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Anti-periodic boundary value problems of fractional differential equations with the Riemann-Liouville fractional derivative

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Abstract

In this paper, the author puts forward a kind of anti-periodic boundary value problems of fractional equations with the Riemann-Liouville fractional derivative. More precisely, the author is concerned with the following fractional equation:

$$D_{0+}^{\alpha}u(t) = f\big(t, u(t), u'(t)\big), \quad t \in (0,1)$$

with the anti-periodic boundary value conditions

$$t^{2-\alpha}u(t)_{|t\to 0^+} = -t^{2-\alpha}u(t)_{|t=1}, \qquad \big(t^{2-\alpha}u(t)\big)'_{|t\to 0^+} = -\big(t^{2-\alpha}u(t)\big)'_{|t=1},$$

where D_{0+}^{α} denotes the standard Riemann-Liouville fractional derivative of order $\alpha \in (1,2)$, and the nonlinear function $f(t,\cdot,\cdot)$ may be singular at $t=0$. By applying the contraction mapping principle and the other fixed point theorem, the author obtains the existence and uniqueness of solutions.

MSC: 34A08; 34B15

Keywords: fractional differential equations; anti-periodic boundary value problems; existence results; fixed point theorem

1 Introduction

In the present paper, we are concerned with the existence of solutions for the fractional differential equation

$$D_{0+}^{\alpha}u(t) = f\big(t, u(t), u'(t)\big), \quad t \in (0,1) \tag{1.1}$$

with anti-periodic boundary value conditions

$$t^{2-\alpha}u(t)_{|t\to 0^+} = -t^{2-\alpha}u(t)_{|t=1}, \quad \big(t^{2-\alpha}u(t)\big)'_{|t\to 0^+} = -\big(t^{2-\alpha}u(t)\big)'_{|t=1}, \tag{1.2}$$

where D_{0+}^{α} denotes the standard Riemann-Liouville fractional derivative of order $\alpha \in (1,2)$, and the nonlinear function $f(t,\cdot,\cdot)$ may be singular at $t=0$.

Differential equations with fractional order are a generalization of ordinary differential equations to non-integer order. This generalization is not a mere mathematical curiosity but rather has interesting applications in many areas of science and engineering such

as electrochemistry, control, porous media, electromagnetism, *etc.* (see [1–4]). There has been a significant development in the study of fractional differential equations in recent years; see, for example, [5–21].

Anti-periodic boundary value problems occur in mathematic modeling of a variety of physical processes and have recently received considerable attention. For examples and details of anti-periodic fractional boundary value problems, see [22–31] and the references therein. However, up to now, almost all studies on anti-periodic fractional boundary value problems have been devoted to fractional equations with the Caputo fractional derivative. Recently, there appeared a paper dealing with anti-periodic boundary value problems of a fractional equation with the Riemann-Liouville fractional derivative (see [22]), which will be formulated later. The main reason is that in general the Riemann-Liouville fractional derivative $u^{(i)}(0)$ $(0 \le i < \alpha)$ does not exist unless $u^{(i)}(0) = 0$, and therefore there is some difficulty in expressing anti-periodic boundary value conditions with the Riemann-Liouville fractional derivative. So, no matter which kind of anti-periodic boundary value conditions we shall propose, first of all, we must ensure that the limits exist taken on the left-hand side of the formula on the anti-periodic boundary value conditions when variable tends to zero. In the paper [22] mentioned above, Ahmad and Nieto put forward a kind of boundary value problems as follows:

$$\begin{cases} D^{\alpha}u(t) = f(t,u(t)), \quad t \in [0,T], 0 < \alpha \le 2, \\ I_{0+}^{2-\alpha}u(0^+) = b_0 I_{0+}^{2-\alpha}u(T^-), \qquad D_{0+}^{\alpha-1}u(0^+) = b_1 D_{0+}^{\alpha-1}u(T^-), \quad b_0 \ne 1, b_1 \ne 1. \end{cases}$$

It is well known that the limits $I_{0+}^{2-\alpha}u(0^+)$ and $D_{0+}^{\alpha-1}u(0^+)$ exist. Moreover, it can be verified that $t^{2-\alpha}u(t)_{|t\to 0^+} = \frac{1}{\Gamma(\alpha-1)} I_{0+}^{2-\alpha}u(0^+)$, $(t^{2-\alpha}u(t))'_{|t\to 0^+} = \frac{1}{\Gamma(\alpha)} D_{0+}^{\alpha-1}u(0^+)$. Especially, comparing with the recent article (see [32]) dealing with the following periodic boundary value problems with Riemann-Liouville fractional derivative

$$\begin{cases} D^{2\alpha}u(t) = f(t,u,D^{\alpha}u), \quad t \in (0,1]\backslash\{t_1,t_2,\dots,t_m\}, 0 < \alpha \le 1, \\ t^{1-\alpha}u(t)|_{t\to 0+} = u(1), \qquad t^{1-\alpha}D^{\alpha}u(t)|_{t\to 0} = D^{\alpha}u(1), \\ (t-t_j)^{1-\alpha}(u(t)-u(t_j))|_{t\to t_j^+} = I_j(u(t_j)), \\ (t-t_j)^{1-\alpha}(D^{\alpha}u(t)-D^{\alpha}u(t_j))|_{t\to t_j^+} = \bar{I}_j(u(t_j)), \end{cases}$$

and taking into account the consistency with integer order anti-periodic boundary value problems, we consider the anti-periodic boundary value condition (1.2) in the present paper to be more natural and suitable. It is noteworthy that such a form of anti-periodic boundary value conditions (1.2) is very convenient to construct an appropriate Banach space which coordinates the feature of the solution u because of the fact that $\lim u(t) = \infty$ $(t \to 0^+)$ may occur and function $f(t,\cdot,\cdot)$ may be singular at $t = 0$. Moreover, when $\alpha \to 2$ in (1.2), the anti-periodic boundary value conditions in (1.2) are changed into $u(0) = -u(1)$, $u'(0) = -u'(1)$, which are coincident with anti-periodic boundary value conditions of second-order differential equations (see [33]).

The rest of this paper is organized as follows. In Section 2, we present some necessary definitions and preliminary results that will be used to prove our main results. In Section 3, we put forward and prove our main results. By applying the contraction mapping principle and the other fixed point theorem, we obtain the existence and uniqueness of solutions for boundary value problems (simply denoted by BVP). Finally, we give two examples to demonstrate our main results.

2 Preliminaries

In this section, we introduce some preliminary facts which are used throughout this paper.

Let $\mathbb{N}$ be the set of positive integers, $\mathbb{R}$ be the set of real numbers.

Definition 2.1 ([3]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (a, b] \to \mathbb{R}$ is given by

$$I_{a+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds, \quad t \in (a, b].$$

Definition 2.2 ([3]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : (a, b] \to \mathbb{R}$ is given by

$$D_{a+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}\, ds, \quad t \in (a, b],$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of α.

Lemma 2.1 ([34]) *Let $\alpha > 0$. If $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order α that belongs to $C(0,1) \cap L(0,1)$, then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, where $n = [\alpha] + 1$.

Let

$$X_1 = \left\{ u : u \in C'(0,1],\ \lim_{t \to 0^+} t^{2-\alpha} u(t) \text{ and } \lim_{t \to 0^+} \left(t^{2-\alpha} u(t)\right)' \text{ exist} \right\}, \quad \text{and}$$

$$X_0 = \left\{ u : u \in C(0,1] \cap L(0,1),\ \sup_{t \in (0,1]} \left| t^{2-\alpha} u(t) \right| < \infty \right\}.$$

It is easy to verify that X_1 is a normed linear space with the norm

$$\|u\|_1 = \max\left\{ \sup_{t \in (0,1]} \left| t^{2-\alpha} u(t) \right|,\ \sup_{t \in (0,1]} \left| \left(t^{2-\alpha} u(t)\right)' \right| \right\}, \quad u \in X_1, \tag{2.1}$$

and X_0 is a normed linear space with the norm

$$\|u\|_0 = \sup_{t \in (0,1]} \left| t^{2-\alpha} u(t) \right|, \quad u \in X_0,$$

respectively. Moreover, we have the following lemma.

Lemma 2.2 *X_1 is a Banach space with the norm $\|u\|_1$.*

Proof In fact, let $\{u_n\}$ be any sequence in X_1 with $\|u_n - u_m\|_1 \to 0$. Then, for arbitrary $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|u_n - u_m\|_1 < \varepsilon,$$

when $n, m \geq N$. So,

$$\left|t^{2-\alpha}u_n(t) - t^{2-\alpha}u_m(t)\right| < \varepsilon, \quad t \in (0,1], \tag{2.2}$$

$$\left|\left(t^{2-\alpha}u_n(t)\right)' - \left(t^{2-\alpha}u_m(t)\right)'\right| < \varepsilon, \quad t \in (0,1], \tag{2.3}$$

when $n, m \geq N$.

In view of (2.2), there exists $v \in C(0,1]$ such that $t^{2-\alpha}u_n \to v$ uniformly on $(0,1]$, and so

$$u_n(t) \to v(t)t^{\alpha-2}, \quad t \in (0,1].$$

Set $u = v(t)t^{\alpha-2}$. Then $u_n(t) \to u(t)$, $t \in (0,1]$ and $u \in C(0,1]$. So, letting $m \to +\infty$ in (2.2), we have

$$\left|t^{2-\alpha}u_n(t) - t^{2-\alpha}u(t)\right| \leq \varepsilon, \quad t \in (0,1],$$

when $n \geq N$. That is, $t^{2-\alpha}u_n \to t^{2-\alpha}u$ uniformly on $(0,1]$. Because $\lim_{t\to 0^+} t^{2-\alpha}u_n(t)$ exists for all $n \in \mathbb{N}$, by applying the theorem to the limit convergence of function sequences, we know that $\lim_{n\to\infty}\lim_{t\to 0^+} t^{2-\alpha}u_n(t)$ exists, and

$$\lim_{t\to 0^+} t^{2-\alpha}u(t) = \lim_{t\to 0^+}\lim_{n\to\infty} t^{2-\alpha}u_n(t) = \lim_{n\to\infty}\lim_{t\to 0^+} t^{2-\alpha}u_n(t).$$

So, $\lim_{t\to 0^+} t^{2-\alpha}u(t)$ exists.

Similarly, applying the theorem to the differentiability of function sequences and letting $m \to \infty$ in (2.3), we obtain

$$\left|\left(t^{2-\alpha}u_n(t)\right)' - \left(t^{2-\alpha}u(t)\right)'\right| < \varepsilon, \quad t \in (0,1], \tag{2.4}$$

whenever $n \geq N$.

Let $f_n(t) = (t^{2-\alpha}u_n(t))'$ and $f(t) = (t^{2-\alpha}u(t))'$. Then (2.4) means that $f_n \to f$ uniformly on (0.1], and so

$$\lim_{t\to 0^+} f(t) = \lim_{n\to\infty}\lim_{t\to 0^+} f_n(t) \quad \text{exists}$$

by applying the theorem to the limit convergence of function sequences again. That is,

$$\lim_{t\to 0^+}\left(t^{2-\alpha}u(t)\right)' = \lim_{n\to\infty}\lim_{t\to 0^+}\left(t^{2-\alpha}u_n(t)\right)' \quad \text{exists.}$$

Thus, $\lim_{t\to 0^+}(t^{2-\alpha}u(t))'$ exists.

To summarize, $\|u_n - u\|_1 \to 0$ with $u \in X_1$. □

Now, we consider the following auxiliary boundary value problem:

$$\begin{cases} D^{\alpha}_{0+}u(t) = h(t), & t \in (0,1], \\ t^{2-\alpha}u(t)_{|t\to 0^+} = -t^{2-\alpha}u(t)_{|t=1}, \\ (t^{2-\alpha}u(t))'_{|t\to 0^+} = -(t^{2-\alpha}u(t))'_{|t=1}, \end{cases} \tag{2.5}$$

where $h \in X_0$.

We have the following lemma.

Lemma 2.3 *For a given $h \in X_0$, if $u \in X_1$ is a solution of BVP* (2.5), *then u is given by*

$$u(t) = -t^{\alpha-1}\int_0^1 G_1(s)h(s)\,ds + t^{\alpha-2}\int_0^1 G_2(s)h(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds, \quad t \in (0,1], \tag{2.6}$$

where

$$G_1(s) = \frac{1}{2\Gamma(\alpha)}\big[(2-\alpha)(1-s)^{\alpha-1} + (\alpha-1)(1-s)^{\alpha-2}\big], \quad s \in (0,1), \tag{2.7}$$

$$G_2(s) = \frac{1}{4\Gamma(\alpha)}\big[(\alpha-1)(1-s)^{\alpha-2} - \alpha(1-s)^{\alpha-1}\big], \quad s \in (0,1). \tag{2.8}$$

Proof In fact, owing to the fact that $h \in X_0$, in view of Lemma 2.1, we have

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + I_{0+}^{\alpha}h(t), \quad t \in (0,1] \tag{2.9}$$

for some $c_1, c_2 \in \mathbb{R}$.

So,

$$t^{2-\alpha}u(t) = c_1 t + c_2 + t^{2-\alpha}I_{0+}^{\alpha}h(t), \quad t \in (0,1], \tag{2.10}$$

and

$$\big(t^{2-\alpha}u(t)\big)' = c_1 + (2-\alpha)t^{1-\alpha}I_{0+}^{\alpha}h(t) + t^{2-\alpha}I_{0+}^{\alpha-1}h(t), \quad t \in (0,1]. \tag{2.11}$$

Because $|s^{2-\alpha}h(s)| \le \|h\|_0$, $s \in (0,1]$, we have

$$\begin{aligned}
\big|I_{0+}^{\alpha}h(t)\big| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big|h(s)\big|\,ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-2}\big|s^{2-\alpha}h(s)\big|\,ds \\
&\le \frac{\|h\|_0}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-2}\,ds \\
&\overset{s=t\tau}{=} \frac{\|h\|_0}{\Gamma(\alpha)}t^{2(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-1}\tau^{\alpha-2}\,d\tau \\
&= \frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}\|h\|_0 t^{2(\alpha-1)}, \quad t \in (0,1].
\end{aligned} \tag{2.12}$$

Thus,

$$\begin{aligned}
&\big|t^{2-\alpha}I_{0+}^{\alpha}h(t)\big| \le \frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}\|h\|_0 t^{\alpha}, \\
&\big|t^{1-\alpha}I_{0+}^{\alpha}h(t)\big| \le \frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}\|h\|_0 t^{\alpha-1}, \quad t \in (0,1].
\end{aligned} \tag{2.13}$$

Similarly, we have

$$\left|t^{2-\alpha}I_{0+}^{\alpha-1}h(t)\right| \leq \frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-2)}\|h\|_0 t^{\alpha-1}, \quad t \in (0,1]. \tag{2.14}$$

From (2.13)-(2.14), it follows that

$$\lim_{t\to 0+} t^{2-\alpha}I_{0+}^{\alpha}h(t) = 0, \tag{2.15}$$

$$\lim_{t\to 0+} t^{1-\alpha}I_{0+}^{\alpha}h(t) = 0, \tag{2.16}$$

and

$$\lim_{t\to 0+} t^{2-\alpha}I_{0+}^{\alpha-1}h(t) = 0. \tag{2.17}$$

So, by the boundary value conditions in (2.5) together with (2.10)-(2.11), from (2.15)-(2.17) it follows that

$$c_1 + 2c_2 + I_{0+}^{\alpha}h(1) = 0, \tag{2.18}$$

$$2c_1 + (2-\alpha)I_{0+}^{\alpha}h(1) + I_{0+}^{\alpha-1}h(1) = 0. \tag{2.19}$$

Thus, (2.18)-(2.19) imply that

$$c_1 = -\frac{2-\alpha}{2}I_{0+}^{\alpha}h(1) - \frac{1}{2}I_{0+}^{\alpha-1}h(1), \tag{2.20}$$

$$c_2 = -\frac{\alpha}{4}I_{0+}^{\alpha}h(1) + \frac{1}{4}I_{0+}^{\alpha-1}h(1). \tag{2.21}$$

Substituting (2.20)-(2.21) into (2.9), we have

$$\begin{aligned} u(t) &= -\frac{1}{2}\left[(2-\alpha)I_{0+}^{\alpha}h(1) + I_{0+}^{\alpha-1}h(1)\right]t^{\alpha-1} + \frac{1}{4}\left[I_{0+}^{\alpha-1}h(1) - \alpha I_{0+}^{\alpha}h(1)\right]t^{\alpha-2} + I_{0+}^{\alpha}h(t) \\ &= -t^{\alpha-1}\int_0^1 G_1(s)h(s)\,ds + t^{\alpha-2}\int_0^1 G_2(s)h(s)\,ds + I_{0+}^{\alpha}h(t), \quad t \in (0,1], \end{aligned}$$

where G_1, G_2 are given by (2.7) and (2.8), respectively.

The proof is complete. □

We definite an operator T on X_0 as follows:

$$Th(t) = -t^{\alpha-1}\int_0^1 G_1(s)h(s)\,ds + t^{\alpha-2}\int_0^1 G_2(s)h(s)\,ds + I_{0+}^{\alpha}h(t), \quad t \in (0,1] \tag{2.22}$$

for $h \in X_0$.

We first establish the following lemma.

Lemma 2.4 $TX_0 \subset X_1$.

Proof For any $h \in X_0$, from (2.12), we have that

$$\int_0^1 (1-s)^{\alpha-1}\big|h(s)\big|\,ds \le \frac{\Gamma(\alpha)\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}\|h\|_0.$$

Again,

$$\begin{aligned}\int_0^1 (1-s)^{\alpha-2}\big|h(s)\big|\,ds &= \int_0^1 (1-s)^{\alpha-2}s^{\alpha-2}\big|s^{2-\alpha}h(s)\big|\,ds\\ &\le \|h\|_0\int_0^1 (1-s)^{\alpha-2}s^{\alpha-2}\,ds\\ &= \|h\|_0 B(\alpha-1,\alpha-1).\end{aligned}$$

Thus, from (2.7)-(2.8), we know that the integrals $\int_0^1 G_1(s)h(s)\,ds$ and $\int_0^1 G_2(s)h(s)\,ds$ converge. So, from (2.12) and (2.22), it follows that $Th(t)$ exists on $(0,1]$. That is, the operator T is well defined.

In what follows, we show that $Th \in X_1$.

From $h \in X_0$, we know that $h \in C(0,1]$ and $|s^{2-\alpha}h(s)| \le \|h\|_0$. Let $\phi(s) = s^{2-\alpha}h(s)$, $s \in (0,1]$. Then $\phi \in C(0,1]$ and $|\phi(s)| \le \|h\|_0$, $s \in (0,1]$, and so $\psi \in C((0,1]\times(0,1])$ and $|\psi(t,s)| \le \|h\|_0$ by setting $\psi(t,\tau) = \phi(t\tau)$, $(t,\tau) \in (0,1]\times(0,1]$.

Since

$$\begin{aligned}\int_0^t (t-s)^{\alpha-2}h(s)\,ds &= \int_0^t (t-s)^{\alpha-2}s^{2-\alpha}\phi(s)\,ds\\ &\overset{s=t\tau}{=} t^{2\alpha-3}\int_0^1 (1-\tau)^{\alpha-2}\tau^{\alpha-2}\psi(t,\tau)\,d\tau,\end{aligned} \tag{2.23}$$

and $|\psi(t,\tau)| \le \|h\|_0$, $\psi \in C((0,1]\times(0,1])$, we know that $\int_0^t (t-s)^{\alpha-2}h(s)\,ds$ is continuous on $(0,1]$ from (2.23) and applying the Lebesgue convergence theorem. Consequently, $(Th)' \in C(0,1]$ according to

$$\begin{aligned}(Th)'(t) = {}&-(\alpha-1)t^{\alpha-2}\int_0^1 G_1(s)h(s)\,ds + (\alpha-2)t^{\alpha-3}\int_0^1 G_2(s)h(s)\,ds\\ &+\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}h(s)\,ds,\quad t\in(0,1]\end{aligned} \tag{2.24}$$

by (2.22).

Finally, in view of (2.22), we have

$$t^{2-\alpha}(Th)(t) = -t\int_0^1 G_1(s)h(s)\,ds + \int_0^1 G_2(s)h(s)\,ds + t^{2-\alpha}I_{0+}^{\alpha}h(t),\quad t\in(0,1], \tag{2.25}$$

and

$$\begin{aligned}\big(t^{2-\alpha}(Th)(t)\big)' = {}&-\int_0^1 G_1(s)h(s)\,ds + (2-\alpha)t^{1-\alpha}I_{0+}^{\alpha}h(t)\\ &+\frac{1}{\Gamma(\alpha-1)}t^{2-\alpha}\int_0^t (t-s)^{\alpha-2}h(s)\,ds,\quad t\in(0,1].\end{aligned} \tag{2.26}$$

Again,

$$\begin{aligned}\frac{t^{2-\alpha}}{\Gamma(\alpha-1)}\left|\int_0^t (t-s)^{\alpha-2}h(s)\,ds\right| &\le \frac{\|h\|_0 t^{2-\alpha}}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}s^{\alpha-2}\,ds \\ &\overset{s=t\tau}{=} \frac{\|h\|_0 t^{\alpha-1}}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\tau^{\alpha-2}\,d\tau \\ &= \frac{\Gamma(\alpha-1)\|h\|_0}{\Gamma(2\alpha-2)}t^{\alpha-1}, \quad t\in(0,1]. \end{aligned} \tag{2.27}$$

Hence, formula (2.25) together with (2.13) implies that $\lim_{t\to 0^+} t^{2-\alpha}(Th)(t) = \int_0^1 G_2(s)\,ds$ exists, and formula (2.26) together with (2.13)-(2.27) implies that $\lim_{t\to 0^+}(t^{2-\alpha}(Th)(t))' = -\int_0^1 G_1(s)\,ds$ exists.

Summing up the above analysis, we obtain that $Th \in X_1$. The proof is complete. □

We need the following lemma, which is important in establishing our main result in the next section.

Lemma 2.5 *$T : X_0 \to X_1$ is completely continuous.*

Proof We divide the proof into two parts.

Part 1. First, we show that T is a continuous operator.

Let $\{u_n\}$ be an arbitrary sequence in X_0 with $u_n \to u \in X_0$. Then

$$\left|s^{2-\alpha}\big(u_n(s)-u(s)\big)\right| \le \|u_n-u\|_0, \quad s\in(0,1].$$

Thus, from (2.25), it follows that

$$\begin{aligned} t^{2-\alpha}\left|Tu_n(t)-Tu(t)\right| &\le t\int_0^1 G_1(s)\left|u_n(s)-u(s)\right|ds + \int_0^1 \left|G_2(s)\right|\cdot\left|u_n(s)-u(s)\right|ds \\ &\quad + \frac{t^{2-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|u_n(s)-u(s)\right|ds \\ &\le \left[t\int_0^1 G_1(s)s^{\alpha-2}\,ds + \int_0^1 \left|G_2(s)\right|s^{\alpha-2}\,ds \right. \\ &\quad \left. + \frac{t^{2-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-2}\,ds\right]\cdot\|u_n-u\|_0 \\ &\le \frac{(5\alpha+2)\Gamma(\alpha-1)}{4\Gamma(2\alpha-1)}\|u_n-u\|_0, \quad t\in(0,1], \end{aligned}$$

noting that

$$\int_0^1 G_1(s)s^{\alpha-2}\,ds = \frac{\alpha\Gamma(\alpha-1)}{2\Gamma(2\alpha-1)}, \qquad \int_0^1 \left|G_2(s)\right|s^{\alpha-2}\,ds \le \frac{(3\alpha-2)\Gamma(\alpha-1)}{4\Gamma(2\alpha-1)}, \tag{2.28}$$

and

$$\begin{aligned}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-2}\,ds &\overset{s=t\tau}{=} \frac{t^{2(\alpha-1)}}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau^{\alpha-2}\,d\tau \\ &= \frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}t^{2(\alpha-1)}, \quad t\in(0,1]. \end{aligned} \tag{2.29}$$

So,

$$\sup_{t\in(0,1]} t^{2-\alpha}\big|Tu_n(t)-Tu(t)\big| \le \frac{(5\alpha+2)\Gamma(\alpha-1)}{4\Gamma(2\alpha-1)}\|u_n-u\|_0. \tag{2.30}$$

Similarly, from (2.26), (2.28) and (2.29), we have

$$\begin{aligned}\big|(t^{2-\alpha}\big((Tu_n)(t)-(Tu)(t)\big)'\big| &\le \bigg[\int_0^1 G_1(s)s^{\alpha-2}\,ds+\frac{2-\alpha}{\Gamma(\alpha)}t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^{\alpha-2}\,ds\\ &\quad+\frac{1}{\Gamma(\alpha-1)}t^{2-\alpha}\int_0^t (t-s)^{\alpha-2}s^{\alpha-2}\,ds\bigg]\|u_n-u\|_0\\ &\le \frac{3\alpha\Gamma(\alpha-1)}{2\Gamma(2\alpha-1)}\|u_n-u\|_0,\quad t\in(0,1]\end{aligned}$$

noting that

$$\begin{aligned}\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}s^{\alpha-2}\,ds &\overset{s=t\tau}{=} \frac{t^{2\alpha-3}}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\tau^{\alpha-2}\,d\tau\\ &= \frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-2)}t^{2\alpha-3},\quad t\in(0,1].\end{aligned} \tag{2.31}$$

So,

$$\sup_{t\in(0,1]}\big|(t^{2-\alpha}\big(Tu_n(t)-Tu(t)\big)'\big| \le \frac{3\alpha\Gamma(\alpha-1)}{2\Gamma(2\alpha-1)}\|u_n-u\|_0. \tag{2.32}$$

From (2.30)-(2.31), we have that

$$\|Tu_n-Tu\|_1 \le D\|u_n-u\|_0,$$

where

$$D=\max\bigg\{\frac{(5\alpha+2)\Gamma(\alpha-1)}{4\Gamma(2\alpha-1)},\frac{3\alpha\Gamma(\alpha-1)}{2\Gamma(2\alpha-1)}\bigg\}=\frac{(5\alpha+2)\Gamma(\alpha-1)}{4\Gamma(2\alpha-1)}. \tag{2.33}$$

So, T is a continuous operator on X_0.

Part 2. Now we show that T is a compact operator.

Assume that Ω is an arbitrary bounded set in X_0. Then there exists $M>0$ such that $\|u\|_0\le M$ for all $u\in\Omega$. By an argument similar to (2.30) and (2.32) in Part 1, from (2.25)-(2.26), we can obtain

$$\begin{aligned}&\sup_{t\in(0,1]}\big|t^{2-\alpha}Tu(t)\big| \le \frac{(5\alpha+2)\Gamma(\alpha-1)}{4\Gamma(2\alpha-1)}\|u\|_0,\\ &\sup_{t\in(0,1]}\big|\big(t^{2-\alpha}Tu(t)\big)'\big| \le \frac{3\alpha\Gamma(\alpha-1)}{2\Gamma(2\alpha-1)}\|u\|_0,\end{aligned}$$

and so $\|Tu\|_1\le D\|u\|_0\le DM$, where D is as in (2.33). It means that $T\Omega$ is bounded.

Now, we show that the set of functions B is equicontinuous on (0.1], where $B=\{t^{2-\alpha}Tu(t):u\in\Omega\}$.

In fact, for any $t_1, t_2 \in (0,1]$ with $t_1 < t_2$ and $u \in \Omega$, from (2.25), we have

$$\begin{aligned}
&\left|t_2^{2-\alpha} Tu(t_2) - t_1^{2-\alpha} Tu(t_1)\right| \\
&\quad \le (t_2 - t_1) \int_0^1 G_1(s)\left|u(s)\right| ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \left| t_2^{2-\alpha} \int_0^{t_2} (t_2 - s)^{\alpha-1} u(s)\, ds - t_1^{2-\alpha} \int_0^{t_1} (t_1 - s)^{\alpha-1} u(s)\, ds \right| \\
&\quad \le (t_2 - t_1) \int_0^1 G_1(s)\left|u(s)\right| ds + \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} \left[t_2^{2-\alpha} (t_2 - s)^{\alpha-1} - t_1^{2-\alpha} (t_1 - s)^{\alpha-1} \right] u(s)\, ds \right| \\
&\qquad + \frac{1}{\Gamma(\alpha)} t_2^{2-\alpha} \left| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} u(s)\, ds \right|. \qquad (2.34)
\end{aligned}$$

Again,

$$\begin{aligned}
&\int_0^1 G_1(s)\left|u(s)\right| ds = \int_0^1 G_1(s) s^{\alpha-2} \left| s^{2-\alpha} u(s) \right| ds \le M \int_0^1 G_1(s) s^{2-\alpha}\, ds = \frac{M\alpha\Gamma(\alpha-1)}{2\Gamma(2\alpha-1)}, \\
&\left| \int_0^{t_1} \left[t_2^{2-\alpha} (t_2 - s)^{\alpha-1} - t_1^{2-\alpha} (t_1 - s)^{\alpha-1} \right] u(s)\, ds \right| \\
&\quad \le \left(t_2^{2-\alpha} - t_1^{2-\alpha} \right) \int_0^{t_1} (t_2 - s)^{\alpha-1} \left| u(s) \right| ds + t_1^{2-\alpha} \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] \left| u(s) \right| ds \\
&\quad \le M \left(t_2^{2-\alpha} - t_1^{2-\alpha} \right) \int_0^{t_1} (t_2 - s)^{\alpha-1} s^{\alpha-2}\, ds \\
&\qquad + M t_1^{2-\alpha} \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] s^{\alpha-2}\, ds,
\end{aligned}$$

and

$$\begin{aligned}
t_2^{2-\alpha} \left| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} u(s)\, ds \right| &\le M \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} s^{\alpha-2}\, ds \\
&\le M \int_{t_1}^{t_2} s^{\alpha-2}\, ds = \frac{M}{\alpha - 1} \left(t_2^{\alpha-1} - t_1^{\alpha-1} \right).
\end{aligned}$$

Thus, from (2.34), it follows that

$$\begin{aligned}
&\left|t_2^{2-\alpha} Tu(t_2) - t_1^{2-\alpha} Tu(t_1)\right| \\
&\quad \le \frac{M\alpha\Gamma(\alpha-1)}{2\Gamma(2\alpha-1)} (t_2 - t_1) + \frac{M(t_2^{2-\alpha} - t_1^{2-\alpha})}{\Gamma(\alpha)} \int_0^{t_1} (t_2 - s)^{\alpha-1} s^{\alpha-2}\, ds \\
&\qquad + \frac{M t_1^{2-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] s^{\alpha-2}\, ds + \frac{M(t_2^{\alpha-1} - t_1^{\alpha-1})}{(\alpha-1)\Gamma(\alpha)}. \qquad (2.35)
\end{aligned}$$

Again,

$$\int_0^{t_1} (t_2 - s)^{\alpha-1} s^{\alpha-2}\, ds \le \int_0^1 s^{\alpha-2}\, ds = \frac{1}{\alpha - 1}, \qquad (2.36)$$

$$
\begin{aligned}
&t_1^{2-\alpha}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]s^{\alpha-2}\,ds\\
&\quad=t_1^{2-\alpha}\left[\int_0^{t_1}(t_2-s)^{\alpha-1}s^{\alpha-2}\,ds-\int_0^{t_1}(t_1-s)^{\alpha-1}s^{\alpha-2}\,ds\right]\\
&\quad=t_1^{2-\alpha}\left[t_2^{2(\alpha-1)}\int_0^{\frac{t_1}{t_2}}(1-\tau)^{\alpha-1}\tau^{\alpha-2}\,d\tau-t_1^{2(\alpha-1)}\int_0^1(1-\tau)^{\alpha-1}\tau^{\alpha-2}\,d\tau\right]\\
&\quad\le t_1^{2-\alpha}\left(t_2^{2(\alpha-1)}-t_1^{2(\alpha-1)}\right)\int_0^1(1-\tau)^{\alpha-1}\tau^{\alpha-2}\,d\tau\\
&\quad=\left(t_1^{2-\alpha}\cdot t_2^{2(\alpha-1)}-t_1^{\alpha}\right)B(\alpha,\alpha-1)\\
&\quad\le\left(t_2^{\alpha}-t_1^{\alpha}\right)B(\alpha,\alpha-1),
\end{aligned}
\tag{2.37}
$$

because

$$
\begin{aligned}
&\int_0^{t_1}(t_2-s)^{\alpha-1}s^{2-\alpha}\,ds\overset{s=t_2\tau}{=}t_2^{2(\alpha-1)}\int_0^{\frac{t_1}{t_2}}(1-\tau)^{\alpha-1}\tau^{\alpha-2}\,d\tau,\\
&\int_0^{t_1}(t_1-s)^{\alpha-1}s^{\alpha-2}\,ds\overset{s=t_1\tau}{=}t_1^{2(\alpha-1)}\int_0^{1}(1-\tau)^{\alpha-1}\tau^{\alpha-2}\,d\tau,
\end{aligned}
$$

and $t_1<t_2$, $1<\alpha<2$.

Substituting (2.36)-(2.37) into (2.35), we have

$$
\begin{aligned}
\left|t_2^{2-\alpha}Tu(t_2)-t_1^{2-\alpha}Tu(t_1)\right|\le{}&\frac{M\alpha\Gamma(\alpha-1)}{2\Gamma(2\alpha-1)}(t_2-t_1)+\frac{M(t_2^{2-\alpha}-t_1^{2-\alpha})}{(\alpha-1)\Gamma(\alpha)}\\
&+\frac{M\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}\left(t_2^{\alpha}-t_1^{\alpha}\right)+\frac{M(t_2^{\alpha-1}-t_1^{\alpha-1})}{(\alpha-1)\Gamma(\alpha)}.
\end{aligned}
$$

The above inequality shows that the set $B=\{t^{2-\alpha}Tu(t)|u\in\Omega\}$ is equicontinuous on $(0,1]$.

Finally, we show that the set B' is also equicontinuous on $(0,1]$, where

$$
B'=\left\{\left(t^{2-\alpha}Tu(t)\right)'|u\in\Omega\right\}.
$$

As before, for any $t_1,t_2\in(0,1]$ with $t_1<t_2$ and $u\in\Omega$, in view of (2.26), we have

$$
\begin{aligned}
&\left|\left(t_2^{2-\alpha}Tu(t_2)\right)'-\left(t_1^{2-\alpha}Tu(t_1)\right)'\right|\\
&\quad\le\frac{2-\alpha}{\Gamma(\alpha)}\left|t_2^{1-\alpha}\int_0^{t_2}(t_2-s)^{\alpha-1}u(s)\,ds-t_1^{1-\alpha}\int_0^{t_1}(t_1-s)^{\alpha-1}u(s)\,ds\right|\\
&\qquad+\frac{1}{\Gamma(\alpha-1)}\left|t_2^{2-\alpha}\int_0^{t_2}(t_2-s)^{\alpha-2}u(s)\,ds-t_1^{2-\alpha}\int_0^{t_1}(t_1-s)^{\alpha-2}u(s)\,ds\right|\\
&\quad\le\frac{2-\alpha}{\Gamma(\alpha)}\left[\int_0^{t_1}\left|t_2^{1-\alpha}(t_2-s)^{\alpha-1}-t_1^{1-\alpha}(t_1-s)^{\alpha-1}\right|\left|u(s)\right|ds\right.\\
&\qquad\left.+t_2^{1-\alpha}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left|u(s)\right|ds\right]\\
&\qquad+\frac{1}{\Gamma(\alpha-1)}\left[\int_0^{t_1}\left|t_2^{2-\alpha}(t_2-s)^{\alpha-2}-t_1^{2-\alpha}(t_1-s)^{\alpha-2}\right|\left|u(s)\right|ds\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ t_2^{2-\alpha} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-2} \big|u(s)\big|\, ds \Bigg] \\
&\le \frac{(2-\alpha)M}{\Gamma(\alpha)} \Bigg[\int_0^{t_1} \left(\left(1 - \frac{s}{t_2}\right)^{\alpha-1} - \left(1 - \frac{s}{t_1}\right)^{\alpha-1} \right) s^{\alpha-2}\, ds + \int_{t_1}^{t_2} \left(1 - \frac{s}{t_2}\right)^{\alpha-1} s^{\alpha-2}\, ds \Bigg] \\
&\quad + \frac{M}{\Gamma(\alpha-1)} \Bigg[\int_0^{t_1} \left(\left(1 - \frac{s}{t_1}\right)^{\alpha-2} - \left(1 - \frac{s}{t_2}\right)^{\alpha-2} \right) s^{\alpha-2}\, ds \\
&\quad + \int_{t_1}^{t_2} \left(1 - \frac{s}{t_2}\right)^{\alpha-2} s^{\alpha-2}\, ds \Bigg], \qquad (2.38)
\end{aligned}
$$

noting that $|u(s)| = s^{\alpha-2} \cdot |s^{2-\alpha} u(s)| \le s^{\alpha-2} M$, $s \in (0,1]$, and

$$
\left(1 - \frac{s}{t_2}\right)^{\alpha-1} > \left(1 - \frac{s}{t_1}\right)^{\alpha-1}, \qquad \left(1 - \frac{s}{t_1}\right)^{\alpha-2} > \left(1 - \frac{s}{t_2}\right)^{\alpha-2}
$$

keeping in mind that $1 < \alpha < 2$.

For an arbitrary $\varepsilon > 0$, take $\gamma = \frac{1}{2}\left(\frac{\Gamma(2\alpha-2)\varepsilon}{3M\Gamma(\alpha-1)}\right)^{\frac{1}{\alpha-1}}$. We can assume that ε is small enough that $\gamma < 1$. Since $t^{\alpha-1}$ is uniformly continuous on $[0,1]$, there exists $\delta_1 > 0$ such that

$$
0 < t_2^{\alpha-1} - t_1^{\alpha-1} < \frac{(\alpha-1)\Gamma(\alpha)\varepsilon}{3(2-\alpha)M[(\alpha-1)B(\alpha,\alpha-1)+1]},
$$

when $0 < t_2 - t_1 < \delta_1$, $t_1, t_2 \in (0,1]$.

Moreover, because

$$
\begin{aligned}
&\int_0^{t_1} \left[\left(1 - \frac{s}{t_2}\right)^{\alpha-1} - \left(1 - \frac{s}{t_1}\right)^{\alpha-1} \right] s^{\alpha-2}\, ds \\
&= \int_0^{t_1} \left(1 - \frac{s}{t_2}\right)^{\alpha-1} s^{\alpha-2}\, ds - \int_0^{t_1} \left(1 - \frac{s}{t_1}\right)^{\alpha-1} s^{\alpha-2}\, ds \\
&= t_2^{\alpha-1} \int_0^{\frac{t_1}{t_2}} (1-\tau)^{\alpha-1} \tau^{\alpha-2}\, d\tau - t_1^{\alpha-1} \int_0^1 (1-\tau)^{\alpha-1} \tau^{\alpha-2}\, d\tau \\
&\le \left(t_2^{\alpha-1} - t_1^{\alpha-1}\right) \int_0^1 (1-\tau)^{\alpha-1} \tau^{\alpha-2}\, d\tau = \left(t_2^{\alpha-1} - t_1^{\alpha-1}\right) B(\alpha, \alpha-1)
\end{aligned}
$$

and

$$
\int_{t_1}^{t_2} \left(1 - \frac{s}{t_2}\right)^{\alpha-1} s^{\alpha-2}\, ds < \int_{t_1}^{t_2} s^{\alpha-2}\, ds = \frac{1}{\alpha-1}\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right),
$$

we obtain

$$
\begin{aligned}
&\frac{(2-\alpha)M}{\Gamma(\alpha)} \Bigg[\int_0^{t_1} \left(\left(1 - \frac{s}{t_2}\right)^{\alpha-1} - \left(1 - \frac{s}{t_1}\right)^{\alpha-1} \right) s^{\alpha-2}\, ds + \int_{t_1}^{t_2} \left(1 - \frac{s}{t_2}\right)^{\alpha-1} s^{\alpha-2}\, ds \Bigg] \\
&\quad \le \frac{(2-\alpha)M}{\Gamma(\alpha)} \left[B(\alpha, \alpha-1) + \frac{1}{\alpha-1} \right] \left(t_2^{\alpha-1} - t_1^{\alpha-1}\right) < \frac{\varepsilon}{3} \qquad (2.39)
\end{aligned}
$$

when $0 < t_2 - t_1 < \delta_1$, $t_1, t_2 \in (0,1]$.

(1) If $0 < t_1 < \gamma$ and $t_1 < t_2$ with $t_2 - t_1 < \gamma$, then

$$\begin{aligned}
&\frac{M}{\Gamma(\alpha-1)}\int_0^{t_1}\left[\left(1-\frac{s}{t_1}\right)^{\alpha-2}-\left(1-\frac{s}{t_2}\right)^{\alpha-2}\right]s^{\alpha-2}\,ds \\
&\quad<\frac{M}{\Gamma(\alpha-1)}\int_0^{t_1}\left(1-\frac{s}{t_1}\right)^{\alpha-2}s^{\alpha-2}\,ds \\
&\quad\overset{s=t_1\tau}{=}\frac{Mt_1^{\alpha-1}}{\Gamma(\alpha-1)}\int_0^1(1-\tau)^{\alpha-2}\tau^{\alpha-2}\,d\tau \\
&\quad\le\frac{M\Gamma(\alpha-1)}{\Gamma(2\alpha-2)}\gamma^{\alpha-1}<\frac{\varepsilon}{3}
\end{aligned}\tag{2.40}$$

and

$$\begin{aligned}
\frac{M}{\Gamma(\alpha-1)}\int_{t_1}^{t_2}\left(1-\frac{s}{t_2}\right)^{\alpha-2}s^{\alpha-2}\,ds &\overset{s=t_2\tau}{=}\frac{Mt_2^{\alpha-1}}{\Gamma(\alpha-1)}\int_{\frac{t_1}{t_2}}^1(1-\tau)^{\alpha-2}\tau^{\alpha-2}\,d\tau \\
&<\frac{Mt_2^{\alpha-1}}{\Gamma(\alpha-1)}\int_0^1(1-\tau)^{\alpha-2}\tau^{\alpha-2}\,d\tau \\
&<\frac{2^{\alpha-1}M\Gamma(\alpha-1)}{\Gamma(2\alpha-2)}\gamma^{\alpha-1}=\frac{\varepsilon}{3}
\end{aligned}\tag{2.41}$$

because $0 < t_2 < 2\gamma$.

Consequently, by (2.40)-(2.41) we have

$$\begin{aligned}
&\frac{M}{\Gamma(\alpha-1)}\left[\int_0^{t_1}\left(\left(1-\frac{s}{t_1}\right)^{\alpha-2}-\left(1-\frac{s}{t_2}\right)^{\alpha-2}\right)s^{\alpha-2}\,ds+\int_{t_1}^{t_2}\left(1-\frac{s}{t_2}\right)^{\alpha-2}s^{\alpha-2}\,ds\right] \\
&\quad<\frac{2\varepsilon}{3}
\end{aligned}\tag{2.42}$$

when $0 < t_1 < \gamma$ with $0 < t_2 - t_1 < \gamma$.

(2) If $\gamma \le t_1 < t_2$, then

$$\begin{aligned}
&\int_0^{t_1}\left[\left(1-\frac{s}{t_1}\right)^{\alpha-2}-\left(1-\frac{s}{t_2}\right)^{\alpha-2}\right]s^{\alpha-2}\,ds \\
&\quad=t_1^{\alpha-1}\int_0^1(1-\tau)^{\alpha-2}\tau^{\alpha-2}\,d\tau-t_2^{\alpha-1}\int_0^{\frac{t_1}{t_2}}(1-\tau)^{\alpha-2}\tau^{\alpha-2}\,d\tau \\
&\quad=\left(t_1^{\alpha-1}-t_2^{\alpha-1}\right)\int_0^1(1-\tau)^{\alpha-2}\tau^{\alpha-2}\,d\tau+t_2^{\alpha-1}\int_{\frac{t_1}{t_2}}^1(1-\tau)^{\alpha-2}\tau^{\alpha-2}\,d\tau \\
&\quad<t_2^{\alpha-1}\int_{\frac{t_1}{t_2}}^1(1-\tau)^{\alpha-2}\tau^{\alpha-2}\,d\tau<\int_{\frac{t_1}{t_2}}^1(1-\tau)^{\alpha-2}\tau^{\alpha-2}\,d\tau,
\end{aligned}\tag{2.43}$$

and

$$\int_{t_1}^{t_2}\left(1-\frac{s}{t_2}\right)^{\alpha-2}s^{\alpha-2}\,ds=t_2^{\alpha-1}\int_{\frac{t_1}{t_2}}^1(1-\tau)^{\alpha-2}\tau^{\alpha-2}\,d\tau\le\int_{\frac{t_1}{t_2}}^1(1-\tau)^{\alpha-2}\tau^{\alpha-2}.\tag{2.44}$$

Hence, (2.43)-(2.44) imply that

$$\frac{M}{\Gamma(\alpha-1)}\left[\int_0^{t_1}\left(\left(1-\frac{s}{t_2}\right)^{\alpha-2}-\left(1-\frac{s}{t_1}\right)^{\alpha-2}\right)s^{\alpha-2}\,ds+\int_{t_1}^{t_2}\left(1-\frac{s}{t_2}\right)^{\alpha-1}s^{\alpha-2}\,ds\right]$$
$$<\frac{2M}{\Gamma(\alpha-1)}\int_{\frac{t_1}{t_2}}^{1}(1-\tau)^{\alpha-2}\tau^{\alpha-2}\,d\tau. \tag{2.45}$$

Because $\int_0^1(1-\tau)^{\alpha-2}\tau^{\alpha-2}\,d\tau=B(\alpha-1,\alpha-1)<\infty$, there exists $\sigma\in(0,1)$ such that

$$\int_{\eta}^{1}(1-\tau)^{\alpha-2}\tau^{\alpha-2}\,d\tau<\frac{\Gamma(\alpha-1)}{3M}\varepsilon, \tag{2.46}$$

when $0<1-\eta<\sigma$.

Take $0<\delta_2\le\gamma\sigma$, then from (2.46), it follows that

$$\int_{\frac{t_1}{t_2}}^{1}(1-\tau)^{\alpha-2}\tau^{\alpha-2}\,d\tau<\frac{\Gamma(\alpha-1)}{3M}\varepsilon, \tag{2.47}$$

when $0<t_2-t_1<\delta_2$, noting that $0<1-\frac{t_1}{t_2}=\frac{t_2-t_1}{t_2}<\frac{\delta_2}{\gamma}\le\sigma$ because $t_2>t_1\ge\gamma$.

So, from (2.45) together with (2.47), it follows that

$$\frac{M}{\Gamma(\alpha-1)}\left[\int_0^{t_1}\left(\left(1-\frac{s}{t_2}\right)^{\alpha-2}-\left(1-\frac{s}{t_1}\right)^{\alpha-2}\right)s^{\alpha-2}\,ds+\int_{t_1}^{t_2}\left(1-\frac{s}{t_2}\right)^{\alpha-1}s^{\alpha-2}\,ds\right]$$
$$<\frac{2\varepsilon}{3}, \tag{2.48}$$

when $\gamma\le t_1$ and $0<t_2-t_1<\delta_2$.

Now, take $\delta=\min\{\delta_1,\delta_2,\gamma\}$. Then, when $0<t_2-t_1<\delta$, by (2.38), (2.39), (2.42) and (2.48), we have

$$\left|\left(t_2^{2-\alpha}Tu(t_2)\right)'-\left(t_1^{2-\alpha}Tu(t_1)\right)'\right|<\varepsilon. \tag{2.49}$$

The above inequality (2.49) shows that the set B' is equicontinuous on $(0,1]$. As a consequence of the Arzela-Ascoli theorem, we have that $T\Omega$ is a compact set in X_1. The proof is complete. □

Finally, for the remainder of this section, we give the following lemma, which will be used to obtain our main results.

Lemma 2.6 ([35]) *Let E be a Banach space. Assume that Ω is an open bounded subset of E with $\theta\in\Omega$, and let $A:\bar{\Omega}\to E$ be a completely continuous operator such that*

$$\|Au\|\le\|u\|,\quad \forall u\in\partial\Omega.$$

Then A has a fixed point in Ω.

3 Main results

Let us introduce some assumptions which will be used throughout this paper.

(H_1) $f \in C((0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

(H_2) $\sup_{t\in(0,1]} |t^{2-\alpha} f(t,0,0)| < \infty$.

(H_3) There exist two constants $L_1 > 0$, $L_2 > 0$ such that

$$\left|f(t,x_2,y_2) - f(t,x_1,y_1)\right| \le L_1|x_2 - x_1| + L_2 t|y_2 - y_1|$$

for all $x_i, y_i \in \mathbb{R}$, $i = 1,2$, and $t \in (0,1]$.

(H_4) There exist a function $\phi \in X_0$ and constants $N_1 > 0$, $N_2 > 0$, $\theta_1, \theta_2 \in (0,1)$ such that

$$\left|f(t,x,y)\right| \le \phi(t) + N_1|x|^{\theta_1} + N_2 t|y|^{\theta_2}$$

for all $x_i, y_i \in \mathbb{R}$, $i = 1,2$, and $t \in (0,1]$.

We define two operators F, T on X_1 as follows:

$$Au = TFu, \qquad Fu = f\left(t,u,u'\right), \quad t \in (0,1], \text{ for } u \in X_1,$$

where T is defined as before.

We first establish the following lemma to obtain our results.

Lemma 3.1 *Suppose that* (H_1), (H_2), (H_3) *hold. Then the operator A maps X_1 into X_1.*

Proof For any $u \in X_1$, by (H_3) , we have

$$\left|f\left(t,u,u'\right) - f(t,0,0)\right| \le L_1\left|u(t)\right| + L_2 t\left|u'(t)\right|, \quad t \in (0,1].$$

So,

$$\left|f\left(t,u(t),u'(t)\right)\right| \le \left|f(t,0,0)\right| + L_1\left|u(t)\right| + L_2 t\left|u'(t)\right|, \quad t \in (0,1]. \tag{3.1}$$

Again,

$$\left|u(t)\right| = \left|t^{\alpha-2}t^{2-\alpha}u(t)\right| \le t^{\alpha-2}\|u\|_1, \quad t \in (0,1], \tag{3.2}$$

and so

$$\begin{aligned}\left|u'(t)\right| &= \left|t^{\alpha-2}\left(t^{2-\alpha}u(t)\right)' + (\alpha-2)t^{-1}u(t)\right| \\ &\le t^{\alpha-2}\left|\left(t^{2-\alpha}u(t)\right)'\right| + (2-\alpha)t^{-1}\left|u(t)\right| \\ &\le t^{\alpha-2}\|u\|_1 + (2-\alpha)t^{\alpha-3}\|u\|_1 \\ &< t^{\alpha-3}(3-\alpha)\|u\|_1, \quad t \in (0,1].\end{aligned} \tag{3.3}$$

Hence, by (3.1)-(3.3), we have

$$\left|f\left(t,u(t),u'(t)\right)\right| \le \left|f(t,0,0)\right| + \left(L_1 + (3-\alpha)L_2\right)t^{\alpha-2}\|u\|_1, \quad t \in (0,1], \tag{3.4}$$

and so

$$\left|t^{2-\alpha}f\left(t,u(t),u'(t)\right)\right| \le \left|t^{2-\alpha}f(t,0,0)\right| + \left(L_1 + (3-\alpha)L_2\right)\|u\|_1, \quad t \in (0,1]. \tag{3.5}$$

Again, from (H_2) and by letting $K = \sup_{t\in(0,1]} |t^{2-\alpha}f(t,0,0)|$, we have

$$\left|f(t,0,0)\right| = t^{\alpha-2}\left|t^{2-\alpha}f(t,0,0)\right| \le Kt^{\alpha-2}, \quad t \in (0,1],$$

and so $f(t,0,0) \in L(0,1)$. Thus, according to hypothesis (H_1) and (3.4)-(3.5), we know that $f(t,u,u') \in C(0,1] \cap L(0,1)$ and $\sup_{t\in(0,1]} |t^{2-\alpha}f(t,u,u')| < \infty$. That is, $Fu \in X_0$. Therefore, in view of Lemma 2.5, it follows that $Au \in X_1$. Thus, $A : X_1 \to X_1$. The proof is complete. □

The following lemma is significant to obtaining the result in this article.

Lemma 3.2 *Suppose that* (H_1) *and* (H_4) *hold. Then the operator* $A : X_1 \to X_1$ *is completely continuous.*

Proof We first show that the operator F maps X_1 into X_0 under (H_4).

In fact, for any $u \in X_1$, by an argument similar to (3.4)-(3.5), from (H_4) combined with (3.2)-(3.3), it follows that

$$\begin{aligned}\left|f\left(t,u(t),u'(t)\right)\right| &\le \phi(t) + N_1\left|u(t)\right|^{\theta_1} + N_2 t\left|u'(t)\right|^{\theta_2} \\ &\le \phi(t) + N_1 t^{(\alpha-2)\theta_1}\|u\|_1^{\theta_1} + N_2 t(3-\alpha)^{\theta_2} t^{(\alpha-3)\theta_2}\|u\|_1^{\theta_2} \\ &\le \phi(t) + t^{(\alpha-2)}\left(N_1\|u\|_1^{\theta_1} + N_2(3-\alpha)\|u\|_1^{\theta_2}\right), \quad t \in (0,1],\end{aligned} \tag{3.6}$$

and so

$$\left|t^{2-\alpha}f\left(t,u(t),u'(t)\right)\right| \le t^{2-\alpha}\phi(t) + N_1\|u\|_1^{\theta_1} + N_2(3-\alpha)\|u\|_1^{\theta_2}, \quad t \in (0,1] \tag{3.7}$$

noting that $\theta_1, \theta_2 \in (0,1)$ and $1 < \alpha < 2$.

Thus, according to hypothesis (H_1), formulae (3.6)-(3.7) ensure that $Fu \in X_0$, noting that $\phi \in X_0$.

Now, we prove that the operator $A : X_1 \to X_1$ is completely continuous.

First of all, in view of Lemma 2.5, we know that the operator $A : X_1 \to X_1$ because of the fact that $F : X_1 \to X_0$ and $A = TF$.

It remains to show that the operator A is completely continuous. The following proof is divided into two steps.

Step 1. We show that the operator A is compact on X_1.

In fact, assuming that Ω is an arbitrary set in X_1, there exists $M > 0$ such that $\|u\|_1 \le M$ for all $u \in \Omega$. Thus, from (3.7), it follows that

$$\|Fu\|_0 \le \sup_{t\in(0,1]} t^{2-\alpha}\phi(t) + N_1 M^{\theta_1} + N_2(3-\alpha)M^{\theta_2} \triangleq N.$$

Therefore, the set $B = \{h | h = Fu, u \in \Omega\}$ is bounded in X_0, and therefore, TB is a compact set in view of Lemma 2.5. That is, $A\Omega$ is a compact set, owing to the fact that $A = TF$ and $B = F\Omega$. Hence, A is a compact operator.

Step 2. We show that the operator A is continuous on X_1.

Let $\{u_n\}_{n=1}^{\infty}$ be an arbitrary sequence in X_1 with $u_n \to u_0 \in X_1$. Then there exists $M > 0$ such that $\|u_n\|_1 \le M$, $n = 0, 1, 2, \ldots$. Thus, from (3.7), it follows that

$$\begin{aligned}\left|t^{2-\alpha}Fu_n(t)\right| &\le \left|t^{2-\alpha}\phi(t)\right| + N_1\|u_n\|_1^{\theta_1} + N_2(3-\alpha)\|u_n\|_1^{\theta_2} \\ &\le \sup_{t\in(0,1]}\left|t^{2-\alpha}\phi(t)\right| + N_1M^{\theta_1} + N_2(3-\alpha)M^{\theta_2} \triangleq N \end{aligned} \tag{3.8}$$

for $n = 0, 1, 2, \ldots$.

The following proof is divided into two parts.

Part 1. By (2.25), we have

$$\begin{aligned}&\left|t^{2-\alpha}\big(Au_n(t) - Au_0(t)\big)\right| \\ &\quad\le t\int_0^1 G_1(s)\left|Fu_n(s) - Fu_0(s)\right| ds \\ &\qquad + \int_0^1 \left|G_2(s)\right|\cdot\left|Fu_n(s) - Fu_0(s)\right| ds + \frac{1}{\Gamma(\alpha)}t^{2-\alpha}\int_0^t (t-s)^{\alpha-1}\left|Fu_n(s) - Fu_0(s)\right| ds \\ &\quad\le \int_0^1 \left(G_1(s) + \left|G_2(s)\right| + \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}\right)\left|Fu_n(s) - u_0(s)\right| ds, \quad t \in (0,1]. \end{aligned} \tag{3.9}$$

By (2.28)-(2.29), we get

$$\int_0^1 \left(G_1(s) + \left|G_2(s)\right| + \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}\right)s^{\alpha-2}\,ds \le \frac{5\alpha+2}{4\Gamma(2\alpha-1)} < +\infty.$$

So, for any $\varepsilon > 0$, by the absolute continuity of Lebesgue integral, there exists $0 < \delta < 1$ such that

$$2N\int_0^{\delta} \left(G_1(s) + \left|G_2(s)\right| + \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}\right)s^{\alpha-2}\,ds < \frac{\varepsilon}{2},$$

and therefore,

$$\int_0^{\delta} \left(G_1(s) + \left|G_2(s)\right| + \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}\right)\left|Fu_n(s) - Fu_0(s)\right| ds < \frac{\varepsilon}{2} \tag{3.10}$$

holds for $n = 1, 2, \ldots$, because $|Fu_n(s) - Fu_0(s)| \le 2Ns^{\alpha-2}$, $s \in (0, \delta]$, $n = 1, 2, \ldots$, observing (3.8).

On the other hand, for all $t \in [\delta, 1]$, formulae (3.2)-(3.3) imply that

$$\begin{aligned}&\left|u_n(t)\right| \le t^{\alpha-2}\|u_n\|_1 \le \delta^{\alpha-2}M, \quad n = 0, 1, 2, \ldots, \\ &\left|u_n'(t)\right| \le (3-\alpha)t^{\alpha-3}\|u_n\|_1 \le 2\delta^{\alpha-3}M, \quad n = 0, 1, 2, \ldots, \end{aligned}$$

and

$$\left|u_n(s) - u_0(s)\right| \le \delta^{\alpha-2}\|u_n - u_0\|_1, \quad n \ge 1, \tag{3.11}$$

$$\left|\big(u_n(t) - u_0(t)\big)'\right| \le 2\delta^{\alpha-3}\|u_n - u_0\|_1, \quad n \ge 1. \tag{3.12}$$

Since $f(t,x,y)$ is uniformly continuous on $[\delta,1]\times[-\delta^{\alpha-2}M,\delta^{\alpha-2}M]\times[-2\delta^{\alpha-3}M,2\delta^{\alpha-3}M]$, from (H_1), for ε given as before, there exists $K_1\geq 1$ such that

$$\left|Fu_n(s)-Fu_0(s)\right|<\frac{d}{2}\varepsilon,\quad s\in[\delta,1], \tag{3.13}$$

when $n\geq K_1$, noting that (3.11)-(3.12), where $d=(\int_\delta^1 (G_1(s)+|G_2(s)|+\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)})\,ds)^{-1}$.

Thus, when $n\geq K_1$, formula (3.13) implies that

$$\int_\delta^1\left(G_1(s)+\left|G_2(s)\right|+\frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}\right)\left|Fu_n(s)-Fu_0(s)\right|ds<\frac{\varepsilon}{2} \tag{3.14}$$

holds for $n=1,2,\ldots$.

Substituting (3.10) and (3.14) into (3.9), we have that

$$\left|t^{2-\alpha}\left(Au_n(t)-Au(t)\right)\right|<\varepsilon,\quad t\in(0,1],$$

when $n\geq K_1$, and so

$$\sup_{t\in(0,1]}\left|t^{2-\alpha}\left(Au_n(t)-Au(t)\right)\right|\leq\varepsilon \tag{3.15}$$

when $n\geq K_1$.

Part 2. By (2.26), we have

$$\begin{aligned}
&\left|\left(t^{2-\alpha}\left(Au_n(t)-Au_0(t)\right)\right)'\right|\\
&\quad\leq\int_0^1 G_1(s)\left|Fu_n(s)-Fu_0(s)\right|ds\\
&\qquad+\frac{2-\alpha}{\Gamma(\alpha)}t^{1-\alpha}\int_0^t(t-s)^{\alpha-1}\left|Fu_n(s)-Fu_0(s)\right|ds\\
&\qquad+\frac{t^{2-\alpha}}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\left|Fu_n(s)-Fu_0(s)\right|ds\\
&\quad=\int_0^1 G_1(s)\left|Fu_n(s)-Fu_0(s)\right|ds+\frac{2-\alpha}{\Gamma(\alpha)}\int_0^t\left(1-\frac{s}{t}\right)^{\alpha-1}\left|Fu_n(s)-Fu_0(s)\right|ds\\
&\qquad+\frac{1}{\Gamma(\alpha-1)}\int_0^t\left(1-\frac{s}{t}\right)^{\alpha-2}\left|Fu_n(s)-Fu_0(s)\right|ds\\
&\quad<\int_0^1 G_1(s)\left|Fu_n(s)-Fu_0(s)\right|ds+\frac{1}{\Gamma(\alpha)}\int_0^t\left(1-\frac{s}{t}\right)^{\alpha-2}\left|Fu_n(s)-Fu_0(s)\right|ds,
\end{aligned} \tag{3.16}$$

because of the fact that $\frac{2-\alpha}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}=\frac{1}{\Gamma(\alpha)}$ and $(1-\frac{s}{t})^{\alpha-1}<(1-\frac{s}{t})^{\alpha-2}$.

For ε given as before, take $0<\delta_1<\min\{1,(\frac{\Gamma(\alpha)\varepsilon}{6N\cdot B(\alpha-1,\alpha-1)})^{\frac{1}{\alpha-1}}\}$, where N is given by (3.8).

(i) If $t\in(0,\delta_1]$, then from (3.8) it follows that

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_0^t\left(1-\frac{s}{t}\right)^{\alpha-2}\left|Fu_n(s)-Fu_0(s)\right|ds\\
&\quad=\frac{1}{\Gamma(\alpha)}\int_0^t\left(1-\frac{s}{t}\right)^{\alpha-2}s^{\alpha-2}\left|s^{2-\alpha}\left(Fu_n(s)-Fu_0(s)\right)\right|ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{2N}{\Gamma(\alpha)} \int_0^t \left(1-\frac{s}{t}\right)^{\alpha-2} s^{\alpha-2}\,ds \\
&\overset{s=t\tau}{=} \frac{2N}{\Gamma(\alpha)} t^{\alpha-1} \int_0^1 (1-\tau)^{\alpha-2}\tau^{\alpha-2}\,d\tau \\
&\leq \frac{2N}{\Gamma(\alpha)} \delta_1^{\alpha-1} B(\alpha-1,\alpha-1) < \frac{\varepsilon}{3}.
\end{aligned} \tag{3.17}$$

(ii) If $t \in (\delta_1, 1]$, then

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_0^t \left(1-\frac{s}{t}\right)^{\alpha-2} \left|Fu_n(s) - Fu_0(s)\right| ds \\
&\quad = \frac{1}{\Gamma(\alpha)} \int_0^{\delta_1} \left(1-\frac{s}{t}\right)^{\alpha-2} \left|Fu_n(s) - Fu_0(s)\right| ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_{\delta_1}^t \left(1-\frac{s}{t}\right)^{\alpha-2} \left|Fu_n(s) - Fu_0(s)\right| ds.
\end{aligned} \tag{3.18}$$

Again,

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_0^{\delta_1} \left(1-\frac{s}{t}\right)^{\alpha-2} \left|Fu_n(s) - Fu_0(s)\right| ds \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \int_0^{\delta_1} \left(1-\frac{s}{\delta_1}\right)^{\alpha-2} \left|Fu_n(s) - Fu_0(s)\right| ds \\
&\quad \leq \frac{2N}{\Gamma(\alpha)} \int_0^{\delta_1} \left(1-\frac{s}{\delta_1}\right)^{\alpha-2} s^{\alpha-2}\,ds \\
&\quad \overset{s=\delta_1\tau}{=} \frac{2N}{\Gamma(\alpha)} \delta_1^{\alpha-1} \int_0^1 (1-\tau)^{\alpha-2}\tau^{\alpha-2}\,ds \\
&\quad = \frac{2N}{\Gamma(\alpha)} {\delta_1}^{\alpha-1} B(\alpha-1,\alpha-1) < \frac{\varepsilon}{3}.
\end{aligned} \tag{3.19}$$

By an argument similar to (3.13), we can know that there exists $K_2 \geq 1$ such that

$$\left|Fu_n(s) - Fu_0(s)\right| < \frac{d_1}{3}\varepsilon, \quad s \in [\delta_1, 1], \tag{3.20}$$

when $n \geq K_2$, where $d_1 = (\alpha-1)\Gamma(\alpha)$.

Thus, from (3.20), for $t \in (\delta_1, 1]$, it follows that

$$\begin{aligned}
\frac{1}{\Gamma(\alpha)} \int_{\delta_1}^t \left(1-\frac{s}{t}\right)^{\alpha-2} \left|Fu_n(s) - Fu_0(s)\right| ds &< \frac{d_1\varepsilon}{3\Gamma(\alpha)} \int_{\delta_1}^t \left(1-\frac{s}{t}\right)^{\alpha-2} ds \\
&< \frac{d_1\varepsilon}{3(\alpha-1)\Gamma(\alpha)} = \frac{\varepsilon}{3}
\end{aligned} \tag{3.21}$$

when $n \geq K_2$.

By (3.18) together with (3.19) and (3.21), we have

$$\frac{1}{\Gamma(\alpha)} \int_0^t \left(1-\frac{s}{t}\right)^{\alpha-2} \left|Fu_n(s) - Fu_0(s)\right| ds < \frac{2\varepsilon}{3}, \quad t \in (0, 1], \tag{3.22}$$

when $n \geq K_2$.

On the other hand, from the proof in Part 1 (see the deducing on (3.10) and (3.14)), there exists K_3 such that the inequality

$$\int_0^1 G_1(s)\big|Fu_n(s) - Fu_0(s)\big|\,ds < \frac{\varepsilon}{3} \tag{3.23}$$

holds when $n \geq K_3$.

Define $L = \max\{K_2, K_3\}$. From (3.16) together with (3.17), (3.22) and (3.23), we have

$$\big|\big(t^{2-\alpha}\big(Au_n(t) - Au_0(t)\big)\big)'\big| < \varepsilon, \quad t \in (0,1]$$

when $n \geq L$, and so

$$\sup_{t\in(0,1]}\big|\big(t^{2-\alpha}\big(Au_n(t) - Au_0(t)\big)\big)'\big| \leq \varepsilon \tag{3.24}$$

when $n \geq L$.

Formulae (3.15), (3.24) yield that $Au_n \to Au_0$ in X_1. That is, the operator A is continuous on X_1. The proof is complete. □

In view of Lemma 2.3, it is easy to know that $u \in X_1$ is a solution of BVP (1.1)-(1.2) if and only if $u \in X_1$ is a fixed point of the operator A. Therefore, we can focus on seeking the existence of a fixed point of A in X_1.

Let $D = \frac{(5\alpha+2)\Gamma(\alpha-1)}{4\Gamma(2\alpha-1)}(L_1 + (3-\alpha)L_2)$, where L_1, L_2 is given by (H_3).

We are now in a position to state the first theorem in the present paper.

Theorem 3.1 *Suppose that* (H_1)-(H_3) *hold. If* $D < 1$, *then BVP* (1.1)-(1.2) *has a unique solution.*

Proof According to (H_1)-(H_2), we know that $A : X_1 \to X_1$ by Lemma 3.1. By an argument similar to (2.30), for any $u, v \in X_1$, in view of (H_3) together with (2.25), (2.28) and (2.29), we have

$$\begin{aligned}
&\big|t^{2-\alpha}\big(Au(t) - Av(t)\big)\big| \\
&\quad\leq t\int_0^1 G_1(s)\big|Fu(s) - Fv(s)\big|\,ds \\
&\qquad + \int_0^1 \big|G_2(s)\big|\cdot\big|Fu(s) - Fv(s)\big|\,ds + \frac{t^{2-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big|Fu(s) - Fv(s)\big|\,ds, \\
&\quad\leq \left[t\int_0^1 G_1(s)s^{\alpha-2}\,ds + \int_0^1 \big|G_2(s)\big|s^{\alpha-2}\,ds + \frac{t^{2-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-2}\,ds\right] \\
&\qquad\cdot\big(L_1 + (3-\alpha)L_2\big)\|u - v\|_1 \\
&\quad\leq \frac{(5\alpha+2)\Gamma(\alpha-1)(L_1 + (3-\alpha)L_2)}{4\Gamma(2\alpha-1)}\|u - v\|_1 \\
&\quad\doteq D_1\|u - v\|_1, \quad t \in (0,1],
\end{aligned}$$

because

$$\begin{aligned}\bigl|Fu(s)-Fv(s)\bigr| &\le L_1\bigl|u(s)-v(s)\bigr| + L_2 s\bigl|\bigl(u(s)-v(s)\bigr)'\bigr| \\ &\le s^{\alpha-2}\bigl(L_1+(3-\alpha)L_2\bigr)\|u-v\|_1, \quad s\in(0,1]\end{aligned}$$

by (3.2)-(3.3).

Thus,

$$\sup_{t\in(0,1]}\bigl|t^{2-\alpha}\bigl(Au(t)-Av(t)\bigr)\bigr| \le D_1\|u-v\|_1. \tag{3.25}$$

Similarly, from (2.26) together with (2.28), (2.31), we have

$$\sup_{t\in(0,1]}\bigl|\bigl(t^{2-\alpha}\bigl(Au(t)-Av(t)\bigr)\bigr)'\bigr| \le D_2\|u-v\|_1, \tag{3.26}$$

where $D_2=\frac{3\alpha(\Gamma\alpha-1)}{2\Gamma(2\alpha-1)}(L_1+(3-\alpha)L_2)$.

From the fact that $\frac{5\alpha+2}{4}>\frac{3\alpha}{2}$, by (3.25) and (3.26), we immediately have that

$$\|Au-Av\|_1 \le D\|u-v\|_1.$$

As $D<1$, A is a contraction mapping. So, by the contraction mapping principle, A has a unique fixed point $\bar{u}$. That is, $\bar{u}$ is the unique solution of BVP (1.1)-(1.2). □

We give another result in this paper as follows.

Theorem 3.2 *If* (H_1), (H_4) *hold, then BVP* (1.1)-(1.2) *has at least one solution.*

Proof First, by Lemma 3.2, we know that $A: X_1\to X_1$ is completely continuous.

Let $p_0=\frac{(5\alpha+2)\Gamma(\alpha-1)}{4\Gamma(2\alpha-1)}$, $e_0=\frac{3\alpha\Gamma(\alpha-1)}{2\Gamma(2\alpha-1)}$, $q_0=\sup_{t\in(0,1]}t^{2-\alpha}\phi(t)$, $d_0=p_0q_0$, $d_1=e_0q_0$, and $\theta=\max\{\theta_1,\theta_2\}$. Put $D=p_0(N_1+(3-\alpha)N_2)$, $G=e_0(N_1+(3-\alpha)N_2)$.

Take $R>\max\{1,2d_0,2d_1,(2D)^{\frac{1}{1-\theta}},(2G)^{\frac{1}{1-\theta}}\}$, and set $\Omega=\{u:\|u\|_1<R\}$. Now, we prove that

$$\|Au\|_1\le\|u\|_1, \quad \forall u\in\partial\Omega. \tag{3.27}$$

In fact, for any $u\in\partial\Omega$, by (H_4) and (2.25) together with (3.7), we have

$$\begin{aligned}\bigl|t^{2-\alpha}Au(t)\bigr| &\le t\int_0^1 G_1(s)\bigl|Fu(s)\bigr|\,ds+\int_0^1\bigl|G_2(s)\bigr|\cdot\bigl|Fu(s)\bigr|\,ds \\ &\quad+\frac{t^{2-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\bigl|Fu(s)\bigr|\,ds \\ &\le\left[\int_0^1 G_1(s)s^{\alpha-2}\,ds+\int_0^1\bigl|G_2(s)\bigr|s^{\alpha-2}\,ds+\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}s^{\alpha-2}\,ds\right] \\ &\quad\cdot\Bigl(\sup_{t\in(0,1]}t^{2-\alpha}\phi(t)+N_1\|u\|_1^{\theta_1}+N_2(3-\alpha)\|u\|_1^{\theta_2}\Bigr) \\ &\le\frac{(5\alpha+2)\Gamma(\alpha-1)}{4\Gamma(2\alpha-1)}\Bigl(\sup_{t\in(0,1]}t^{2-\alpha}\phi(t)+N_1\|u\|_1^{\theta_1}+N_2(3-\alpha)\|u\|_1^{\theta_2}\Bigr)\end{aligned}$$

$$
\begin{aligned}
&= p_0\bigl(q_0 + N_1R^{\theta_1} + N_2(3-\alpha)R^{\theta_2}\bigr)\\
&< p_0\bigl(q_0 + \bigl(N_1 + (3-\alpha)N_2\bigr)R^{\theta}\bigr)\\
&= d_0 + DR^{\theta} < R,
\end{aligned}
$$

because

$$
\int_0^1 G_1(s)s^{\alpha-2}\,ds + \int_0^1 \bigl|G_1(s)\bigr|s^{\alpha-2}\,ds + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}s^{\alpha-2}\,ds < p_0,
$$

noting that (2.28), (2.29) and keeping in mind the choice of R and $\|u\|_1 = R$.

Thus,

$$
\sup_{t\in(0,1]}\bigl|t^{2-\alpha}Au(t)\bigr| \le R. \tag{3.28}
$$

Similarly, by (H_4) and (2.26) together with (3.7), we have

$$
\begin{aligned}
\bigl|\bigl(t^{2-\alpha}Au(t)\bigr)'\bigr| &\le \int_0^1 G_1(s)\bigl|Fu(s)\bigr|\,ds + \frac{2-\alpha}{\Gamma(\alpha)}t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\bigl|Fu(s)\bigr|\,ds\\
&\quad + \frac{1}{\Gamma(\alpha-1)}t^{2-\alpha}\int_0^t (t-s)^{\alpha-2}\bigl|Fu(s)\bigr|\,ds\\
&\le \biggl(\int_0^1 G_1(s)s^{\alpha-2}\,ds + \frac{2-\alpha}{\Gamma(\alpha)}t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^{\alpha-2}\,ds\\
&\quad + \frac{1}{\Gamma(\alpha-1)}t^{2-\alpha}\int_0^t (t-s)^{\alpha-2}s^{\alpha-2}\,ds\biggr)\\
&\quad \cdot \bigl(q_0 + N_1R^{\theta_1} + N_2(3-\alpha)R^{\theta_2}\bigr)\\
&= \biggl(\frac{\alpha\Gamma(\alpha-1)}{2\Gamma(2\alpha-1)} + \frac{(2-\alpha)\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}t^{\alpha-1} + \frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-2)}t^{\alpha-1}\biggr)\\
&\quad \cdot \bigl(q_0 + N_1R^{\theta_1} + N_2(3-\alpha)R^{\theta_2}\bigr)\\
&< e_0\bigl(q_0 + N_1R^{\theta_1} + N_2(3-\alpha)R^{\theta_2}\bigr)\\
&< e_0\bigl(q_0 + \bigl(N_1 + (3-\alpha)N_2\bigr)R^{\theta}\bigr)\\
&= d_1 + GR^{\theta} < \biggl(\frac{1}{2} + GR^{\theta-1}\biggr)R < R
\end{aligned}
$$

noting that (2.28), (2.29), (2.31) and the choice of R as well as $\|u\|_1 = R$.

Thus,

$$
\sup_{t\in(0,1]}\bigl|\bigl(t^{2-\alpha}Au(t)\bigr)'\bigr| \le R. \tag{3.29}
$$

Summing up (3.28) and (3.29), we have that $\|Au\|_1 \le R$. That is, $\|Au\|_1 \le \|u\|_1$ because $\|u\|_1 = R$. So, the relation (3.27) holds. As a consequence of Lemma 2.6, the operator A has at least one point $\bar{u}$. That is, $\bar{u}$ is a solution of BVP (1.1)-(1.2). □

Example 3.1 Consider the following anti-periodic boundary value problem:

$$
\begin{cases}
D_{0+}^{\frac{3}{2}}u(t) = at^{\beta} + \frac{1}{10\sqrt{\pi}(1+t)}\frac{|u(t)|}{1+|u(t)|} - \frac{t}{5\sqrt{\pi}}\ln(1+(u'(t))^2),\\
t^{\frac{1}{2}}u(t)_{|t\to 0^+} = -t^{\frac{1}{2}}u(t)_{|t=1}, \qquad (t^{\frac{1}{2}}u(t))'_{|t\to 0^+} = -(t^{\frac{1}{2}}u(t))'_{|t=1},
\end{cases} \tag{3.30}
$$

where $\alpha = \frac{3}{2}$, $\beta \geq -\frac{1}{2}$ and $a \in \mathbb{R}$. Clearly, the function $f(t,x,y) = at^{\beta} + \frac{1}{10\sqrt{\pi}(t+1)}\frac{|x|}{1+|x|} - \frac{t}{5\sqrt{\pi}}\ln(1+y^2)$ satisfies $|f(t,x_2,y_2) - f(t,x_1,y_1)| \leq \frac{1}{10\sqrt{\pi}}|x_2 - x_1| + \frac{t}{5\sqrt{\pi}}|y_2 - y_1|$, and $\sup_{t\in(0,1]}|t^{2-\alpha}f(t,0,0)| < \infty$. Further, $L_1 = \frac{1}{10\sqrt{\pi}}$, $L_2 = \frac{1}{5\sqrt{\pi}}$, and $D = \frac{19}{20} < 1$. As a consequence of Theorem 3.1, BVP (3.30) has a unique solution.

Example 3.2 Consider the following anti-periodic boundary value problem:

$$\begin{cases} D_{0+}^{\frac{3}{2}}u(t) = (1+e^{Mt})t^{-\frac{1}{2}} + (\tan t)\sin\frac{u(t)}{1+(\sqrt[3]{u(t)})^2} + 2(\sin\pi t)\ln(1+(\sqrt[5]{u'(t)})^2), \\ \quad t \in (0,1), \\ t^{\frac{1}{2}}u(t)_{|t\to 0^+} = -t^{\frac{1}{2}}u(t)_{|t=1}, \qquad (t^{\frac{1}{2}}u(t))'_{|t\to 0^+} = -(t^{\frac{1}{2}}u(t))'_{|t=1}, \end{cases} \tag{3.31}$$

where $\alpha = \frac{3}{2}$, $M \in \mathbb{R}$. Clearly, the function $f(t,x,y) = (1+e^{Mt})t^{-\frac{1}{2}} + (\tan t)\frac{x}{1+(\sqrt[3]{x})^2} + 2(\sin\pi t)\ln(1+(\sqrt[5]{y})^2)$ satisfies $|f(t,x,y)| \leq \phi(t) + (\tan 1)|x|^{\frac{1}{3}} + 2|y|^{\frac{2}{5}}$, where $\phi = (1+e^{Mt})t^{-\frac{1}{2}} \in X_0$, So, all the assumptions of Theorem 3.2 are satisfied. Hence BVP (3.16) has at least one solution.

Competing interests
The author declares that he has no competing interests.

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