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Multiple positive solutions of boundary value problems for fractional order integro-differential equations in a Banach space

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Abstract

In this paper, we obtain the existence of multiple positive solutions of a boundary value problem for α-order nonlinear integro-differential equations in a Banach space by means of fixed point index theory of completely continuous operators.

MSC: 26A33; 34B15

Keywords: fractional order; integro-differential equation; measure of noncompactness; fixed point index; boundary value problem

1 Introduction

Fractional differential equations (FDEs) have been of great interest for the last three decades [1–11]. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity [12], electrochemistry [13], control, porous media [14], *etc.* Therefore, the theory of FDEs has been developed very quickly. Many qualitative theories of FDEs have been obtained. Many important results can be found in [15–19] and references cited therein.

In this paper, we shall use the fixed point index theory of completely continuous operators to investigate the multiple positive solutions of a boundary value problem for a class of α order nonlinear integro-differential equations in a Banach space.

Let E be a real Banach space, P be a cone in E and P^0 denote the interior points of P. A partial ordering in E is introduced by $x \leq y$ if and only if $y - x \in P$. P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where θ denotes the zero element of E, and the smallest constant N is called the normal constant of P. P is called solid if P^0 is nonempty. If $x \leq y$ and $x \neq y$, we write $x < y$. If P is solid and $y - x \in P^0$, we write $x \ll y$. For details on cone theory, see [1].

For the application in the sequel, we first state the following lemmas and definitions which can be found in [1, 10, 20].

Lemma 1.1 *Let P be a cone in a real Banach space E, and let Ω be a nonempty bounded open convex subset of P. Suppose that $A : \overline{\Omega} \to P$ is completely continuous and $A(\overline{\Omega}) \subset \Omega$,* *where* $\overline{\Omega}$ *denotes the closure of* Ω *in* P. *Then the fixed point index*

$$i(A,\Omega,P) = 1.$$

Lemma 1.2 *Let* P *be a cone in a real Banach space* E, *and let* $\Omega = \Omega_1 \cup \Omega_2$, *where* Ω_i ($i = 1,2$) *are nonempty bounded open convex subsets of* P *and* $\Omega_1 \cap \Omega_2 = \emptyset$. *Suppose that* $A : \overline{\Omega} \to P$ *is a strict set contraction and* $A(\overline{\Omega}) \subset \Omega$. *Then*

$$i(A,\Omega,P) = i(A,\Omega_1,P) + i(A,\Omega_2,P).$$

Lemma 1.3 *If* $U \subset C[I,E]$ *is bounded and equicontinuous, then* $\alpha_E(U(t))$ *is continuous on* I, *and set*

$$\alpha_C(U) = \max_{t\in I} \alpha_E\bigl(U(t)\bigr), \qquad \alpha_E\left(\int_I u(t)\,\mathrm{d}t : u \in U\right) \le \int_I \alpha_E\bigl(U(t)\bigr)\,\mathrm{d}t,$$

where $I = [a,b]$, $U(t) = \{u(t) : u \in U\}$.

Definition 1.1 The fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is given by

$$I^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

Definition 1.2 The fractional derivative of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is given by

$$D^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s,$$

where $n = [\alpha] + 1$, provided the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 1.4 *Let* $\alpha > 0$, *then*

$$I^{\alpha}_{0+}D^{\alpha}_{0+}x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$$

for some $c_i \in E$, $i = 0,1,2,\ldots,n-1$, $n = -[-\alpha]$.

In this article, let $J = [0,+\infty)$, $BC[J,E] = \{u \in C[J,E] : \sup_{t\in J} \frac{\|u(t)\|}{1+t^{\alpha-1}} < \infty\}$. It is easy to see that $BC[J,E]$ is a Banach space with the norm

$$\|u\|_B = \sup_{t\in J} \frac{\|u(t)\|}{1+t^{\alpha-1}}.$$

Consider the boundary value problem (BVP) for a fractional nonlinear integro-differential equation of mixed type in E:

$$\begin{cases} D^{\alpha}_0 u(t) + f(t,u(t),(Tu)(t),(Su)(t)) = \theta \quad \forall t \in J, \\ u(0) = u'(0) = \theta, \qquad D^{\alpha-1}_0 u(+\infty) = \sum_{i=1}^m \beta_i u(\eta_i), \end{cases} \tag{1}$$

where D_0^α is the standard Riemann-Liouville fractional derivative of order $2 < \alpha < 3$, $f \in C[J \times P \times P \times P, P]$, $\beta_i > 0$ $(i = 1, 2, \ldots, m)$, $0 < \eta_1 < \eta_2 < \cdots < \eta_m$, $\sum_{i=1}^m \beta_i \eta_i^{\alpha-1} < \Gamma(\alpha)$ and

$$(Tu)(t) = \int_0^t K(t,s)u(s)\,\mathrm{d}s, \qquad (Su)(t) = \int_0^{+\infty} H(t,s)u(s)\,\mathrm{d}s, \tag{2}$$

$K \in C[D, R^+]$, $D = \{(t,s) \in J \times J : t \geq s\}$, $H \in C[J \times J, R^+]$, R^+ denotes the set of all nonnegative real numbers.

2 Several lemmas

To establish the existence of multiple positive solutions in $BC[J, P]$ of (1), let us list the following assumptions.

(H_1) $k^* = \sup_{t\in J} \int_0^t K(t,s)\,\mathrm{d}s < \infty$, $h^* = \sup_{t\in J} \frac{1}{1+t^{\alpha-1}} \int_0^{+\infty} H(t,s)(1 + s^{\alpha-1})\,\mathrm{d}s < \infty$, $\int_0^{+\infty}(H(t',s) - H(t,s))(1 + s^{\alpha-1})\,\mathrm{d}s \to 0$, as $t' \to t$ $(t \in J)$.

(H_2) There exist $a, b \in C[J, R^+]$ and $g \in C[J \times J \times J, R^+]$ such that

$$\bigl\|f(t,u,v,w)\bigr\| \leq a(t) + b(t)g\bigl(\|u\|, \|v\|, \|w\|\bigr) \quad \forall t \in J, u, v, w \in P.$$

(H_3) There exists $c \in C[J, R^+]$ such that

$$\frac{\|f(t,u,v,w)\|}{c(t)(\|u\| + \|v\| + \|w\|)} \to 0, \quad \text{as } u, v, w \in P, \|u\| + \|v\| + \|w\| \to \infty$$

uniformly for $t \in J$, and

$$c^* = \int_0^{+\infty} c(t)\bigl(1 + t^{\alpha-1}\bigr)\,\mathrm{d}t < \infty.$$

(H_4) There exists $d \in C[J, R^+]$ such that

$$\frac{\|f(t,(1+t^{\alpha-1})u,(1+t^{\alpha-1})v,(1+t^{\alpha-1})w)\|}{d(t)(\|u\| + \|v\| + \|w\|)} \to 0, \quad \text{as } u, v, w \in P, \|u\| + \|v\| + \|w\| \to 0$$

uniformly for $t \in J$, and

$$d^* = \int_0^{+\infty} d(t)\,\mathrm{d}t < \infty.$$

(H_5) For any $t \in J$ and $r > 0$, $f(t, P_r, P_r, P_r) = \{f(t,u,v,w) : u, v, w \in P_r\}$ is relatively compact in E, where $P_r = \{u \in P : \|u\| \leq r\}$.

(H_6) P is normal and solid, and there exist $u_0 \gg \theta$, $0 < t_* < t^* < \infty$ and $\sigma \in C[I, R^+]$ such that

$$f(t,u,v,w) \geq \sigma(t)u_0 \quad \forall t \in I, u \geq u_0, v \geq \theta, w \geq \theta$$

and

$$\int_{t_*}^{t^*} \gamma(s)\sigma(s) > 1,$$

where $I = [t_*, t^*]$, $\gamma(s) = \min_{t\in I} G(t,s)$.

(H_7) There exist $u_0 > \theta$, $0 < t_* < t^* < \infty$ and $\sigma \in C[I, R^+]$ such that

$$f(t,u,v,w) \geq \sigma(t)u_0 \quad \forall t \in I, u \geq u_0, v \geq \theta, w \geq \theta$$

and

$$\int_{t_*}^{t^*} \gamma(s)\sigma(s) \geq 1,$$

where $I = [t_*, t^*]$, $\gamma(s) = \min_{t\in I} G(t,s)$.

Remark 2.1 It is clear that (H_5) is satisfied automatically when E is finite dimensional.

Remark 2.2 It is clear that assumption (H_7) is weaker than assumption (H_6).

We shall reduce BVP (1) to an integral equation in E. To this end, we first consider the operator A defined by

$$\begin{aligned}(Au)(t) = {}& -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\bigl(s,u(s),(Tu)(s),(Su)(s)\bigr)\,\mathrm{d}s \\ & - \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^m \beta_i t^{\alpha-1}\int_0^{\eta_i}(\eta_i - s)^{\alpha-1} f\bigl(s,u(s),(Tu)(s),(Su)(s)\bigr)\,\mathrm{d}s \\ & + \lambda t^{\alpha-1}\int_0^{+\infty} f\bigl(s,u(s),(Tu)(s),(Su)(s)\bigr)\,\mathrm{d}s, \end{aligned} \tag{3}$$

where $\lambda = \frac{1}{\Gamma(\alpha)-\sum_{i=1}^m \beta_i \eta_i^{\alpha-1}}$.

In our main results, we make use of the following lemmas.

Lemma 2.1 *Let assumption* (H_1) *be satisfied, then the operators T and S defined by* (2) *are bounded linear operators from $BC[J,E]$ into $BC[J,P]$, and*

$$\|T\| \leq k^*, \qquad \|S\| \leq h^*. \tag{4}$$

Moreover,

$$T : BC[J,P] \to BC[J,P], \qquad S : BC[J,P] \to BC[J,P]. \tag{5}$$

Proof Inequalities (4) follow from two simple inequalities:

$$\begin{aligned}\frac{\|(Tu)(t)\|}{1+t^{\alpha-1}} &\leq \int_0^t K(t,s)\frac{1+s^{\alpha-1}}{1+t^{\alpha-1}}\frac{\|u(s)\|}{1+s^{\alpha-1}}\,\mathrm{d}s \leq k^*\|u\|_B, \\ \frac{\|(Su)(t)\|}{1+t^{\alpha-1}} &\leq \int_0^{+\infty} H(t,s)\frac{1+s^{\alpha-1}}{1+t^{\alpha-1}}\frac{\|u(s)\|}{1+s^{\alpha-1}}\,\mathrm{d}s \leq h^*\|u\|_B,\end{aligned}$$

and (5) is obvious. □

Lemma 2.2 *Let assumptions* (H_1), (H_2) *and* (H_3) *be satisfied, then the operator A defined by* (3) *is a continuous operator from $BC[J,E]$ into $BC[J,E]$.*

Proof Let

$$\varepsilon_1 = \frac{1}{2(1+k^*+h^*)}\left[\left(\frac{1}{\Gamma(\alpha)}+\lambda\right)c^* + \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{m}\beta_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-1}\bigl(a(s)+Mb(s)\bigr)\,\mathrm{d}s\right]^{-1},$$

where λ is defined in the operator A.

By virtue of assumptions (H_2) and (H_3), there exists an $R>0$ such that

$$\begin{aligned}&\bigl\|f(t,u,v,w)\bigr\| \le \varepsilon_1 c(t)\bigl(\|u\|+\|v\|+\|w\|\bigr)\\ &\quad \forall t\in J, u,v,w\in P, \|u\|+\|v\|+\|w\|>R\end{aligned} \tag{6}$$

and

$$\begin{aligned}&\bigl\|f(t,u,v,w)\bigr\| \le a(t)+Mb(t)\\ &\quad \forall t\in J, u,v,w\in P, \|u\|+\|v\|+\|w\|\le R,\end{aligned} \tag{7}$$

where

$$M=\max\bigl\{g(x_1,x_2,x_3): 0\le x_1,x_2,x_3\le R\bigr\}.$$

It follows from (6) and (7) that for $t\in J$, $u,v,w\in P$, we have

$$\bigl\|f(t,u,v,w)\bigr\| \le \varepsilon_1 c(t)\bigl(\|u\|+\|v\|+\|w\|\bigr)+a(t)+Mb(t). \tag{8}$$

Let $u\in BC[J,P]$, we have, by (8) and Lemma 2.1,

$$\bigl\|f\bigl(t,u,(Tu)(t),(Su)(t)\bigr)\bigr\| \le \varepsilon_1 c(t)\bigl(1+t^{\alpha-1}\bigr)\bigl(1+k^*+h^*\bigr)\|u\|_B + a(t)+Mb(t), \tag{9}$$

which implies the convergence of the infinite integral

$$\int_0^{+\infty} f\bigl(t,u,(Tu)(t),(Su)(t)\bigr)\,\mathrm{d}s$$

and

$$\int_0^{+\infty}\bigl\|f\bigl(t,u,(Tu)(t),(Su)(t)\bigr)\bigr\|\,\mathrm{d}s \le c^*\varepsilon_1\bigl(1+k^*+h^*\bigr)\|u\|_B + a^* + Mb^*. \tag{10}$$

Thus, we have, by (3), (9) and (10),

$$\begin{aligned}\frac{\|(Au)(t)\|}{1+t^{\alpha-1}} \le{}& \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}\bigl\|f\bigl(s,u(s),(Tu)(s),(Su)(s)\bigr)\bigr\|\,\mathrm{d}s\\ &+\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{m}\beta_i\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\int_0^{\eta_i}(\eta_i-s)^{\alpha-1}\bigl\|f\bigl(s,u(s),(Tu)(s),(Su)(s)\bigr)\bigr\|\,\mathrm{d}s\\ &+\frac{\lambda t^{\alpha-1}}{1+t^{\alpha-1}}\int_0^{+\infty}\bigl\|f\bigl(s,u(s),(Tu)(s),(Su)(s)\bigr)\bigr\|\,\mathrm{d}s\end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{1}{\Gamma(\alpha)} + \lambda\right) \int_0^{+\infty} \left\| f\big(s, u(s), (Tu)(s), (Su)(s)\big)\right\| \,\mathrm{d}s \\
&\quad + \frac{\lambda}{\Gamma(\alpha)} \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} \left\| f\big(s, u(s), (Tu)(s), (Su)(s)\big)\right\| \,\mathrm{d}s \\
&\leq \left(\frac{1}{\Gamma(\alpha)} + \lambda\right) \big(c^* \varepsilon_1 \big(1 + k^* + h^*\big) \|u\|_B + a^* + Mb^*\big) \\
&\quad + \frac{\lambda \varepsilon_1}{\Gamma(\alpha)} \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} \varepsilon_1 c(s) \big(1 + s^{\alpha-1}\big)\big(1 + k^* + h^*\big) \|u\|_B \,\mathrm{d}s \\
&\quad + \frac{\lambda}{\Gamma(\alpha)} \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} \big(a(s) + Mb(s)\big) \,\mathrm{d}s \\
&\leq \frac{1}{2} \|u\|_B + \left(\frac{1}{\Gamma(\alpha)} + \lambda\right) \big(a^* + Mb^*\big) \\
&\quad + \frac{\lambda}{\Gamma(\alpha)} \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} \big(a(s) + Mb(s)\big) \,\mathrm{d}s. \qquad (11)
\end{aligned}$$

It follows from (11) that

$$\begin{aligned}
\|Au\|_B &\leq \frac{1}{2} \|u\|_B + \left(\frac{1}{\Gamma(\alpha)} + \lambda\right) \big(a^* + Mb^*\big) \\
&\quad + \frac{\lambda}{\Gamma(\alpha)} \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} \big(a(s) + Mb(s)\big) \,\mathrm{d}s. \qquad (12)
\end{aligned}$$

Thus, we have $A(BC[J,E]) \subset BC[J,E]$.

Finally, we show that A is continuous. Let $u_n, \tilde{u} \in BC[J,E]$, $\|u_n - \tilde{u}\|_B \to 0$ ($n \to \infty$). Then $r = \sup_n \|u_n\| < \infty$ and $\|\tilde{u}\|_B \leq r$. By (3), we have

$$\begin{aligned}
&\left\| \frac{(Au_n)(t)}{1 + t^{\alpha-1}} - \frac{(A\tilde{u})(t)}{1 + t^{\alpha-1}} \right\| \qquad (13) \\
&\quad \leq \int_0^t \frac{(t-s)^{\alpha-1}}{1 + t^{\alpha-1}} \left\| f\big(s, u_n(s), (Tu_n)(s), (Su_n)(s)\big) - f\big(s, \tilde{u}(s), (T\tilde{u})(s), (S\tilde{u})(s)\big)\right\| \,\mathrm{d}s \\
&\qquad + \frac{\lambda}{\Gamma(\alpha)} \sum_{i=1}^{m} \beta_i \frac{t^{\alpha-1}}{1 + t^{\alpha-1}} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} \left\| f\big(s, u_n(s), (Tu_n)(s), (Su_n)(s)\big) \right. \\
&\qquad \left. - f\big(s, \tilde{u}(s), (T\tilde{u})(s), (S\tilde{u})(s)\big)\right\| \,\mathrm{d}s \\
&\qquad + \frac{\lambda t^{\alpha-1}}{1 + t^{\alpha-1}} \int_0^{+\infty} \left\| f\big(s, u_n(s), (Tu_n)(s), (Su_n)(s)\big) - f\big(s, \tilde{u}(s), (T\tilde{u})(s), (S\tilde{u})(s)\big)\right\| \,\mathrm{d}s \\
&\quad \leq \left(\frac{1}{\Gamma(\alpha)} + \lambda\right) \int_0^{+\infty} \left\| f\big(s, u_n(s), (Tu_n)(s), (Su_n)(s)\big) - f\big(s, \tilde{u}(s), (T\tilde{u})(s), (S\tilde{u})(s)\big)\right\| \,\mathrm{d}s \\
&\qquad + \frac{\lambda}{\Gamma(\alpha)} \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} \left\| f\big(s, u_n(s), (Tu_n)(s), (Su_n)(s)\big) \right. \\
&\qquad \left. - f\big(s, \tilde{u}(s), (T\tilde{u})(s), (S\tilde{u})(s)\big)\right\| \,\mathrm{d}s. \qquad (14)
\end{aligned}$$

It is clear that

$$f\big(t, u_n(t), (Tu_n)(t), (Su_n)(t)\big) \to f\big(t, \tilde{u}(t), (T\tilde{u})(t), (S\tilde{u})(t)\big), \quad n \to \infty, \tag{15}$$

and by (9),

$$\begin{aligned} &\big\| f\big(t, u_n(t), (Tu_n)(t), (Su_n)(t)\big) - f\big(t, \tilde{u}(t), (T\tilde{u})(t), (S\tilde{u})(t)\big) \big\| \\ &\quad \le 2\varepsilon_1 c(t)\big(1 + t^{\alpha-1}\big)\big(1 + k^* + h^*\big)\|u\|_B + 2a(t) + 2Mb(t) = \mu(t) \\ &\qquad \forall t \in J, n = 1, 2, 3, \ldots, \mu \in L\big[J, R^+\big]. \end{aligned} \tag{16}$$

It follows from (15) and (16) and the dominated convergence theorem that

$$\lim_{n\to\infty} \int_0^{+\infty} \big\| f\big(t, u_n(t), (Tu_n)(t), (Su_n)(t)\big) - f\big(t, \tilde{u}(t), (T\tilde{u})(t), (S\tilde{u})(t)\big) \big\| \, \mathrm{d}s = 0 \tag{17}$$

and

$$\begin{aligned} &\lim_{n\to\infty} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} \big\| f\big(t, u_n(t), (Tu_n)(t), (Su_n)(t)\big) - f\big(t, \tilde{u}(t), (T\tilde{u})(t), (S\tilde{u})(t)\big) \big\| \, \mathrm{d}s \\ &\quad = 0, \quad i = 1, 2, \ldots, m. \end{aligned} \tag{18}$$

It follows from (14), (17) and (18) that $\|Au_n - A\tilde{u}\|_B \to 0$ $(n \to \infty)$, and the continuity of A is proved. □

Lemma 2.3 *Let assumptions* (H_1), (H_2) *and* (H_3) *be satisfied, then* $u \in BC[J, E]$ *is a solution of BVP* (1) *if and only if* $u \in BC[J, E]$ *is a solution of the following integral equation*:

$$\begin{aligned} u(t) = {} & -\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f\big(s, u(s), (Tu)(s), (Su)(s)\big) \, \mathrm{d}s \\ & - \frac{\lambda}{\Gamma(\alpha)} \sum_{i=1}^{m} \beta_i t^{\alpha-1} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} f\big(s, u(s), (Tu)(s), (Su)(s)\big) \, \mathrm{d}s \\ & + \lambda t^{\alpha-1} \int_0^{+\infty} f\big(s, u(s), (Tu)(s), (Su)(s)\big) \, \mathrm{d}s, \end{aligned} \tag{19}$$

i.e., *u is a fixed point of the operator A defined by* (3) *in* $BC[J, E]$.

Proof If $u \in BC[J, E]$ is a solution of BVP (1), then by applying Lemma 1.4 we reduce $D_0^\alpha u(t) + f(t, u(t), (Tu)(t), (Su)(t)) = \theta$ to an equivalent integral equation

$$u(t) = -I_{0+}^\alpha f\big(t, u(t), (Tu)(t), (Su)(t)\big) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} \tag{20}$$

for some c_1, c_2, c_3. (20) can be rewritten

$$\begin{aligned} u(t) = {} & -\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f\big(s, u(s), (Tu)(s), (Su)(s)\big) \, \mathrm{d}s \\ & + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}. \end{aligned} \tag{21}$$

By $u(0) = u'(0) = \theta$, we have

$$c_2 = c_3 = 0. \tag{22}$$

By $D_0^{\alpha-1}u(+\infty) = \sum_{i=1}^{m} \beta_i u(\eta_i)$, we obtain

$$\begin{aligned} c_1 &= \lambda \int_0^{+\infty} f\bigl(s, u(s), (Tu)(s), (Su)(s)\bigr)\,\mathrm{d}s \\ &\quad - \frac{\lambda}{\Gamma(\alpha)} \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} f\bigl(s, u(s), (Tu)(s), (Su)(s)\bigr)\,\mathrm{d}s. \end{aligned} \tag{23}$$

Now, substituting (22) and (23) into (21), we see that $u(t)$ satisfies integral equation (19). Conversely, if u is a solution of (19), the direct differentiation of (19) gives

$$\begin{aligned} u'(t) &= -\frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f\bigl(s, u(s), (Tu)(s), (Su)(s)\bigr)\,\mathrm{d}s \\ &\quad - \frac{\lambda}{\Gamma(\alpha-1)} \sum_{i=1}^{m} \beta_i t^{\alpha-2} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} f\bigl(s, u(s), (Tu)(s), (Su)(s)\bigr)\,\mathrm{d}s \\ &\quad + \lambda(\alpha-1)t^{\alpha-2} \int_0^{+\infty} f\bigl(s, u(s), (Tu)(s), (Su)(s)\bigr)\,\mathrm{d}s \end{aligned} \tag{24}$$

and

$$D_{0+}^{\alpha-1}u(t) = \int_0^t f\bigl(s, u(s), (Tu)(s), (Su)(s)\bigr)\,\mathrm{d}s. \tag{25}$$

Consequently, $u \in BC[J, E]$, and by (19), (24) and (25), it is easy to see that $u(t)$ satisfies BVP (1). □

Lemma 2.4 *Integral equation* (19) *can be expressed as*

$$u(t) = \int_0^{+\infty} G(t,s) f\bigl(s, u(s), (Tu)(s), (Su)(s)\bigr)\,\mathrm{d}s, \tag{26}$$

and $G(t,s) > 0$ *for any* $t, s \in (0, \infty)$, *where*

$$G(t,s) = \begin{cases} \frac{-(t-s)^{\alpha-1}(\Gamma(\alpha) - \sum_{i=1}^{m} \beta_i \eta_i^{\alpha-1}) - \sum_{i=j}^{m} \beta_i t^{\alpha-1}(\eta_i - s)^{\alpha-1} + \Gamma(\alpha) t^{\alpha-1}}{(\Gamma(\alpha) - \sum_{i=1}^{m} \beta_i \eta_i^{\alpha-1})\Gamma(\alpha)}, \\ \quad \eta_{k-1} \le t \le \eta_k, \eta_{j-1} \le s \le \eta_j, k = 1, 2, \ldots, m, \\ \quad j = 1, 2, \ldots, k-1 \text{ or} \\ \quad \eta_{k-1} \le t \le \eta_k, s \le t, k = 1, 2, \ldots, m; \\ \frac{-\sum_{i=j}^{m} \beta_i t^{\alpha-1}(\eta_i - s)^{\alpha-1} + \Gamma(\alpha) t^{\alpha-1}}{(\Gamma(\alpha) - \sum_{i=1}^{m} \beta_i \eta_i^{\alpha-1})\Gamma(\alpha)}, \\ \quad \eta_{k-1} \le t \le \eta_k, \eta_{j-1} \le s \le \eta_j, k = 1, 2, \ldots, m, \\ \quad j = k+1, \ldots, m \text{ or} \\ \quad \eta_{k-1} \le t \le \eta_k, t \le s, k = 1, 2, \ldots, m; \\ \frac{-(t-s)^{\alpha-1}(\Gamma(\alpha) - \sum_{i=1}^{m} \beta_i \eta_i^{\alpha-1}) + \Gamma(\alpha) t^{\alpha-1}}{(\Gamma(\alpha) - \sum_{i=1}^{m} \beta_i \eta_i^{\alpha-1})\Gamma(\alpha)}, \quad \eta_m \le s \le t; \\ \frac{t^{\alpha-1}}{\Gamma(\alpha) - \sum_{i=1}^{m} \beta_i \eta_i^{\alpha-1}}, \quad t \le \eta_m \le s \text{ or } \eta_m \le t \le s. \end{cases} \tag{27}$$

Proof Let $h(t) = f(t, u(t), (Tu)(t), (Su)(t))$. For $t \leq \eta_1$, one has

$$\begin{aligned}
u(t) = {} & -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,\mathrm{d}s \\
& - \frac{\lambda}{\Gamma(\alpha)} \beta_1 t^{\alpha-1} \left(\int_0^t (\eta_1 - s)^{\alpha-1} h(s)\,\mathrm{d}s + \int_t^{\eta_1} (\eta_1 - s)^{\alpha-1} h(s)\,\mathrm{d}s \right) \\
& - \frac{\lambda}{\Gamma(\alpha)} \beta_2 t^{\alpha-1} \left(\int_0^t (\eta_2 - s)^{\alpha-1} h(s)\,\mathrm{d}s + \int_t^{\eta_1} (\eta_2 - s)^{\alpha-1} h(s)\,\mathrm{d}s \right. \\
& \left. + \int_{\eta_1}^{\eta_2} (\eta_2 - s)^{\alpha-1} h(s)\,\mathrm{d}s \right) \\
& \cdots \\
& - \frac{\lambda}{\Gamma(\alpha)} \beta_m t^{\alpha-1} \left(\int_0^t (\eta_m - s)^{\alpha-1} h(s)\,\mathrm{d}s + \int_t^{\eta_1} (\eta_m - s)^{\alpha-1} h(s)\,\mathrm{d}s \right. \\
& \left. + \cdots + \int_{\eta_{m-1}}^{\eta_m} (\eta_m - s)^{\alpha-1} h(s)\,\mathrm{d}s \right) \\
& + \lambda t^{\alpha-1} \left(\int_0^t h(s)\,\mathrm{d}s + \int_t^{\eta_1} h(s)\,\mathrm{d}s + \int_{\eta_1}^{\eta_2} h(s)\,\mathrm{d}s \right. \\
& \left. + \cdots + \int_{\eta_{m-1}}^{\eta_m} h(s)\,\mathrm{d}s + \int_{\eta_m}^{+\infty} h(s)\,\mathrm{d}s \right) \\
= {} & \int_0^{+\infty} G(t,s) h(s)\,\mathrm{d}s,
\end{aligned}$$

$0 < s \leq t$

$$\begin{aligned}
G(t,s) &= \frac{\lambda}{\Gamma(\alpha)} \left[-(t-s)^{\alpha-1} \left(\Gamma(\alpha) - \sum_{i=1}^m \beta_i \eta_i^{\alpha-1} \right) - \sum_{i=1}^m \beta_i (\eta_i - s)^{\alpha-1} t^{\alpha-1} + \Gamma(\alpha) t^{\alpha-1} \right] \\
&\geq \frac{\lambda}{\Gamma(\alpha)} \left[-t^{\alpha-1} \left(\Gamma(\alpha) - \sum_{i=1}^m \beta_i \eta_i^{\alpha-1} \right) - \sum_{i=1}^m \beta_i (\eta_i - s)^{\alpha-1} t^{\alpha-1} + \Gamma(\alpha) t^{\alpha-1} \right] \\
&= \frac{\lambda}{\Gamma(\alpha)} \sum_{i=1}^m \beta_i \bigl(\eta_i^{\alpha-1} - (\eta_i - s)^{\alpha-1} \bigr) t^{\alpha-1} > 0,
\end{aligned}$$

$0 < t \leq s \leq \eta_1$

$$\begin{aligned}
G(t,s) &= \frac{\lambda}{\Gamma(\alpha)} \left[-\sum_{i=1}^m \beta_i (\eta_i - s)^{\alpha-1} t^{\alpha-1} + \Gamma(\alpha) t^{\alpha-1} \right] \\
&\geq \frac{\lambda}{\Gamma(\alpha)} \left(\Gamma(\alpha) - \sum_{i=1}^m \beta_i \eta_i^{\alpha-1} \right) t^{\alpha-1} \geq 0,
\end{aligned}$$

$\eta_{j-1} \leq s \leq \eta_j$, $j = 2, 3, \ldots, m$

$$\begin{aligned}
G(t,s) &= \frac{\lambda}{\Gamma(\alpha)} \left[-\sum_{i=j}^m \beta_i (\eta_i - s)^{\alpha-1} t^{\alpha-1} + \Gamma(\alpha) t^{\alpha-1} \right] \\
&\geq \frac{\lambda}{\Gamma(\alpha)} \left(\Gamma(\alpha) - \sum_{i=j}^m \beta_i \eta_i^{\alpha-1} \right) t^{\alpha-1} > 0,
\end{aligned}$$

$\eta_m \le s$

$$G(t,s) = \frac{\lambda}{\Gamma(\alpha)} t^{\alpha-1} > 0.$$

By simple calculation, we can prove the rest of the lemma. □

Lemma 2.5 *Let assumptions* (H_1), (H_2) *and* (H_3) *be satisfied, and let* U *be a bounded subset of* $BC[J,E]$. *Then* $\{\frac{(Au)(t)}{1+t_2^{\alpha-1}} : u \in U\}$ *is equicontinuous on any finite subinterval of* J, *and for any given* $\varepsilon > 0$, *there exists* $\tau > 0$ *such that*

$$\left\| \frac{Au(t_1)}{1+t_1^{\alpha-1}} - \frac{Au(t_2)}{1+t_2^{\alpha-1}} \right\| < \varepsilon$$

uniformly with respect to $u \in U$, *as* $t_1, t_2 \ge \tau$.

Proof For $u \in U$, $t_1 < t_2$, by using (3), we have

$$\begin{aligned}
&\left\| \frac{Au(t_1)}{1+t_1^{\alpha-1}} - \frac{Au(t_2)}{1+t_2^{\alpha-1}} \right\| \\
&\quad \le \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left| \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} \right| \bigl\| f\bigl(s,u(s),(Tu)(s),(Su)(s)\bigr) \bigr\| \,\mathrm{d}s \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} \bigl\| f\bigl(s,u(s),(Tu)(s),(Su)(s)\bigr) \bigr\| \,\mathrm{d}s \\
&\qquad + \left| \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} \right| \Biggl(\lambda \int_0^{+\infty} \bigl\| f\bigl(t,u,(Tu)(t),(Su)(t)\bigr) \bigr\| \,\mathrm{d}s \\
&\qquad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} \bigl\| f\bigl(s,u(s),(Tu)(s),(Su)(s)\bigr) \bigr\| \,\mathrm{d}s \Biggr).
\end{aligned} \tag{28}$$

This, together with (9) and (10), implies that $\{\frac{Au(t_1)}{1+t_1^{\alpha-1}} : u \in U\}$ are equicontinuous on any finite subinterval of J.

Now, we are going to prove that for any given $\varepsilon > 0$, there exists sufficiently large $\tau > 0$, which satisfies

$$\left\| \frac{Au(t_1)}{1+t_1^{\alpha-1}} - \frac{Au(t_2)}{1+t_2^{\alpha-1}} \right\| \le \varepsilon$$

for all $u \in U$ and $t_1, t_2 \ge \tau$.

Together with (28), we need only to show that for any given $\varepsilon > 0$, there exists sufficiently large $\tau > 0$ such that

$$\begin{aligned}
&\left\| \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} f\bigl(s,u(s),(Tu)(s),(Su)(s)\bigr) \,\mathrm{d}s \right. \\
&\quad \left. - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} f\bigl(s,u(s),(Tu)(s),(Su)(s)\bigr) \,\mathrm{d}s \right\| < \varepsilon.
\end{aligned}$$

It follows from (10) that for any given $\varepsilon > 0$, there exists a sufficiently large $L > 0$ such that

$$\int_L^{+\infty} \left\| f\big(t, u, (Tu)(t), (Su)(t)\big) \right\| \, \mathrm{d}s < \frac{\varepsilon}{3} \quad \forall u \in U, \tag{29}$$

and there exists $K > 0$ such that

$$\int_0^{+\infty} \left\| f\big(t, u, (Tu)(t), (Su)(t)\big) \right\| \, \mathrm{d}s \le K \quad \forall u \in U. \tag{30}$$

On the other hand, let $g(t,s) = \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}$, $s \in [0,L]$, $t \in [L,+\infty)$, then we have

$$\lim_{t\to\infty} \sup_{s\in[0,L]} \left|g(t,s) - 1\right| \le \lim_{t\to\infty} g(t,L) = 0.$$

Thus, there exists $\tau > 0$ such that for $t_1, t_2 \ge \tau$,

$$\begin{aligned} &\sup_{s\in[0,L]} \left|g(t_1,s) - g(t_2,s)\right| \\ &\quad \le \sup_{s\in[0,L]} \left|g(t_1,s) - 1\right| + \sup_{s\in[0,L]} \left|g(t_2,s) - 1\right| \\ &\quad < \frac{\varepsilon}{3K}. \end{aligned} \tag{31}$$

Therefore, from (29), (30) and (31) we have

$$\begin{aligned} &\left\| \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} f\big(s, u(s), (Tu)(s), (Su)(s)\big) \, \mathrm{d}s \right. \\ &\qquad \left. - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} f\big(s, u(s), (Tu)(s), (Su)(s)\big) \, \mathrm{d}s \right\| \\ &\quad \le \int_0^L \left| \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} \right| \left\| f\big(s, u(s), (Tu)(s), (Su)(s)\big) \right\| \, \mathrm{d}s \\ &\qquad + \int_L^{t_1} \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} \left\| f\big(s, u(s), (Tu)(s), (Su)(s)\big) \right\| \, \mathrm{d}s \\ &\qquad + \int_L^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} \left\| f\big(s, u(s), (Tu)(s), (Su)(s)\big) \right\| \, \mathrm{d}s \\ &\quad \le \frac{\varepsilon}{3K} \int_0^L \left\| f\big(s, u(s), (Tu)(s), (Su)(s)\big) \right\| \, \mathrm{d}s + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon. \end{aligned}$$

Consequently, the proof is complete. □

Lemma 2.6 *Let assumptions* (H_1), (H_2) *and* (H_3) *be satisfied, and let* U *be a bounded subset of* $BC[J,E]$. *Then*

$$\alpha_B(AU) = \sup_{t\in J} \alpha_E\left(\frac{(Au)(t)}{1+t^{\alpha-1}}\right).$$

Proof By Lemma 2.2, we know AU is a bounded subset of $BC[J,E]$. Thus,

$$\varrho =: \sup_{t\in J} \alpha_E\left(\frac{(AU)(t)}{1+t^{\alpha-1}}\right) < \infty.$$

First, we claim that $\alpha_B(AU) \le \varrho$.

In fact, by Lemma 2.5, we know that for any given $\varepsilon > 0$, there exists a $\tau > 0$ such that

$$\left\| \frac{(Au)(t_1)}{1+t_1^{\alpha-1}} - \frac{(Au)(t_2)}{1+t_2^{\alpha-1}} \right\| < \varepsilon \tag{32}$$

uniformly with respect to $u \in U$ and $t_1, t_2 \ge \tau$.

Since $\{\frac{(Au)(t)}{1+t^{\alpha-1}} : u \in U\}$ is equicontinuous on $[0,\tau]$, by Lemma 1.3, we know

$$\alpha_B(AU|_{[0,\tau]}) = \max_{t\in[0,\tau]} \alpha_E\left(\frac{(Au)(t)}{1+t^{\alpha-1}}\right),$$

where

$$AU|_{[0,\tau]} = \{u(t) : t \in [0,\tau], u \in U\},$$

that is, $AU|_{[0,\tau]}$ is the restriction of AU on $[0,\tau]$. Therefore, there exists $U_1, U_2, \ldots, U_k \subset U$ such that

$$U = \bigcup_{i=1}^{k} U_i$$

satisfying

$$AU|_{[0,\tau]} = \bigcup_{i=1}^{k} AU_i|_{[0,\tau]}, \qquad \operatorname{diam}_B(AU_i) < \varrho + \varepsilon, \quad i = 1, 2, 3, \ldots, k, \tag{33}$$

where $\operatorname{diam}_B(\cdot)$ denote the diameters of bounded subsets of $BC[J,E]$.

At the same time, for any $Au_1, Au_2 \in AU_i$, by (32) and (33), we obtain

$$\begin{aligned} \left\| \frac{(Au_1)(t)}{1+t^{\alpha-1}} - \frac{(Au_2)(t)}{1+t^{\alpha-1}} \right\| &\le \left\| \frac{(Au_1)(t)}{1+t^{\alpha-1}} - \frac{(Au_2)(t)}{1+t^{\alpha-1}} \right\| + \left\| \frac{(Au_1)(t)}{1+t^{\alpha-1}} - \frac{(Au_2)(t)}{1+t^{\alpha-1}} \right\| \\ &\quad + \left\| \frac{(Au_1)(t)}{1+t^{\alpha-1}} - \frac{(Au_2)(t)}{1+t^{\alpha-1}} \right\| \\ &\le \varepsilon + \varrho + \varepsilon + \varepsilon = \varrho + 3\varepsilon \quad \forall t \in [\tau, +\infty). \end{aligned} \tag{34}$$

It follows from (33) and (34) that

$$\operatorname{diam}_B(AU_i) \le \varrho + 3\varepsilon, \quad i = 1, 2, 3, \ldots, k.$$

Then, by using $AU = \bigcup_{i=1}^{k} AU_i$, we have

$$\alpha_B(AU) \le \varrho.$$

On the other hand, for any given $\varepsilon > 0$, there exist $V_i \subset U$, $i = 1, 2, 3, \ldots, l$, such that

$$AU = \bigcup_{i=1}^{l} AV_i \quad \text{and} \quad \operatorname{diam}_B(AV_i) \le \alpha_B(AU) + \varepsilon.$$

Hence, for $\forall t \in J$, $\forall u_1, u_2 \in U_i$, $i = 1, 2, 3, \ldots, l$, we have

$$\left\| \frac{(Au_1)(t)}{1+t^{\alpha-1}} - \frac{(Au_2)(t)}{1+t^{\alpha-1}} \right\| \le \|Au_1 - Au_2\|_B \le \alpha_B(AU) + \varepsilon. \tag{35}$$

Since $(AU)(t) = \bigcup_{i=1}^{l}(AV_i)(t)$ together with (35), we get

$$\alpha_E\left(\frac{(Au)(t)}{1+t^{\alpha-1}}\right) \le \alpha_B(AU) + \varepsilon,$$

that is,

$$\sup_{t\in J} \alpha_E\left(\frac{(Au)(t)}{1+t^{\alpha-1}}\right) \le \alpha_B(AU) + \varepsilon.$$

Because ε is arbitrary, we obtain

$$\sup_{t\in J} \alpha_E\left(\frac{(Au_1)(t)}{1+t^{\alpha-1}}\right) \le \alpha_B(AU).$$

Consequently, the proof is complete. □

3 Main results

In this section, we give and prove our main results.

Theorem 3.1 *Let* (H_1)-(H_6) *be satisfied. Then BVP* (1) *has at least two positive solutions* $u^*, u^{**} \in BC[J, P]$ *such that* $u^*(t) \gg u_0$ *for* $t \in I$.

Proof By Lemma 2.2 and Lemma 2.4, the operator A defined by (3) is continuous from $BC[J, P]$ into $BC[J, P]$, and by Lemma 2.3, we need only to show that A has two positive fixed points $u^*, u^{**} \in BC[J, P]$ such that $u^*(t) \gg u_0$ for $t \in I$.

First, we shall prove A is compact.

Let $U = \{u_n\} \subset BC[J, E]$ be bounded and $\|u_n\| \le K$ $(n = 1, 2, 3, \ldots)$. From (9), we can choose a sufficiently large $\tau > 0$ such that for all $u \in U$

$$\int_\tau^{+\infty} \left\| f\big(s, u(s), (Tu)(s), (Su)(s)\big) \right\| \,\mathrm{d}s < \varepsilon. \tag{36}$$

It follows from Lemma 2.5 that

$$\left\{ \frac{(Au_n)(t)}{1+t^{\alpha-1}} : n = 1, 2, 3, \ldots \right\} \tag{37}$$

is equicontinuous on $[0, \tau]$. Thus, by (3), (36) and (37), we have

$$\begin{aligned} \alpha_E\left(\frac{AU(t)}{1+t^{\alpha-1}}\right) \le{}& \frac{1}{\Gamma(\alpha)} \int_0^\tau \alpha_E\Big(f\big(s, U(s), (TU)(s), (SU)(s)\big)\Big) \,\mathrm{d}s + 2\varepsilon \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} \alpha_E\Big(f\big(s, U(s), (TU)(s), (SU)(s)\big)\Big) \,\mathrm{d}s \\ &+ \int_0^\tau \alpha_E\Big(f\big(s, U(s), (TU)(s), (SU)(s)\big)\Big) \,\mathrm{d}s + 2\lambda\varepsilon, \end{aligned} \tag{38}$$

where $\frac{AU(t)}{1+t^{\alpha-1}} = \{\frac{Au_n(t)}{1+t^{\alpha-1}} : n = 1,2,3,\ldots\}$, $U(s) = \{u_n(s) : n = 1,2,3,\ldots\}$, $(TU)(s) = \{(Tu_n)(s) : n = 1,2,3,\ldots\}$, $(SU)(s) = \{(Su_n)(s) : n = 1,2,3,\ldots\}$.

Since $U(s), (TU)(s), (SU)(s) \subset P_{r^*}$ for $s \in J$, where $r^* = \max\{r, k^*r, h^*r\}$, we see that, by virtue of assumption (H_2),

$$\alpha_E\big(f\big(s, U(s), (TU)(s), (SU)(s)\big)\big) = 0 \quad \forall t \in J. \tag{39}$$

It follows from (38) and (39) that

$$\alpha_E\left(\frac{AU(t)}{1+t^{\alpha-1}}\right) \le 2(1+\lambda)\varepsilon,$$

which implies, by virtue of the arbitrariness of ε, that

$$\alpha_E\left(\frac{AU(t)}{1+t^{\alpha-1}}\right) = 0 \quad \forall t \in J.$$

Using Lemma 2.6, we have

$$\alpha_B(AU) = \sup_{t\in J}\left(\frac{AU(t)}{1+t^{\alpha-1}}\right) = 0.$$

Thus, we can conclude that AU is relatively compact in $BC[J,E]$, *i.e.*, A is compact.

As in the proof of Lemma 2.2, (12) holds. Choose

$$\begin{aligned} R^* > \bigg\{ & 2\|u_0\|, 2\left(\frac{1}{\Gamma(\alpha)} + \lambda\right)\big(a^* + Mb^*\big) \\ & + \frac{2\lambda}{\Gamma(\alpha)} \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1}\big(a(s) + Mb(s)\big)\,\mathrm{d}s \bigg\}, \end{aligned} \tag{40}$$

where $u_0 \gg \theta$ is given in assumption (H_6), and let $\Omega_1 = \{u \in BC[J,P] : \|u\| < R^*\}$. Then $\overline{\Omega}_1 = \{u \in BC[J,P] : \|u\| \le R^*\}$ and, by (12) and (40), we have

$$A(\overline{\Omega}_1) \subset \Omega_1. \tag{41}$$

By virtue of (H_4), there exists an $r_1 > 0$ such that

$$\begin{aligned} & \big\|f\big(t, \big(1+t^{\alpha-1}\big)u, \big(1+t^{\alpha-1}\big)v, \big(1+t^{\alpha-1}\big)w\big)\big\| \le \varepsilon_2 d(t)\big(\|u\| + \|v\| + \|w\|\big) \\ & \quad \forall t \in J, u, v, w \in P, \|u\| + \|v\| + \|w\| \le r_1, \end{aligned} \tag{42}$$

where

$$\varepsilon_2 = \frac{1}{2(1+k^*+h^*)}\left[\left(\frac{1}{\Gamma(\alpha)} + \lambda\right)d^* + \frac{\lambda}{\Gamma(\alpha)} \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} d(s)\,\mathrm{d}s\right]^{-1}. \tag{43}$$

Let

$$r_2 = \frac{r_1}{1+k^*+h^*}.$$

Then, for $u \in BC[J,P]$ with $\|u\|_B \le r_2$, we have by (42)

$$\begin{aligned}
&\big\|f\big(t,u(t),(Tu)(t),(Su)(t)\big)\big\| \\
&\quad= \left\|f\left(t,\big(1+t^{\alpha-1}\big)\frac{u(t)}{1+t^{\alpha-1}},\big(1+t^{\alpha-1}\big)\frac{(Tu)(t)}{1+t^{\alpha-1}},\big(1+t^{\alpha-1}\big)\frac{(Su)(t)}{1+t^{\alpha-1}}\right)\right\| \\
&\quad\le \varepsilon_2 d(t)\left(\frac{\|u(t)\|}{1+t^{\alpha-1}}+\frac{\|(Tu)(t)\|}{1+t^{\alpha-1}}+\frac{\|(Su)(t)\|}{1+t^{\alpha-1}}\right) \\
&\quad\le \varepsilon_2 d(t)\big(1+k^*+h^*\big)\|u\|_B \quad \forall t\in J.
\end{aligned} \tag{44}$$

It follows from (3), (43) and (44) that

$$\begin{aligned}
\frac{\|(Au)(t)\|}{1+t^{\alpha-1}} &\le \frac{1}{\Gamma(\alpha)}\int_0^t \big\|f\big(s,u(s),(Tu)(s),(Su)(s)\big)\big\|\,\mathrm{d}s \\
&\quad+\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^m \beta_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-1}\big\|f\big(s,u(s),(Tu)(s),(Su)(s)\big)\big\|\,\mathrm{d}s \\
&\quad+\lambda\int_0^{+\infty}\big\|f\big(s,u(s),(Tu)(s),(Su)(s)\big)\big\|\,\mathrm{d}s \\
&\le \left(\frac{1}{\Gamma(\alpha)}+\lambda\right)\varepsilon_2 d^*\big(1+k^*+h^*\big)\|u\|_B \\
&\quad+\frac{\varepsilon_2(1+k^*+h^*)}{\Gamma(\alpha)}\sum_{i=1}^m \beta_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-1}d(s)\,\mathrm{d}s \\
&= \frac{1}{2}\|u\|_B,
\end{aligned}$$

which implies

$$\|Au\|_B \le \frac{1}{2}\|u\|_B, \quad u\in BC[J,P], \|u\|_B\le r_2. \tag{45}$$

Choose

$$0<r<\min\left\{\frac{\|u_0\|}{N(1+t_*^{\alpha-1})}, r_2, R\right\}. \tag{46}$$

Let $\Omega_2=\{u\in BC[J,P]:\|u\|_B<r\}$. Then $\overline{\Omega}_2=\{u\in BC[J,P]:\|u\|_B\le r\}$, and we have, by (45) and (46),

$$A(\overline{\Omega}_2)\subset\Omega_2. \tag{47}$$

Let $\Omega_3=\{u\in BC[J,P]:\|u\|_B<R, u(t)\gg u_0, t\in I\}$, and we are going to show that Ω_3 is an open set of $BC[J,P]$. It is clear that we need only to show the following: for any $\bar{u}\in\Omega_3$, there exists $\eta>0$ such that $u\in BC[J,P]$, $\|u-\bar{u}\|_B<\eta$ implies that $u(t)\gg u_0$ for $t\in I$. We have $\bar{u}(t)\gg u_0$ for $t\in I$. So, for any $s\in I$, there exists a $\varepsilon=\varepsilon(s)>0$ such that

$$\bar{u}(s)\ge(1+3\varepsilon)u_0. \tag{48}$$

Since $u_0 \gg \theta$ and $\bar{u}(t)$ is continuous on J, we can find an open interval $I(s,\delta) = (s-\delta, s+\delta)$ $(\delta > 0)$ such that

$$\varepsilon u_0 + \left[\bar{u}(t) - \bar{u}(s)\right] \geq \theta \quad \forall t \in I(s,\delta),$$

which implies by virtue of (48) that

$$\bar{u}(t) \geq (1+2\varepsilon)u_0 \quad \forall t \in I(s,\delta).$$

Since I is compact, there is a finite collection of such intervals $\{I(s_j,\delta_j)\}$ $(j = 1,2,\dots,k)$ which cover I, and

$$\bar{u}(t) \geq (1+2\varepsilon_j)u_0 \quad \forall t \in I(s_j,\delta_j) \ (j = 1,2,\dots,k),$$

where $\varepsilon_j > 0$ $(j = 1,2,\dots,k)$. Consequently,

$$\bar{u}(t) \geq \left(1+2\varepsilon^*\right)u_0 \quad \forall t \in I, \tag{49}$$

where $\varepsilon^* = \min_{1\leq j\leq k}\{\varepsilon_j\} > 0$. Since $u_0 \gg \theta$, there exists an $\eta = \frac{\|u_0\|}{2N(1+t_*^\alpha)} > 0$ such that

$$\varepsilon^* u_0 + \left[u(t) - \bar{u}(t)\right] \geq \theta \quad \forall t \in I, \tag{50}$$

whenever $u \in BC[J,P]$ satisfying $\|u-\bar{u}\|_B < \eta$, which implies by virtue of (49) and (50) that

$$u(t) \geq \left(1+\varepsilon^*\right)u_0 \gg u_0, \quad u \in BC[J,P], \|u-\bar{u}\|_B < \eta.$$

Thus, we have proved that Ω_3 is open in $BC[J,P]$.

On the other hand, Lemma 2.4 and assumption (H_6) imply

$$\begin{aligned}(Au)(t) &\geq \int_{t_*}^{t^*} G(t,s)f\left(s,u(s),(Tu)(s),(Su)(s)\right)\,\mathrm{d}s \\ &\geq \int_{t_*}^{t^*} G(t,s)\sigma(s)\,\mathrm{d}s\,u_0 \\ &\geq \int_{t_*}^{t^*} \gamma(s)\sigma(s)\,\mathrm{d}s\,u_0 \\ &\gg u_0 \quad \forall t \in I. \end{aligned} \tag{51}$$

Hence

$$A(\overline{\Omega}_3) \subset \Omega_3. \tag{52}$$

Since Ω_1, Ω_2 and Ω_3 are nonempty bounded convex open subsets of $BC[J,P]$, we see that (41), (47) and (52) imply by virtue of Lemma 1.1 the fixed point indices

$$i\left(A,\Omega_i,BC[J,P]\right) = 1 \quad (i = 1,2,3). \tag{53}$$

On the other hand, for $u \in \Omega_3$, we have $u(t) \gg u_0$, and so

$$\|u\|_B \geq \frac{\|u(t_*)\|}{1 + t_*^{\alpha-1}} \geq \frac{\|u_0\|}{N(1 + t_*^{\alpha-1})}.$$

Consequently,

$$\Omega_2 \subset \Omega_1 \subset BC[J,P], \qquad \Omega_3 \subset \Omega_1 \subset BC[J,P], \qquad \Omega_2 \cap \Omega_3 = \emptyset. \tag{54}$$

By (53), (54) and the additivity of the fixed point index (Lemma 1.2), we can obtain

$$\begin{aligned} &i\big(A, \Omega_1/(\overline{\Omega_2 \cup \Omega_3}), BC[J,P]\big) \\ &\quad = i\big(A, \Omega_1, BC[J,P]\big) - i\big(A, \Omega_2, BC[J,P]\big) - i\big(A, \Omega_3, BC[J,P]\big) = -1. \end{aligned} \tag{55}$$

Finally, (53), (54) and (55) imply that A has two fixed points $u^* \in \Omega_3$ and $u^{**} \in \Omega_1/(\overline{\Omega_2 \cup \Omega_3})$. We have, by (51), $u^*(t) \gg u_0$ for $t \in I$. The proof is complete. □

Remark 3.1 Assumption (H_7) and the continuity of f imply that $f(t,\theta,\theta,\theta) = \theta$ for $t \in J$. Hence, under the assumptions of the theorem, BVP (1) has the trivial solution $u(t) \equiv \theta$ besides two positive solutions u^* and u^{**}.

Theorem 3.2 *Let* (H_1)-(H_5) *and* (H_7) *be satisfied. Then BVP* (1) *has at least one positive solution* $\tilde{u}(t) \in BC[J,P]$ *such that* $\tilde{u}(t) \geq u_0$ *for* $t \in I$.

Proof By Lemma 2.2, Lemma 2.4 and the proof of Theorem 3.1, the operator A defined by (3) is completely continuous from $BC[J,P]$ into $BC[J,P]$, and by Lemma 2.3, we need only to show that A has one positive fixed point $\tilde{u} \in BC[J,P]$ such that $\tilde{u}(t) \gg u_0$ for $t \in I$.

As in the proof of Lemma 2.2, (12) holds. Choose R satisfying (40) and let $U = \{u \in BC[J,P] : \|u\| \leq R, u(t) \geq u_0 \ \forall t \in I\}$, where $u_0 > \theta$ is given by assumption (H_7). It is clear that U is a nonempty bounded closed convex subset in $BC[J,P]$ ($U \neq \emptyset$ because $2u_0 \in U$). Let $u \in U$, by (40), we have $\|Au\| \leq R^*$. On the other hand, as in the proof of Theorem 3.1, Lemma 2.4 and assumption (H_7) imply

$$\begin{aligned} (Au)(t) &\geq \int_{t_*}^{t^*} G(t,s) f\big(s, u(s), (Tu)(s), (Su)(s)\big)\, \mathrm{d}s \\ &\geq \int_{t_*}^{t^*} G(t,s)\sigma(s)\, \mathrm{d}s\, u_0 \\ &\geq \int_{t_*}^{t^*} \gamma(s)\sigma(s)\, \mathrm{d}s\, u_0 \\ &\geq u_0 \quad \forall t \in I. \end{aligned} \tag{56}$$

Hence, $Au \in W$, and therefore $AU \subset U$. Thus, the Schauder fixed point theorem implies that A has a fixed point $\tilde{u} \in U$, and by (56) $\tilde{u}(t) \geq u_0$ for $t \in I$. The proof is complete. □

4 Conclusion

In this paper, the issue on the existence of multiple positive solutions of a boundary value problem for α-order nonlinear integro-differential equations in a Banach space has been addressed for the first time. Taking advantage of the fixed point index theory of completely continuous operators, the existence conditions for such boundary value problems have been established.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

RL completed the proof and wrote the initial draft. CK provided the problem and gave some suggestions for amendment. RL then finalized the manuscript. All authors read and approved the final manuscript.

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