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Existence and approximation of solutions for generalized extended nonlinear variational inequalities

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Abstract
In this paper, we consider a new class of generalized extended nonlinear quasi-variational inequality problems involving set-valued relaxed monotone operators and establish its equivalence with the fixed point problem. We study criteria for existence of their solutions. Iterative methods for finding approximate solutions are also proposed and analyzed.
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1 Introduction

Variational inequality theory constitutes significant and novel extensions of the variational principles. It describes a broad spectrum of interesting developments involving a link between various fields of physical, engineering, pure and applied sciences. It has been shown that variational inequality theory provides the unified and efficient framework for a general treatment of a wide class of problems; for details, see Baiocchi and Capelo [1], Fukushima [2], Giannessi and Maugeri [3], Glowinski and Tallec [4], Noor *et al.* [5], Patriksson [6], Kinderlehrer and Stampacchia [7] and references therein. The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. On the one hand, it reveals the fundamental fact on the qualitative aspects of the solutions to important classes of problems; on the other hand, it also enables us to develop highly efficient and powerful new numerical methods for solving various problems. One of the most interesting and important problems in variational inequality theory is the development of efficient numerical methods. There is a substantial number of numerical methods, including the projection methods and their variant forms. The projection method and its variant forms represent important tools for approximate solvability of various kinds of variational inequalities; see [1–34] and references therein. The main idea behind this technique is to establish equivalence between the variational inequalities and the fixed point problem, using the concept of projection. This alternate formulation is used to suggest iterative methods for approximate solvability of variational inequality problems.

In many problems of analysis, one encounters operators who may be split in the form $S = A \pm T$, where A and T satisfy some conditions, and S itself has neither of these properties. An early theorem of this type was given by Krasnoselskii [8], where a complicated operator is split into the sum of two simpler operators. There is another setting arising from perturbation theory. Here, the operator equation $Tx \pm Ax = x$ is considered as a perturbation of $Tx = x$ (or $Ax = x$), and one would like to assert that the original unperturbed equation has a solution. In such a situation, there is, in general, no continuous dependence of solutions on the perturbations. For various results in this direction, please see Browder [9], Fucik [10, 11], Kirk [12], Petryshyn [13], Webb [14]. Another argument is concerned with the approximate solution of the problem: For f in H, find x in H such that $Tx \pm Ax = f$. Here T and A are given self-operators of H. Many boundary value problems for quasi-linear partial differential equations arising in physics, fluid mechanics and other areas of applications can be formulated as the equation $Tx \pm Ax = f$; see, *e.g.*, Zeidler [15]. Combettes and Hirstoaga [16] showed that the finding of zeros of sum of two operators can be solved via the variational inequality involving sum of two operators. Several authors have studied this type of situations; see, *e.g.*, Dhage [17], O'Reagan [18] and references therein.

It is our aim in this paper, to consider a new class of generalized extended nonlinear quasi-variational inequality problems, involving set-valued relaxed monotone operators, and to establish its equivalence with the fixed point problem. Using this framework, we study criteria for existence of their solutions. Iterative methods for finding approximate solutions are also proposed and analyzed. As we shall see, in some circumstances, our results reduce to previous results of Bruck [19], Fang and Peterson [20], Lions and Stampacchia [21], Noor [22–24], Verma [25, 26], Qin and Shang [27], Noor and Noor [28, 29].

2 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let $K : \mathcal{H} \to \mathcal{H}$ be a point to set mapping, which is closed and convex valued. In other words, for every $x \in \mathcal{H}$, the set $K(x)$ is closed and convex.

We consider the problem of finding $x^* \in H$ and $w^* \in T(x^*)$ such that $g(x^*) \in K(x^*)$ and

$$\langle \rho\bigl(Ax^* + w^*\bigr) + g\bigl(x^*\bigr) - h\bigl(x^*\bigr), h\bigl(y^*\bigr) - g\bigl(x^*\bigr)\rangle \geq 0, \quad \forall h\bigl(y^*\bigr) \in K\bigl(x^*\bigr) \tag{2.1}$$

for some $\rho > 0$, where $A : \mathcal{H} \to \mathcal{H}$ and $T : \mathcal{H} \to 2^{\mathcal{H}}$ are nonlinear mappings, while $g, h : \mathcal{H} \to \mathcal{H}$ are any mappings.

We call inequality (2.1) a *generalized extended nonlinear quasi-variational inequality problem*.

We now list some special cases of generalized extended nonlinear quasi-variational inequality problem (2.1).

(1) If we take $T = 0$, then problem (2.1) is equivalent to the extended general quasi-variational inequality problem introduced and studied by Noor *et al.* [29, 30].
(2) If we take $T = 0$ and $g = I$, then problem (2.1) is equivalent to a class of quasi-variational inequality problems introduced by Noor *et al.* [29].
(3) If we take $T = 0$ and $g = h$, then problem (2.1) is equivalent to the general quasi-variational inequality problem studied by Noor *et al.* [30].

(4) If we take $T = 0$ and $h = I$, then problem (2.1) is equivalent to the general quasi-variational inequality problem defined by Noor *et al.* [28].

If $K(x^*) \equiv K$, that is, the convex set $K(x^*)$ is independent of the solution x^*, then generalized extended nonlinear quasi-variational inequality problem (2.1) is equivalent to finding $x^* \in \mathcal{H}$ and $w^* \in T(x^*)$ such that $g(x^*) \in K$ and

$$\langle \rho(Ax^* + w^*) + g(x^*) - h(x^*), h(y^*) - g(x^*) \rangle \geq 0, \quad \forall h(y^*) \in K \tag{2.2}$$

for some $\rho > 0$, where K is a closed and convex subset of a real Hilbert space $\mathcal{H}$.

We call inequality (2.2) a *generalized extended nonlinear variational inequality problem*. Variational inequality problem (2.2) covers several variational inequality problems studied in the literature, to which we now turn:

(1) If we take $T = 0$, then problem (2.2) is equivalent to the extended general variational inequality problem introduced and studied by Noor [31].
(2) If T is single-valued and h is an identity mapping, then problem (2.2) is equivalent to a variational inequality problem studied by Noor and Noor [28].
(3) If we take g, h as identity mappings, then problem (2.2) reduces to a variational inequality problem studied by Verma [26], Qin *et al.* [27].
(4) If we take $T = 0$ and $g = h$, then problem (2.2) is equivalent to the general variational inequality problem studied by Noor [23, 24].
(5) If we take $A = 0$ and h as an identity mapping, then problem (2.2) is equivalent to a variational inequality studied by Verma [25].
(6) If T is single-valued and g, h are identity mappings, then problem (2.2) is equivalent to a variational inequality problem studied by Noor [22].
(7) If $A = 0$ and g, h are identity mappings, then problem (2.2) is equivalent to a variational inequality problem studied by Bruck [19] and Fang *et al.* [20].
(8) If $T = 0$ and g, h are identity mappings, then problem (2.2) is equivalent to a classical variational inequality problem studied by Lions and Stampacchia [21].

Let us recall the following standard and classical result.

Lemma 2.1 *Let $K(x)$ be a closed and convex set in a Hilbert space $\mathcal{H}$. Then, for a given $z \in \mathcal{H}$, $x \in K(x)$ satisfies the inequality*

$$\langle x - z, y - x \rangle \geq 0, \quad \forall y \in K(x),$$

if and only if

$$x = P_{K(x)}z,$$

where $P_{K(x)}$ is the projection of $\mathcal{H}$ onto the closed convex set $K(x)$ in $\mathcal{H}$.

It is important to point out that the implicit projection operator $P_{K(x)}$ is not nonexpansive. We shall assume that the implicit projection operator $P_{K(x)}$ satisfies the Lipschitz-type continuity, which plays an important and fundamental role in the existence theory and in developing numerical methods for solving the quasi-variational inequalities.

Assumption 2.1 For all $x, y, z \in \mathcal{H}$, the implicit projection operator $P_{K(x)}$ satisfies the condition

$$\|P_{K(x)}z - P_{K(y)}z\| \leq \vartheta \|x - y\|, \tag{2.3}$$

where ϑ is a positive constant.

Noor *et al.* [32] showed that Assumption 2.1 holds for certain cases.

We now recall some definitions.

Definition 2.1 A mapping $A : \mathcal{H} \to \mathcal{H}$ is said to be:

(i) *strongly monotone* if there exists a constant $\nu > 0$ such that, for each $x \in \mathcal{H}$,

$$\langle A(x) - A(y), x - y \rangle \geq \nu \|x - y\|^2$$

holds for all $y \in \mathcal{H}$;

(ii) *ϕ-cocoercive* if there exists a constant $\phi > 0$ such that, for each $x \in \mathcal{H}$,

$$\langle A(x) - A(y), x - y \rangle \geq \phi \|A(x) - A(y)\|^2$$

holds for all $y \in \mathcal{H}$;

(iii) *relaxed ϕ-cocoercive* if there exists a constant $\phi > 0$ such that, for each $x \in \mathcal{H}$,

$$\langle A(x) - A(y), x - y \rangle \geq -\phi \|A(x) - A(y)\|^2$$

holds for all $y \in \mathcal{H}$;

(iv) *relaxed (ϕ, γ)-cocoercive* or relaxed cocoercive with constant (ϕ, γ) if there exist constants $\phi > 0$ and $\gamma > 0$ such that, for each $x \in \mathcal{H}$,

$$\langle A(x) - A(y), x - y \rangle \geq -\phi \|A(x) - A(y)\|^2 + \gamma \|x - y\|^2$$

holds for all $y \in \mathcal{H}$;

(v) *μ-Lipschitz continuous* or Lipschitz with constant μ if there exists a constant $\mu > 0$ such that, for each $x, y \in \mathcal{H}$,

$$\|A(x) - A(y)\| \leq \mu \|x - y\|;$$

(vi) *nonexpansive* if for each $x, y \in \mathcal{H}$,

$$\|A(x) - A(y)\| \leq \|x - y\|.$$

A set-valued mapping $T : \mathcal{H} \to 2^{\mathcal{H}}$ is said to be:

(vii) *$\widehat{H}$-Lipschitz continuous* with constant ζ if there exists a constant $\zeta > 0$ such that

$$\widehat{H}\big(T(x), T(y)\big) \leq \zeta \|x - y\|, \quad \forall x, y \in \mathcal{H},$$

where $\widehat{H}$ is the Hausdorff pseudo-metric, *i.e.*, for any two nonempty subsets A and B of $\mathcal{H}$,

$$\widehat{H}(A,B) = \max\Big\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\Big\},$$

where

$$d(x,B) = \inf_{y\in B}\|x-y\| \quad \text{and} \quad d(A,y) = \inf_{x\in A}\|x-y\|.$$

It should be pointed out that if the domain of $\widehat{H}$ is restricted to the family of closed bounded subsets of $\mathcal{H}$ (denoted by $CB(\mathcal{H})$), then $\widehat{H}$ is the Hausdorff metric.

Lemma 2.2 [35] *Let (X,d) be a complete metric space, $T : X \to CB(X)$ be a set-valued mapping. Then, for any $\varepsilon > 0$ and $x, y \in X$, $u \in T(x)$, there exists $v \in T(y)$ such that*

$$d(u,v) \le (1+\varepsilon)\widehat{H}\big(T(x), T(y)\big).$$

Lemma 2.3 [35] *Let (X,d) be a complete metric space, $T : X \to CB(X)$ be a set-valued mapping satisfying*

$$\widehat{H}\big(T(x), T(y)\big) \le kd(x,y), \quad \forall x,y \in X,$$

where $0 \le k < 1$ is a constant. Then the mapping T has a fixed point in X.

3 Existence results

First of all, using Lemma 2.1, we will establish that generalized extended nonlinear quasi-variational inequality problem (2.1) is equivalent to a fixed point problem.

Lemma 3.1 *$x^* \in \mathcal{H}$ and $w^* \in T(x^*)$ such that $g(x^*) \in K(x^*)$ is a solution of generalized extended nonlinear quasi-variational inequality problem* (2.1) *if and only if for some $\rho > 0$, the mapping*

$$F : \mathcal{H} \to 2^{\mathcal{H}}, \qquad F(x^*) = \bigcup_{w^* \in T(x^*)} \big\{x^* - g(x^*) + P_{K(x^*)}\big(h(x^*) - \rho\big(A(x^*) + w^*\big)\big)\big\} \tag{3.1}$$

has a fixed point.

Proof Let $x^* \in \mathcal{H}$ and $w^* \in T(x^*)$ such that $g(x^*) \in K(x^*)$ is a solution of problem (2.1), *i.e.*,

$$\big\langle g(x^*) - \big(h(x^*) - \rho\big(A(x^*) + w^*\big)\big), h(y^*) - g(x^*)\big\rangle \ge 0 \tag{3.2}$$

for all $h(y^*) \in K(x^*)$.

Applying Lemma 2.1 to (3.2), we get

$$g(x^*) = P_{K(x^*)}\big(h(x^*) - \rho\big(A(x^*) + w^*\big)\big),$$

i.e.,

$$\begin{aligned} x^* &= x^* - g(x^*) + P_{K(x^*)}\big(h(x^*) - \rho\big(A(x^*) + w^*\big)\big) \\ &\in \bigcup_{w^* \in T(x^*)} \big\{x^* - g(x^*) + P_{K(x^*)}\big(h(x^*) - \rho\big(A(x^*) + w^*\big)\big)\big\} \\ &\Rightarrow x^* \in F(x^*), \end{aligned}$$

i.e., x^* is a fixed point of F.

Conversely, let x^* be a fixed point of F, *i.e.*, $x^* \in F(x^*)$, then there exists $w^* \in T(x^*)$ such that

$$x^* = x^* - g(x^*) + P_{K(x^*)}\big(h(x^*) - \rho\big(A(x^*) + w^*\big)\big),$$

i.e.,

$$g(x^*) = P_{K(x^*)}\big(h(x^*) - \rho\big(A(x^*) + w^*\big)\big).$$

Hence,

$$\big\langle g(x^*) - \big(h(x^*) - \rho\big(A(x^*) + w^*\big)\big), h(y^*) - g(x^*)\big\rangle \geq 0 \quad \text{for all } h(y^*) \in K(x^*).$$

The proof is complete. □

Lemma 3.1 implies that problem (2.1) is equivalent to fixed point problem (3.1). Using this connection, we will establish the following existence result.

Theorem 3.1 *Let $A, g, h : \mathcal{H} \to \mathcal{H}$ be relaxed cocoercive with constants (ϕ_A, γ_A), (ϕ_g, γ_g), (ϕ_h, γ_h) and Lipschitz continuous mappings with constants μ_A, μ_g, μ_h, respectively. Let $T : \mathcal{H} \to CB(\mathcal{H})$ be an $\widehat{H}$-Lipschitz continuous mapping with constant $\zeta > 0$. Assume that the following assumption holds*:

$$\begin{aligned} &\left|\rho - \frac{\gamma_A - \phi_A\mu_A^2 - \zeta(1-\kappa)}{\mu_A^2 - \zeta^2}\right| < \frac{\sqrt{(\gamma_A - \phi_A\mu_A^2 - \zeta(1-\kappa))^2 - (\mu_A^2 - \zeta^2)\kappa(2-\kappa)}}{(\mu_A^2 - \zeta^2)}, \\ &\mu_A^2 > \zeta^2, \qquad \gamma_A > \zeta(1-\kappa) + \phi_A\mu_A^2 + \sqrt{(\mu_A^2 - \zeta^2)\kappa(2-\kappa)}, \\ &1 + \mu_g^2(1 + 2\phi_g) > 2\gamma_g, \qquad 1 + \mu_h^2(1 + 2\phi_h) > 2\gamma_h, \end{aligned} \tag{3.3}$$

where

$$\kappa = \vartheta + \sqrt{1 - 2\gamma_g + \mu_g^2(1 + 2\phi_g)} + \sqrt{1 - 2\gamma_h + \mu_h^2(1 + 2\phi_h)}.$$

Then problem (2.1) *has a solution*.

Proof In the light of Lemma 3.1, it is enough to show that the mapping F defined by (3.1) has a fixed point. For any $x \neq y \in \mathcal{H}$, $p \in F(x)$ and $q \in F(y)$, there exist $w \in T(x)$ and $u \in T(y)$

such that

$$p = x - g(x) + P_{K(x)}\big[h(x) - \rho\big(A(x) + w\big)\big],$$
$$q = y - g(y) + P_{K(y)}\big[h(y) - \rho\big(A(y) + u\big)\big].$$

Using Assumption 2.1, we have

$$\begin{aligned}
\|p - q\| &\le \big\|x - y - \big(g(x) - g(y)\big)\big\| \\
&\quad + \big\|P_{K(x)}\big[h(x) - \rho\big(A(x) + w\big)\big] - P_{K(y)}\big[h(y) - \rho\big(A(y) + u\big)\big]\big\| \\
&\le \big\|x - y - \big(g(x) - g(y)\big)\big\| \\
&\quad + \big\|P_{K(x)}\big[h(x) - \rho\big(A(x) + w\big)\big] - P_{K(y)}\big[h(x) - \rho\big(A(x) + w\big)\big]\big\| \\
&\quad + \big\|P_{K(y)}\big[h(x) - \rho\big(A(x) + w\big)\big] - P_{K(y)}\big[h(y) - \rho\big(A(y) + u\big)\big]\big\| \\
&\le \big\|x - y - \big(g(x) - g(y)\big)\big\| + \vartheta\|x - y\| + \big\|x - y - \big(h(x) - h(y)\big)\big\| \\
&\quad + \big\|x - y - \rho\big\{A(x) - A(y)\big\}\big\| + \rho\|w - u\|.
\end{aligned} \tag{3.4}$$

Since g is a relaxed (ϕ_g, γ_g)-cocoercive and μ_g-Lipschitz continuous mapping, we find the following:

$$\begin{aligned}
\big\|x - y - \big(g(x) - g(y)\big)\big\|^2 &= \|x - y\|^2 - 2\big\langle g(x) - g(y), x - y\big\rangle + \big\|g(x) - g(y)\big\|^2 \\
&\le \big(1 + \mu_g^2\big)\|x - y\|^2 + 2\phi_g\big\|g(x) - g(y)\big\|^2 - 2\gamma_g\|x - y\|^2 \\
&\le \big(1 - 2\gamma_g + \mu_g^2(1 + 2\phi_g)\big)\|x - y\|^2.
\end{aligned} \tag{3.5}$$

Similarly,

$$\big\|x - y - \big(h(x) - h(y)\big)\big\|^2 \le \big(1 - 2\gamma_h + \mu_h^2(1 + 2\phi_h)\big)\|x - y\|^2. \tag{3.6}$$

Since A is a relaxed (ϕ_A, γ_A)-cocoercive and μ_A-Lipschitz continuous mapping, we have

$$\begin{aligned}
&\big\|x - y - \rho\big\{A(x) - A(y)\big\}\big\|^2 \\
&\quad = \|x - y\|^2 - 2\rho\big\langle A(x) - A(y), x - y\big\rangle + \rho^2\big\|A(x) - A(y)\big\|^2 \\
&\quad \le \|x - y\|^2 - 2\rho\big\{-\phi_A\big\|A(x) - A(y)\big\|^2 + \gamma_A\|x - y\|^2\big\} + \rho^2\mu_A^2\|x - y\|^2 \\
&\quad \le \|x - y\|^2 + 2\rho\phi_A\mu_A^2\|x - y\|^2 - 2\rho\gamma_A\|x - y\|^2 + \rho^2\mu_A^2\|x - y\|^2 \\
&\quad = \big(1 + 2\rho\big(\phi_A\mu_A^2 - \gamma_A\big) + \rho^2\mu_A^2\big)\|x - y\|^2.
\end{aligned} \tag{3.7}$$

Now, since T is an $\widehat{H}$-Lipschitz continuous mapping, we estimate

$$\begin{aligned}
\|w - u\| &\le (1 + \varepsilon)\widehat{H}\big(T(x) - T(y)\big) \\
&\le \zeta(1 + \varepsilon)\|x - y\|.
\end{aligned} \tag{3.8}$$

Substituting (3.5), (3.6), (3.7) and (3.8) into (3.4), we obtain

$$\|p - q\| \le \big[\kappa + f(\rho) + \rho\zeta(1 + \varepsilon)\big]\|x - y\|, \tag{3.9}$$

where

$$\kappa = \vartheta + \sqrt{1 - 2\gamma_g + \mu_g^2(1 + 2\phi_g)} + \sqrt{1 - 2\gamma_h + \mu_h^2(1 + 2\phi_h)}$$

and

$$f(\rho) = \sqrt{1 + 2\rho\left(\phi_A\mu_A^2 - \gamma_A\right) + \rho^2\mu_A^2}.$$

By using (3.9), we get

$$d\bigl(p, F(y)\bigr) = \inf_{q\in F(y)} \|p - q\| \leq \bigl(\kappa + f(\rho) + \rho\zeta(1 + \varepsilon)\bigr)\|x - y\|,$$

since $p \in F(x)$ is arbitrary, we obtain

$$\sup_{p\in F(x)} d\bigl(p, F(y)\bigr) \leq \bigl(\kappa + f(\rho) + \rho\zeta(1 + \varepsilon)\bigr)\|x - y\|. \tag{3.10}$$

Similarly, we get

$$\sup_{q\in F(y)} d\bigl(q, F(x)\bigr) \leq \bigl(\kappa + f(\rho) + \rho\zeta(1 + \varepsilon)\bigr)\|x - y\|. \tag{3.11}$$

From the definition of Hausdorff metric $\widehat{H}$, it follows from (3.10) and (3.11) that

$$\widehat{H}\bigl(F(x), F(y)\bigr) \leq \bigl(\kappa + f(\rho) + \rho\zeta(1 + \varepsilon)\bigr)\|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

Letting $\varepsilon \to 0$, we get that

$$\widehat{H}\bigl(F(x), F(y)\bigr) \leq \bigl(\kappa + f(\rho) + \rho\zeta\bigr)\|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

From (3.3), we get that $(\kappa + f(\rho) + \rho\zeta) < 1$, thus F is a set-valued contraction mapping, by Lemma 2.3 it has a fixed point. Lemma 3.1 implies that it is a solution of variational inequality problem (2.1). □

4 Iterative algorithm and convergence

For given $x_0 \in \mathcal{H}$ and $w_0 \in T(x_0)$, let

$$x_1 = x_0 - g(x_0) + P_{K(x_0)}\bigl(h(x_0) - \rho\bigl(A(x_0) + w_0\bigr)\bigr).$$

By Lemma 2.3 there exists $w_1 \in T(x_1)$ such that

$$\|w_0 - w_1\| \leq (1 + 1)\widehat{H}(Tx_0, Tx_1).$$

Let $x_2 = x_1 - g(x_1) + P_{K(x_1)}(h(x_1) - \rho(A(x_1) + w_1))$, then by Lemma 2.3 there exists $w_2 \in T(x_2)$ such that

$$\|w_1 - w_2\| \leq \left(1 + \frac{1}{2}\right)\widehat{H}(Tx_1, Tx_2).$$

By induction, we can get an iterative algorithm as follows.

Algorithm 1 For given $x_0 \in \mathcal{H}$, $w_0 \in T(x_0)$, define sequences $\{x_n\}$ and $\{w_n\}$ satisfying

$$\left.\begin{aligned} &x_{n+1} = x_n - g(x_n) + P_{K(x_n)}\bigl(h(x_n) - \rho\bigl(A(x_n) + w_n\bigr)\bigr), \\ &w_n \in T(x_n), \quad \|w_n - w_{n+1}\| \le \left(1 + \frac{1}{n+1}\right)\widehat{H}\bigl(T(x_n), T(x_{n+1})\bigr). \end{aligned}\right\} \tag{4.1}$$

Now, we define an Ishikawa-type iterative algorithm [36] for approximate solvability of variational inequality problem (2.2).

Algorithm 2 For a given $x_0 \in \mathcal{H}$, compute x_{n+1} by the scheme

$$\begin{aligned} &y_n = (1 - \beta_n)x_n + \beta_n\bigl[x_n - g(x_n) + P_{K(x_n)}\bigl(h(x_n) - \rho\bigl(A(x_n) + w_n\bigr)\bigr)\bigr], \\ &x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\bigl[x_n - g(x_n) + P_{K(y_n)}\bigl(h(y_n) - \rho\bigl(A(y_n) + u_n\bigr)\bigr)\bigr], \end{aligned} \tag{4.2}$$

where $w_n \in T(x_n)$, $u_n \in T(y_n)$, $n = 0, 1, 2, \dots$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$, satisfying certain conditions.

To prove the next result, we need the following.

Lemma 4.1 [37] *Let* $\{a_n\}$ *be a nonnegative sequence satisfying*

$$a_{n+1} \le (1 - c_n)a_n + b_n,$$

with $c_n \in [0, 1]$, $\sum_{n=0}^{\infty} c_n = \infty$, $b_n = o(c_n)$. *Then* $\lim_{n\to\infty} a_n = 0$.

Theorem 4.1 *Let* A, T, g, h *satisfy all the assumptions of Theorem* 3.1, *and let* $\{\alpha_n\}$, $\{\beta_n\}$ *be sequences in* $[0, 1]$, *for all* $n \ge 0$, *such that* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Then the approximate sequences* $\{x_n\}$, $\{w_n\}$ *constructed by Algorithm* 2 *converge strongly to a solution of problem* (2.1).

Proof By Theorem 3.1, generalized extended nonlinear quasi-variational inequality problem (2.1) has a solution. Let $x^* \in \mathcal{H}$, $w^* \in T(x^*)$ such that $g(x^*) \in K(x^*)$ be a solution of (2.1). By Lemma 3.1, we have

$$x^* = x^* - g\bigl(x^*\bigr) + P_{K(x^*)}\bigl(h\bigl(x^*\bigr) - \rho\bigl(A\bigl(x^*\bigr) - w^*\bigr)\bigr).$$

Using (4.2), we have

$$\begin{aligned} \bigl\|x_{n+1} - x^*\bigr\| &= \bigl\|(1 - \alpha_n)x_n + \alpha_n\bigl[x_n - g(x_n) + P_{K(y_n)}\bigl(h(y_n) - \rho\bigl(A(y_n) + u_n\bigr)\bigr)\bigr] - x^*\bigr\| \\ &\le (1 - \alpha_n)\bigl\|x_n - x^*\bigr\| \\ &\quad + \alpha_n\bigl\|\bigl[x_n - g(x_n) + P_{K(y_n)}\bigl(h(y_n) - \rho\bigl(A(y_n) + u_n\bigr)\bigr)\bigr] - x^*\bigr\| \\ &\le (1 - \alpha_n)\bigl\|x_n - x^*\bigr\| + \alpha_n\bigl\|x_n - x^* - \bigl(g(x_n) - g\bigl(x^*\bigr)\bigr)\bigr\| \\ &\quad + \alpha_n\bigl\|P_{K(y_n)}\bigl[h(y_n) - \rho\bigl(A(y_n) + u_n\bigr)\bigr] - P_{K(x^*)}\bigl[h\bigl(x^*\bigr) - \rho\bigl(A\bigl(x^*\bigr) + w^*\bigr)\bigr]\bigr\| \\ &\le (1 - \alpha_n)\bigl\|x_n - x^*\bigr\| + \alpha_n\bigl\|x_n - x^* - \bigl(g(x_n) - g\bigl(x^*\bigr)\bigr)\bigr\| \\ &\quad + \alpha_n\bigl\|P_{K(y_n)}\bigl[h(y_n) - \rho\bigl(A(y_n) + u_n\bigr)\bigr] - P_{K(y_n)}\bigl[h\bigl(x^*\bigr) - \rho\bigl(A\bigl(x^*\bigr) + w^*\bigr)\bigr]\bigr\| \end{aligned}$$

$$
\begin{aligned}
&+ \alpha_n \| P_{K(y_n)}[h(x^*) - \rho(A(x^*) + w^*)] - P_{K(x^*)}[h(x^*) - \rho(A(x^*) + w^*)] \| \\
&\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \| x_n - x^* - (g(x_n) - g(x^*)) \| \\
&\quad + \alpha_n \| h(y_n) - h(x^*) - \rho\{(A(y_n) + u_n) - (A(x^*) + w^*)\} \| \\
&\quad + \alpha_n \vartheta \| y_n - x^* \| \\
&\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \| x_n - x^* - (g(x_n) - g(x^*)) \| \\
&\quad + \alpha_n \| y_n - x^* - (h(y_n) - h(x^*)) \| \\
&\quad + \alpha_n \| y_n - x^* - \rho(A(y_n) - A(x^*)) \| \\
&\quad + \alpha_n \rho \| u_n - w^* \| + \alpha_n \vartheta \| y_n - x^* \| \\
&\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \sqrt{1 - 2\gamma_g + \mu_g^2(1 + 2\phi_g)} \| x_n - x^* \| \\
&\quad + \alpha_n \sqrt{1 - 2\gamma_h + \mu_h^2(1 + 2\phi_h)} \| y_n - x^* \| \\
&\quad + \alpha_n \sqrt{1 + 2\rho(\phi_A \mu_A^2 - \gamma_A) + \rho^2 \mu_A^2} \| y_n - x^* \| \\
&\quad + \alpha_n \rho \zeta (1 + \varepsilon) \| y_n - x^* \| + \alpha_n \vartheta \| y_n - x^* \| \\
&= (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \theta_g \| x_n - x^* \| \\
&\quad + \alpha_n (\theta_h + f(\rho) + \rho\zeta(1 + \varepsilon) + \vartheta) \| y_n - x^* \|, \qquad (4.3)
\end{aligned}
$$

where

$$\theta_g = \sqrt{1 - 2\gamma_g + \mu_g^2(1 + 2\phi_g)}, \qquad \theta_h = \sqrt{1 - 2\gamma_h + \mu_h^2(1 + 2\phi_h)}$$

and

$$f(\rho) = \sqrt{1 + 2\rho(\phi_A \mu_A^2 - \gamma_A) + \rho^2 \mu_A^2}.$$

Similarly, we have

$$
\begin{aligned}
\| y_n - x^* \| &\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n \| x_n - x^* - (g(x_n) - g(x^*)) \| \\
&\quad + \beta_n \| P_{K(x_n)}[h(x_n) - \rho(A(x_n) + w_n)] - P_{K(x^*)}[h(x^*) - \rho(A(x^*) - w^*)] \| \\
&\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n \theta_g \| x_n - x^* \| \\
&\quad + \beta_n \| P_{K(x_n)}[h(x_n) - \rho(A(x_n) + w_n)] - P_{K(x_n)}[h(x^*) - \rho(A(x^*) - w^*)] \| \\
&\quad + \beta_n \| P_{K(x_n)}[h(x^*) - \rho(A(x^*) - w^*)] - P_{K(x^*)}[h(x^*) - \rho(A(x^*) - w^*)] \| \\
&\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n \theta_g \| x_n - x^* \| \\
&\quad + \beta_n \| h(x_n) - h(x^*) - \rho\{(A(x_n) + w_n) - (A(x^*) - w^*)\} \| \\
&\quad + \beta_n \vartheta \| x_n - x^* \| \\
&\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n \theta_g \| x_n - x^* \| + \beta_n \vartheta \| x_n - x^* \| \\
&\quad + \beta_n \| x_n - x^* - (h(x_n) - h(x^*)) \| \\
&\quad + \beta_n \| x_n - x^* - \rho\{A(x_n) - A(x^*)\} \| + \beta_n \rho \| w_n - w^* \|
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\theta_g\|x_n - x^*\| \\
&\quad + \beta_n\big(\theta_h + f(\rho) + \rho\zeta(1+\varepsilon) + \vartheta\big)\|x_n - x^*\| \\
&= (1-\beta_n)\|x_n - x^*\| + \beta_n\big(\kappa + f(\rho) + \rho\zeta(1+\varepsilon)\big)\|x_n - x^*\|.
\end{aligned} \tag{4.4}
$$

Substituting (4.4) into (4.3) yields that

$$
\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_g\|x_n - x^*\| \\
&\quad + \alpha_n\big(\theta_h + f(\rho) + \rho\zeta(1+\varepsilon) + \vartheta\big) \\
&\quad \times \big\{1 - \beta_n\big(1 - \big(\kappa + f(\rho) + \rho\zeta(1+\varepsilon)\big)\big)\big\}\|x_n - x^*\|.
\end{aligned} \tag{4.5}
$$

Letting $\varepsilon \to 0$, we get from (4.5) that

$$
\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_g\|x_n - x^*\| \\
&\quad + \alpha_n\big(\theta_h + f(\rho) + \rho\zeta + \vartheta\big)\big\{1 - \beta_n\big(1 - \big(\kappa + f(\rho) + \rho\zeta\big)\big)\big\}\|x_n - x^*\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\big(\theta_g + \theta_h + f(\rho) + \rho\zeta + \vartheta\big)\|x_n - x^*\| \\
&= (1-\alpha_n)\|x_n - x^*\| + \alpha_n\big(\kappa + f(\rho) + \rho\zeta\big)\|x_n - x^*\| \\
&= \big[1 - \alpha_n\big\{1 - \big(\kappa + f(\rho) + \rho\zeta\big)\big\}\big]\|x_n - x^*\|.
\end{aligned} \tag{4.6}
$$

By virtue of Lemma 4.1, we get from (4.6) that $\lim_{n\to\infty}\|x_{n+1} - x^*\| = 0$, *i.e.*, $x_n \to x^*$, as $n \to \infty$. Since

$$
\|w_n - w^*\| \leq (1+\varepsilon)\zeta\|x_n - x^*\|,
$$

letting $n \to \infty$, we get that $w_n \to w^*$. This completes the proof. □

Remark 1 For a suitable and appropriate choice of the operators T, A, g, h and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, one can obtain a number of new and previously known iterative schemes for approximate solvability of variational inequality problems as discussed in special cases. This clearly shows that Algorithm 2 is quite general and unifies several algorithms.

Remark 2 Results presented in the paper are significant improvement and extension of the results obtained previously by many authors. Especially, our Theorem 3.1 extends the existence of solution in the literature to the case of generalized extended nonlinear variational inequality (2.1). Algorithm 2 is a very general and unified algorithm for finding an approximate solution of problem (2.1).

5 Conclusion

In this paper, we have considered a new class of generalized extended nonlinear quasi-variational inequalities, which involves sum of two operators $A : \mathcal{H} \to \mathcal{H}$ and $T : \mathcal{H} \to 2^{\mathcal{H}}$. We have established the equivalence between the generalized extended nonlinear variational inequality and the fixed point problem using projection mapping. Using this equivalence, we have first established criteria for the existence of solution of the proposed variational inequality problem. We have also suggested and analyzed some iterative methods for approximate solvability of generalized extended nonlinear quasi-variational inequalities. Several special cases of the proposed variational inequality problem have also been discussed.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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