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A hybrid iteration scheme for equilibrium problems and common fixed point problems of generalized quasi-φ-asymptotically nonexpansive mappings in Banach spaces

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Abstract

In this article, we introduce an iterative algorithm for finding a common element of the set of common fixed points of a finite family of closed generalized quasi-φ-asymptotically nonexpansive mappings and the set of solutions of equilibrium problem in Banach spaces. Then we study the strong convergence of the algorithm. Our results improve and extend the corresponding results announced by many others.

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1. Introduction and preliminary

Let E be a Banach space with the dual E^*. Let C be a nonempty closed convex subset of E and $f : C \times C \to \mathbb{R}$ a bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for f is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0 \qquad (1.1)$$

for all $y \in C$. The set of solutions of (1.1) is denoted by $EP(f)$. Given a mapping $T : C \to E^*$, let $f(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then $\hat{x} \in EP(f)$ if and only if $\langle T\hat{x}, y - \hat{x} \rangle \geq 0$ for all $y \in C$, i.e., $\hat{x}$ is a solution of the variational inequality. Numerous problems in physics, optimization, engineering and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem; see, for example, Blum-Oettli [1] and Moudafi [2]. For solving the equilibrium problem, let us assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t \to 0} f(tz + (1 - t)x, y) \leq f(x, y)$;

(A4) for each $x \in C$, the function $y \mapsto f(x, y)$ is convex and lower semicontinuous.

Let E be a Banach space with the dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We know that if E is uniformly smooth, strictly convex, and reflexive, then the normalized duality mapping J is single-valued, one-to-one, onto and uniformly norm-to-norm continuous on each bounded subset of E. Moreover, if E is a reflexive and strictly convex Banach space with a strictly convex dual, then J^{-1} is single-valued, one-to-one, surjective, and it is the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$ (see, [3]). It is also well known that if E is uniformly smooth if and only if E^* is uniformly convex.

Let C be a nonempty closed convex subset of a Banach space E and $T : C \to C$ a mapping. A point $x \in C$ is said to be a fixed point of T provided $Tx = x$. In this article, we use $F(T)$ to denote the fixed point set and use $\to$ to denote the strong convergence. Recall that a mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \forall x, y \in C.$$

A mapping $T: C \to C$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of real numbers with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \quad \forall x, y \in C, \quad \forall n \ge 1.$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4] in 1972. They proved that, if C is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping T of C has a fixed point. Further, the set $F(T)$ is closed and convex. Since 1972, a host of authors have studied the weak and strong convergence problems of the iterative algorithms for such a class of mappings (see, e.g., [4-6] and the references therein).

It is well known that if C is a nonempty closed convex subset of a Hilbert space H and $P_C: H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [7] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.$$

Following Alber [7], the generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\varphi(y, x)$, that is, $\prod_C x = \bar{x}$, where $\bar{x}$ is the solution to the following minimization problem:

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x).$$

It follows from the definition of the function φ that

$$(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2, \quad \forall x, y \in E.$$

If E is a Hilbert space, then $\varphi(y, x) = \|y - x\|^2$ and $\Pi_C = P_C$ is the metric projection of H onto C.

Remark 1.1 [8,9] If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\varphi(x, y) = 0$ if and only if $x = y$.

Let C be a nonempty, closed and convex subset of a smooth Banach E and T a mapping from C into itself. The mapping T is said to be φ-nonexpansive if $\varphi(Tx, Ty) \leq \varphi(x, y)$, $\forall x, y \in C$. The mapping T is said to be quasi-φ-nonexpansive if $F(T) \neq \emptyset$, $\varphi(p, Tx) \leq \varphi(p, x)$, $\forall x \in C$, $p \in F(T)$. The mapping T is said to be φ-asymptotically nonexpansive if there exists some real sequence $\{k_n\}$ with $k_n \geq 1$ and $k_n \to 1$ as $n \to \infty$ such that $\varphi(T^n x, T^n y) \leq k_n \varphi(x, y)$, $\forall x, y \in C$. The mapping T is said to be quasi-φ-asymptotically nonexpansive if $F(T) \neq \emptyset$ and there exists some real sequence $\{k_n\}$ with $k_n \geq 1$ and $k_n \to 1$ as $n \to \infty$ such that $\varphi(p, T^n x) \leq k_n \varphi(p, x)$, $\forall x \in C$, $p \in F(T)$. The mapping T is said to be generalized quasi-φ-asymptotically nonexpansive if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{k_n\}$ and $\{c_n\}$ with $k_n \geq 1$, $\lim_{n \to \infty} k_n = 1$ and $\lim_{n \to \infty} c_n = 0$ such that $\varphi(p, T^n x) \leq k_n \varphi(p, x) + c_n$, $\forall x \in C$, $p \in F(T)$. The mapping T is said to be asymptotically regular on C if, for any bounded subset K of C, $\limsup_{n \to \infty} \{\|T^{n+1} x - T^n x\|: x \in K\} = 0$. The mapping T is said to be closed on C if, for any sequence $\{x_n\}$ such that $\lim_{n \to \infty} x_n = x_0$ and $\lim_{n \to \infty} Tx_n = y_0$, then $Tx_0 = y_0$.

We remark that a φ-asymptotically nonexpansive mapping with a nonempty fixed point set $F(T)$ is a quasi-φ-asymptotically nonexpansive mapping, but the converse may be not true. The class of generalized quasi-φ-asymptotically nonexpansive mappings is more general than the class of quasi-φ-asymptotically nonexpansive mappings and φ-asymptotically nonexpansive mappings. The following example shows that the inclusion is proper. Let $K = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and define (see [10]) $Tx = \frac{x}{2} \sin(\frac{1}{x})$ if $x \neq 0$ and $Tx = 0$ if $x = 0$. Then $T^n x \to 0$ uniformly but T is not Lipschitzian. It should be noted that $F(T) = \{0\}$. For each fixed n, define $f_n(x) = \|T^n x\|^2 - \|x\|^2$ for $x \in K$. Set $c_n = \sup_{x \in K} \{f_n(x), 0\}$. Then $\lim_{n \to \infty} c_n = 0$ and

$$\phi(0, T^n x) = \|T^n x\|^2 \leq \|x\|^2 + c_n = \phi(0, x) + c_n.$$

This show that T is a generalized quasi-φ-asymptotically nonexpansive but it is not quasi-φ-asymptotically nonexpansive and φ-asymptotically nonexpansive. Recently, many authors studied the problem of finding a common element of the set of fixed points of nonexpansive or quasi-φ-asymptotically nonexpansive mappings and the set of solutions of an equilibrium problem in the frame work of Hilbert spaces and Banach spaces respectively; see, for instance, [11-15] and the references therein.

In 2009, Cho, Qin and Kang [16] introduced the following iterative scheme on a closed quasi-φ-asymptotically nonexpansive mapping:

$$\begin{cases} x_0 \in E, \ C_1 = C, \quad x_1 = \prod_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_1 + (1 - \alpha_n) J T^n x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) + \alpha_n M\}, \\ x_{n+1} = \prod_{C_{n+1}} x_1, \quad \forall n \geq 0. \end{cases}$$

Strong convergence theorems of fixed points are established in a uniformly smooth and uniformly convex Banach space.

Recently, Takahashi and Zembayashi [17] introduced the following iterative process:

$$
\begin{cases}
x_0 = x \in C, \\
y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S x_n), \\
u_n \in C \text{ such that } f(u_n, \gamma) + \frac{1}{r_n}\langle \gamma - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall \gamma \in C, \\
H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
W_n = \{z \in C : \langle x_n - z, J x - J x_n \rangle \geq 0\}, \\
x_{n+1} = \prod_{H_n \cap W_n} x, \quad \forall n \geq 1,
\end{cases}
\tag{1.2}
$$

where $f{:}C \times C \to \mathbb{R}$ is a bifunction satisfying (A1)-(A4), J is the normalized duality mapping on E and $S : C \to C$ is a relatively nonexpansive mapping. They proved the sequences $\{x_n\}$ defined by (1.2) converge strongly to a common point of the set of solutions of the equilibrium problem (1.1) and the set of fixed points of S provided the control sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy appropriate conditions in Banach spaces.

In this article, inspired and motivated by the works mentioned above, we introduce an iterative process for finding a common element of the set of common fixed points of a finite family of closed generalized quasi-φ-asymptotically nonexpansive mappings and the solution set of equilibrium problem in Banach spaces. In the meantime, the method of the proof is different from the original one. The results presented in this article improve and generalize the corresponding results announced by many others.

Let C_n be a sequence of nonempty closed convex subsets of a reflexive Banach space E. We denote two subsets s - $Li_n C_n$ and w - $Ls_n C_n$ as follows: $x \in s$ - $Li_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and that $x_n \in C_n$ for all $n \geq 0$. Similarly, $y \in w$ - $Ls_n C_n$ if and only if there exists a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and that $y_i \in C_{n_i}$ for all $i \geq 0$. We define the Mosco convergence [18] of $\{C_n\}$ as follows: If C_0 satisfies that $C_0 = s$ - $Li_n C_n = w$ - $Ls_n C_n$, it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco and we write $C_0 = M$ - $\lim_{n \to \infty} C_n$. For more detail, see [19].

In order to obtain the main results of this paper, we need the following lemmas.

Lemma 1.2 [20]*Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n \to \infty} \varphi(x_n, y_n) = 0$, then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.*

Lemma 1.3 [21]*Let E be a smooth, strictly convex and reflexive Banach space having the Kadec-Klee property. Let $\{K_n\}$ be a sequence of nonempty closed convex subsets of E. If $K_0 = M\text{-}\lim_{n \to \infty} K_n$ exists and is nonempty, then $\{\prod_{K_n} x\}$ converges strongly to $\{\prod_{K_0} x\}$ for each $x \in C$.*

Lemma 1.4 [8,22]*Let E be a uniformly convex Banach space, $s > 0$ a positive number and $B_s(0)$ a closed ball of E. Then there exists a strictly increasing, continuous, and convex function $g{:}\ [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$
\left\| \sum_{i=0}^{N} (\alpha_i x_i) \right\|^2 \leq \sum_{i=0}^{N} \alpha_i \|x_i\|^2 - \alpha_k \alpha_l g(\|x_k - x_l\|)
$$

for any k, $l \in \{0, 1, \dots, N\}$, for all $x_0, x_1, \dots, x_N \in B_s(0) = \{x \in E : \|x\| \leq s\}$ and α_0, $\alpha_1, \dots, \alpha_n \in [0, 1]$ such that $\sum_{i=0}^{N} \alpha_i = 1$.

Lemma 1.5 [1]*Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, \gamma) + \frac{1}{r} \langle \gamma - z, Jz - Jx \rangle \geq 0, \quad \forall \gamma \in C.$$

Lemma 1.6 [17]*Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z, \gamma) + \langle Bx, \gamma - z \rangle + \frac{1}{r} \langle \gamma - z, Jz - Jx \rangle \geq 0, \quad \forall \gamma \in C\}$$

for all $x \in E$. Then, the following hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, i.e., for any $x, y \in E$,*

$$\langle T_r x - T_r \gamma, JT_r x - JT_r \gamma \rangle \leq \langle T_r x - T_r \gamma, Jx - J\gamma \rangle;$$

(3) *$F(T_r) = EP(f)$;*

(4) *$EP(f)$ is closed and convex;*

(5) *$\varphi(q, T_r x) + \varphi(T_r x, x) \leq \varphi(q, x), \forall q \in F(T_r).$*

Lemma 1.7 *Let E be a uniformly convex and uniformly smooth Banach space, C a nonempty, closed and convex subset of E and T a closed generalized quasi-φ-asymptotically nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

Proof. We first show that $F(T)$ is closed. To see this, let $\{p_n\}$ be a sequence in $F(T)$ with $p_n \to p$ as $n \to \infty$, we shall prove that $p \in F(T)$. By using the definition of T, we have

$$\phi(p_n, T^n p) \leq k_n \phi(p_n, p) + c_n,$$

which implies that $\varphi(p_n, T^n p) \to 0$ as $n \to \infty$. It follows from Lemma 1.2 that $p_n - T^n p \to 0$ as $n \to \infty$ and hence $T^n p \to p$ as $n \to \infty$. We have $T(T^n p) = T^{n+1} p \to p$ as $n \to \infty$. It follows from the closedness of T that $Tp = p$. We next show that $F(T)$ is convex. To prove this, for arbitrary $p, q \in F(T)$, $t \in (0, 1)$, we set $w = tp + (1 - t)q$. By (1.3), we have

$$\begin{aligned}
&\phi(w, T^n w) \\
&= \|w\|^2 - 2 \langle w, JT^n w \rangle + \|T^n w\|^2 \\
&= \|w\|^2 - 2t \langle p, JT^n w \rangle - 2(1 - t) \langle q, JT^n w \rangle + \|T^n w\|^2 \\
&= \|w\|^2 + t\phi(p, T^n w) + (1 - t)\phi(q, T^n w) - t\|p\|^2 - (1 - t)\|q\|^2 \\
&\leq \|w\|^2 + tk_n\phi(p, w) + tc_n + (1 - t)k_n\phi(q, w) + (1 - t)c_n - t\|p\|^2 - (1 - t)\|q\|^2 \\
&= \|w\|^2 + tk_n\|p\|^2 - 2tk_n \langle p, Jw \rangle + tk_n\|w\|^2 + (1 - t)k_n\|q\|^2 \\
&\quad - 2(1 - t)k_n \langle q, Jw \rangle + (1 - t)k_n\|w\|^2 + c_n - t\|p\|^2 - (1 - t)\|q\|^2 \\
&= (k_n - 1)(t\|p\|^2 + (1 - t)\|q\|^2) + \|w\|^2 + k_n\|w\|^2 - 2k_n(w, Jw) + c_n \\
&= (k_n - 1)(t\|p\|^2 + (1 - t)\|q\|^2) + (k_n + 1)\|w\|^2 - 2k_n\|w\|^2 + c_n \\
&= (k_n - 1)(t\|p\|^2 + (1 - t)\|q\|^2 - \|w\|^2) + c_n,
\end{aligned}$$

which implies that $\varphi(w, T^n w) \to 0$ as $n \to \infty$. By Lemma 1.2, we obtain $T^n w \to w$ as $n \to \infty$, and hence $T(T^n w) = T^{n+1} w \to w$ as $n \to \infty$. Since T is closed, we see that $w = Tw$. This completes the proof.

2. Main results

Theorem 2.1 *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth real Banach space E and let $T_i : C \to C$ be a closed and generalized quasi-φ-asymptotically nonexpansive mapping with real sequences $\{k_{n,i}\} \subseteq [1, \infty)$ and $\{c_{n,i}\} \subseteq [0, \infty)$ such that $\lim_{n \to \infty} k_{n,i} = 1$ and $\lim_{n \to \infty} c_{n,i} = 0$ for each $1 \leq i \leq N$. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Assume that T_i is asymptotically regular on C for each $1 \leq i \leq N$ and $F = (\bigcap_{i=1}^{N} F(T_i)) \bigcap EP(f) \neq \emptyset$. Let $k_n = \max_{1 \leq i \leq N} \{k_{n,i}\}$ and $c_n = \max_{1 \leq i \leq N} \{c_{n,i}\}$. Define a sequence $\{x_n\}$ in C in the following manner:*

$$
\begin{cases}
x_0 \in E \text{ chosen arbitrarily}, \\
C_1 = C, \\
x_1 = \prod_{C_1} x_0, \\
y_n = J^{-1}(\alpha_{n,0} J x_n + \sum_{i=1}^{N} \alpha_{n,i} J T_i^n x_n), \\
u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\
C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq k_n \phi(z, x_n) + c_n\}, \\
x_{n+1} = \prod_{C_{n+1}} x_1
\end{cases}
\tag{2.1}
$$

for every $n \geq 1$, where $\{r_n\}$ is a real sequence in $[a, \infty)$ for some $a > 0$, J is the normalized duality mapping on E. Assume that the control sequences $\{\alpha_{n,0}\}, \{\alpha_{n,1}\}, ..., \{\alpha_{n,N}\}$ are real sequences in (0,1) satisfy $\sum_{i=0}^{N} \alpha_{n,i} = 1$ and $\liminf_{n \to \infty} \alpha_{n,0} \alpha_{n,i} > 0$ for each $i \in \{1, 2, \cdots, N\}$. Then the sequence $\{x_n\}$ converges strongly to $\prod_F x_1$, where $\prod_F$ is the generalized projection from C into F.

Proof. Firstly, by Lemma 1.7, we know that $F(T_i)$ is a closed convex subset of C for every $1 \leq i \leq N$. Hence, $F = (\bigcap_{i=1}^{N} F(T_i)) \bigcap EP(f) \neq \emptyset$ is a nonempty closed convex subset of C and $\prod_F x_1$ is well defined for $x_1 \in C$. Now we show that C_n is closed and convex for each $n \geq 1$. From the definition of C_n, it is obvious that C_n is closed for each $n \geq 1$. We show that C_n is convex for each $n \geq 1$. It is obvious that $C_1 = C$ is convex. Suppose that C_n is convex for some integer n. Observe that the set

$$C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq k_n \phi(z, x_n) + c_n\}$$

can be written to

$$C_{n+1} = \{z \in C_n : (1 - k_n) \|z\|^2 + \|u_n\|^2 - k_n \|x_n\|^2 - c_n \leq 2 \langle z, J u_n - k_n J x_n \rangle\}.$$

For $z_1, z_2 \in C_{n+1} \subseteq C_n$ and $t \in (0,1)$, denote $z = tz_1 + (1 - t)z_2$, we have $z \in C_n$. Setting $A = \|u_n\|^2 - k_n \|x_n\|^2 - c_n$ and $B = J u_n - k_n J x_n$, by noting that $\| \cdot \|^2$ is convex, we have

$$\|z\|^2 \leq t\|z_1\|^2 + (1 - t)\|z_2\|^2.$$

So we obtain

$$
\begin{aligned}
(1 - k_n)\|z\|^2 + A &\leq (1 - k_n)t\|z_1\|^2 + (1 - k_n)(1 - t)\|z_2\|^2 + A \\
&\leq 2t \langle z_1, B \rangle + 2(1 - t) \langle z_2, B \rangle \\
&= 2 \langle z, B \rangle,
\end{aligned}
$$

which implies that $z \in C_{n+1}$, so we get C_{n+1} is convex. Thus, C_n is closed and convex for each $n \geq 1$.

Secondly, we prove that $F \subseteq C_n$ for all $n \geq 1$. We do this by induction. For $n = 1$, we have $F \subseteq C = C_1$. Suppose that $F \subseteq C_n$ for some $n \geq 1$. Let $p \in F \subseteq C$. Putting $u_n = T_{r_n} y_n$ for all $n \geq 1$, we have that T_{r_n} is quasi-φ-nonexpansive from Lemma 1.6.

Since T_i is generalized quasi-φ-asymptotically nonexpansive, by noting that $\|\cdot\|^2$ is convex, we have

$$
\begin{aligned}
&\phi(p, u_n) \\
&= \phi(p, T_{r_n} y_n) \\
&\leq \phi(p, y_n) \\
&= \phi\left(p, J^{-1}\left(\alpha_{n,0} J x_n + \sum_{i=1}^{N} \alpha_{n,i} J T_i^n x_n\right)\right) \\
&= \|p\|^2 - 2\left\langle p, \alpha_{n,0} J x_n + \sum_{i=1}^{N} \alpha_{n,i} J T_i^n x_n\right\rangle + \left\|\alpha_{n,0} J x_n + \sum_{i=1}^{N} \alpha_{n,i} J T_i^n x_n\right\|^2 \\
&\leq \|p\|^2 - 2\alpha_{n,0}\langle p, J x_n\rangle - 2\sum_{i=1}^{N} \alpha_{n,i}\langle p, J T_i^n x_n\rangle + \alpha_{n,0}\|x_n\|^2 + \sum_{i=1}^{N} \alpha_{n,i}\|T_i^n x_n\|^2 \\
&= \alpha_{n,0}\phi(p, x_n) + \sum_{i=1}^{N} \alpha_{n,i}\phi(p, T_i^n x_n) \\
&\leq \alpha_{n,0}\phi(p, x_n) + \sum_{i=1}^{N} \alpha_{n,i} k_{n,i}\phi(p, x_n) + \sum_{i=1}^{N} \alpha_{n,i} c_{n,i} \\
&\leq k_n\phi(p, x_n) + c_n,
\end{aligned}
\tag{2.2}
$$

which infers that $p \in C_{n+1}$, and hence $F \subseteq C_{n+1}$. This proves that $F \subseteq C_n$ for all $n \geq 1$.

Thirdly, we show that $\lim_{n\to\infty} x_n = x^* = \prod_{\overline{C}} x_1$, where $\overline{C} = \cap_{n=1}^{\infty} C_n$. Indeed, since $\{C_n\}$ is a decreasing sequence of closed convex subsets of E such that $\overline{C} = \cap_{n=1}^{\infty} C_n$ is nonempty, it follows that

$$
M - \lim_{n\to\infty} C_n = \overline{C} = \cap_{n=1}^{\infty} C_n \neq \emptyset.
$$

By Lemma 1.3, $\{x_n\} = \{\Pi_{C_n} x_1\}$ converges strongly to $\{x^*\} = \{\Pi_{\overline{C}} x_1\}$ and $\{x_n\}$ is bounded.

Fourthly, we prove that $x^* \in F$.

Since $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1}$, from the definition of C_{n+1}, we get

$$
\phi(x_{n+1}, u_n) \leq k_n\phi(x_{n+1}, x_n) + c_n.
$$

From $\lim_{n\to\infty} x_n = x^*$, one obtain $\varphi(x_{n+1}, x_n) \to 0$ as $n \to \infty$, and it follows from $\lim_{n\to\infty} c_n = 0$ we have

$$
\phi(x_{n+1}, u_n) = 0.
$$

Thus, $\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0$ by Lemma 1.2. It should be noted that

$$
\|x_n - u_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\|
$$

for all $n \geq 1$. It follows that

$$
\lim_{n\to\infty} \|x_n - u_n\| = 0,
\tag{2.3}
$$

which implies that $u_n \to x^*$ as $n \to \infty$. Since J is uniformly norm-to-norm continuous on bounded sets, from (2.3), we have

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{2.4}$$

Let $s = \sup \{ \|x_n\|, \|T_1^n x_n\|, \|T_2^n x_n\|, \cdots, \|T_N^n x_n\| : n \in \mathbb{N} \}$. Since E is uniformly smooth Banach space, we know that E^* is a uniformly convex Banach space. Therefore, from Lemma 1.4 we have, for any $p \in F$, that

$$
\begin{aligned}
&\phi(p, u_n)\\
&= \phi(p, T_{r_n} y_n)\\
&\le \phi(p, y_n)\\
&= \phi\left(p, J^{-1}\left(\alpha_{n,0} Jx_n + \sum_{i=1}^{N} \alpha_{n,i} JT_i^n x_n\right)\right)\\
&= \|p\|^2 - 2\alpha_{n,0}\langle p, Jx_n\rangle - 2\sum_{i=1}^{N} \alpha_{n,i}\langle p, JT_i^n x_n\rangle + \left\|\alpha_{n,0} Jx_n + \sum_{j=1}^{N} \alpha_{n,i} JT_i^n x_n\right\|^2\\
&\le \|p\|^2 - 2\alpha_{n,0}\langle p, Jx_n\rangle - 2\sum_{i=1}^{N} \alpha_{n,i}\langle p, JT_i^n x_n\rangle\\
&\quad + \alpha_{n,0}\|x_n\|^2 + \sum_{i=1}^{N} \alpha_{n,i}\|T_i^n x_n\|^2 - \alpha_{n,0}\alpha_{n,1} g(\|Jx_n - JT_1^n x_n\|)\\
&= \alpha_{n,0}\phi(p, x_n) + \sum_{i=1}^{N} \alpha_{n,i}\phi(p, T_i^n x_n) - \alpha_{n,0}\alpha_{n,1} g(\|Jx_n - JT_1^n x_n\|)\\
&\le \alpha_{n,0}\phi(p, x_n) + \sum_{i=1}^{N} \alpha_{n,i} k_{n,i}\phi(p, x_n) + \sum_{i=1}^{N} \alpha_{n,i} c_{n,i} - \alpha_{n,0}\alpha_{n,1} g(\|Jx_n - JT_1^n x_n\|)\\
&\le k_n \phi(p, x_n) + c_n - \alpha_{n,0}\alpha_{n,1} g(\|Jx_n - JT_1^n x_n\|)\\
&= \phi(p, x_n) + (k_n - 1)\phi(p, x_n) + c_n - \alpha_{n,0}\alpha_{n,1} g(\|Jx_n - JT_1^n x_n\|).
\end{aligned}
$$

Therefore, we have

$$\alpha_{n,0}\alpha_{n,1} g(\|Jx_n - JT_1^n x_n\|) \le \phi(p, x_n) - \phi(p, u_n) + (k_n - 1)\phi(p, x_n) + c_n. \tag{2.5}$$

On the other hand, we have

$$
\begin{aligned}
&\left|\phi(p, x_n) - \phi(p, u_n)\right|\\
&= \left|\|x_n\|^2 - \|u_n\|^2 - 2\langle p, Jx_n - Ju_n\rangle\right|\\
&\le \left|\|x_n\| - \|u_n\|\right|(\|x_n\| + \|u_n\|) + 2\|Jx_n - Ju_n\|\|p\|\\
&\le \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|Jx_n - Ju_n\|\|p\|.
\end{aligned}
$$

It follows from (2.3) and (2.4) that

$$\lim_{n\to\infty}(\phi(p, x_n) - \phi(p, u_n)) = 0. \tag{2.6}$$

Since $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} c_n = 0$ and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,1} > 0$, from (2.5) and (2.6) we have

$$\lim_{n\to\infty} g(\|Jx_n - JT_1^n x_n\|) = 0.$$

Therefore, from the property of g, we obtain

$$\lim_{n\to\infty} \left\| Jx_n - JT_1^n x_n \right\| = 0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \left\| x_n - T_1^n x_n \right\| = 0, \tag{2.7}$$

and hence $T_1^n x_n \to x^*$ as $n \to \infty$. Since $\left\| T_1^{n+1} x_n - x^* \right\| \le \left\| T_1^{n+1} x_n - T_1^n x_n \right\| + \left\| T_1^n x_n - x^* \right\|$, it follows from the asymptotical regularity of T_1 that

$$\lim_{n\to\infty} \left\| T_1^{n+1} x_n - x^* \right\| = 0.$$

That is, $T_1(T_1^n x_n) \to x^*$ as $n \to \infty$. From the closedness of T_1, we get $T_1 x^* = x^*$. Similarly, one can obtain that $T_i x^* = x^*$ for $i = 2,..., N$. So, $x^* \in \cap_{i=1}^N F(T_i)$.

Now we show $x^* \in EP(f) = F(T_r)$. Let $p \in F$. From $u_n = T_{r_n} y_n$, (2.2) and Lemma 1.6, we obtain that

$$\begin{aligned}
\phi(u_n, y_n) &= \phi(T_{r_n} y_n, y_n) \\
&\le \phi(p, y_n) - \phi(p, T_{r_n} y_n) \\
&\le \phi(p, x_n) + (k_n - 1)\phi(p, x_n) + c_n - \phi(p, u_n).
\end{aligned}$$

It follows from (2.6), $k_n \to 1$ and $c_n \to 0$ that $\varphi(u_n, y_n) \to 0$ as $n \to \infty$. Now, by Lemma 1.2, we have that $\|u_n - y_n\| \to 0$ as $n \to \infty$, and hence, $\|Ju_n - Jy_n\| \to 0$ as $n \to \infty$. Since $u_n \to x^*$ as $n \to \infty$, we obtain that $y_n \to x^*$. From the assumption $r_n > a$, we get

$$\lim_{n\to\infty} \frac{\left\| Ju_n - Jy_n \right\|}{r_n} = 0. \tag{2.8}$$

Noting that $u_n = T_{r_n} y_n$, we obtain

$$f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \ge 0, \quad \forall y \in C.$$

From $(A2)$, we have

$$\left\langle y - u_n, \frac{Ju_n - Jy_n}{r_n} \right\rangle \ge -f(u_n, y) \ge f(y, u_n), \quad \forall y \in C. \tag{2.9}$$

Letting $n \to \infty$, we have from $u_n \to x^*$, (2.8) and $(A4)$ that $f(y, x^*) \le 0 (\forall y \in C)$. For t with $0 < t \le 1$ and $y \in C$, let $y_t = ty + (1 - t)x^*$. Since $y \in C$ and $x^* \in C$, we have $y_t \in C$ and hence $f(y_t, x^*) \le 0$. Now, from $(A1)$ and $(A4)$, we have

$$0 = f(y_t, y_t) \le tf(y_t, y) + (1 - t)f(y_t, x^*) \le tf(y_t, y)$$

and hence $f(y_t, y) \ge 0$. Letting $t \to 0$, from $(A3)$, we have $f(x^*, y) \ge 0$. This implies that $x^* \in EP(f)$. Thus, $x^* \in F$.

Finally, since $x^* = \Pi_{\overline{C}} x_1 \in F$ and F is a nonempty closed convex subset of $\overline{C} = \cap_{n=1}^\infty C_n$, we conclude that $x^* = \Pi_F x_1$. This completes the proof.

In Hilbert spaces, Theorem 2.1 reduces to the following theorem.

Theorem 2.2 *Let C be a nonempty, closed and convex subset of a Hilbert space H and let $T_i:C \to C$ be a closed and generalized quasi-φ-asymptotically nonexpansive mapping with real sequences $\{k_{n,i}\} \subset [1, \infty)$ and $\{c_{n,i}\} \subset [0, \infty)$ such that $\lim_{n\to\infty} k_{n,i} = 1$ and $\lim_{n\to\infty} c_{n,i} = 0$ for each $1 \le i \le N$. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Assume that T_i is asymptotically regular on C for each $1 \le i \le N$ and $F = \left(\bigcap_{i=1}^{N} F(T_i)\right) \bigcap EP(f) \neq \emptyset$. Let $k_n = \max_{1\le i\le N}\{k_{n,i}\}$ and $c_n = \max_{1\le i\le N}\{c_{n,i}\}$. Define a sequence $\{x_n\}$ in C in the following manner:*

$$
\begin{cases}
x_0 \in H \text{ chosen arbitrarily,} \\
C_1 = C, \\
x_1 = P_{C_1}x_0, \\
y_n = \alpha_{n,0}x_n + \sum_{i=1}^{N} \alpha_{n,i}T_i^n x_n, \\
u_n \in C \text{ such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \ge 0, \quad \forall y \in C, \\
C_{n+1} = \left\{z \in C_n : \|z - u_n\|^2 \le k_n\|z - x_n\|^2 + c_n\right\}, \\
x_{n+1} = P_{C_{n+1}}x_1
\end{cases}
\tag{2.10}
$$

for every $n \ge 1$, where $\{r_n\}$ is a real sequence in $[a, \infty)$ for some $a > 0$. Assume that the control sequences $\{\alpha_{n,0}\}, \{\alpha_{n,1}\}, \ldots, \{\alpha_{n,N}\}$ are real sequences in $(0,1)$ satisfy $\sum_{i=0}^{N} \alpha_{n,i} = 1$ and $\liminf_{n\to\infty} \alpha_{n,0}\,\alpha_{n,i} > 0$ for each $i \in \{1, 2, \ldots, N\}$. Then the sequence $\{x_n\}$ converges strongly to $P_F x_1$.

Remark 2.3 Theorems 2.1 and 2.2 extend the main results of [16] from quasi-φ-non-expansive mappings to more general generalized quasi-φ-asymptotically nonexpan-sive mappings.

Authors' contributions
ZJ carried out the algorithm design and drafted the manuscript. HS conceived of the study and helped to draft the manuscript. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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