generalize the corresponding conclusions in [15, 21].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in this article. They read and approved the final manuscript.

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