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A characterization of the two-weight inequality for Riesz potentials on cones of radially decreasing functions

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Abstract

We establish necessary and sufficient conditions on a weight pair (v, w) governing the boundedness of the Riesz potential operator I_α defined on a homogeneous group G from $L^p_{\mathrm{dec},r}(w, G)$ to $L^q(v, G)$, where $L^p_{\mathrm{dec},r}(w, G)$ is the Lebesgue space defined for non-negative radially decreasing functions on G. The same problem is also studied for the potential operator with product kernels I_{α_1,α_2} defined on a product of two homogeneous groups $G_1 \times G_2$. In the latter case weights, in general, are not of product type. The derived results are new even for Euclidean spaces. To get the main results we use Sawyer-type duality theorems (which are also discussed in this paper) and two-weight Hardy-type inequalities on G and $G_1 \times G_2$, respectively.

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1 Introduction

A homogeneous group is a simply connected nilpotent Lie group G on a Lie algebra g with the one-parameter group of transformations $\delta_t = \exp(A \log t)$, $t > 0$, where A is a diagonalized linear operator in G with positive eigenvalues. In the homogeneous group G the mappings $\exp o \delta_t o \exp^{-1}$, $t > 0$, are automorphisms in G, which will be again denoted by δ_t. The number $Q = \operatorname{tr} A$ is the homogeneous dimension of G. The symbol e will stand for the neutral element in G.

It is possible to equip G with a homogeneous norm $r : G \to [0, \infty)$ which is continuous on G, smooth on $G \setminus \{e\}$, and satisfies the conditions:

(i) $r(x) = r(x^{-1})$ for every $x \in G$;

(ii) $r(\delta_t x) = t\, r(x)$ for every $x \in G$ and $t > 0$;

(iii) $r(x) = 0$ if and only if $x = e$;

(iv) there exists $c_0 > 0$ such that

$$r(xy) \le c_0\big(r(x) + r(y)\big), \quad x, y \in G.$$

In the sequel we denote by $B(a, t)$ an open ball with the center a and radius $t > 0$, *i.e.*

$$B(a, t) := \big\{ y \in G; r(ay^{-1}) < t \big\}.$$

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It can be observed that $\delta_t B(e,1) = B(e,t)$.

Let us fix a Haar measure $|\cdot|$ in G such that $|B(e,1)| = 1$. Then $|\delta_t E| = t^Q |E|$. In particular, $|B(x,t)| = t^Q$ for $x \in G$, $t > 0$.

Examples of homogeneous groups are: the Euclidean n-dimensional space $\mathbb{R}^n$, the Heisenberg group, upper triangular groups, *etc.* For the definition and basic properties of the homogeneous group we refer to [1, p.12].

An everywhere positive function ρ on G will be called a weight. Denote by $L^p(\rho, G)$ $(1 < p < \infty)$ the weighted Lebesgue space, which is the space of all measurable functions $f : G \to \mathbb{C}$ defined by the norm

$$\|f\|_{L^p(\rho,G)} = \left(\int_G |f(x)|^p \rho(x)\,dx \right)^{\frac{1}{p}} < \infty.$$

If $\rho \equiv 1$, then we use the notation $L^p(G)$.

Denote by $\mathcal{DR}(G)$ the class of all radially decreasing functions on G with values in $\mathbb{R}_+$, *i.e.* the fact that $\phi \in \mathcal{DR}(G)$ means that there is a decreasing $\bar\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\varphi(x) = \bar\phi(r(x))$. In the sequel we will use the symbol ϕ itself for $\bar\phi$; $\phi \in \mathcal{DR}(G)$ will be written also by the symbol $\varphi \downarrow r$. Let G_1 and G_2 be homogeneous groups. We say that a function $\psi : G_1 \times G_2 \mapsto \mathbb{R}_+$ is radially decreasing if it is such in each variable separately uniformly to another one. The fact that ψ is radially decreasing on $G_1 \times G_2$ will be denoted as $\psi \in \mathcal{DR}(G_1 \times G_2)$.

Let

$$(I_\alpha f)(x) = \int_G f(y) \big(r\big(xy^{-1}\big)\big)^{\alpha - Q}\,dy, \quad 0 < \alpha < Q,$$

be the Riesz potential defined on G, where r is the homogeneous norm and dy is the normalized Haar measure on G. The operator I_α plays a fundamental role in harmonic analysis, *e.g.*, in the theory of Sobolev embeddings, in the theory of sublaplacians on nilpotent groups *etc.* Weighted estimates for multiple Riesz potentials can be applied, for example, to establish Sobolev and Poincaré inequalities on product spaces (see, *e.g.*, [2]).

Let G_1 and G_2 be homogeneous groups with homogeneous norms r_1 and r_2 and homogeneous dimensions Q_1 and Q_2, respectively. We define the potential operator on $G_1 \times G_2$ as follows:

$$(I_{\alpha,\beta} f)(x,y) = \iint_{G_1 \times G_2} f(t,\tau) \big(r_1\big(xt^{-1}\big)\big)^{\alpha - Q_1} \big(r_2\big(y\tau^{-1}\big)\big)^{\beta - Q_2}\,dt\,d\tau,$$

$$(x,y) \in G_1 \times G_2, 0 < \alpha < Q_1, 0 < \beta < Q_2.$$

Our aim is to derive two-weight criteria for I_α on the cone of radially decreasing functions on G. The same problem is also studied for the potential operator with product kernels $I_{\alpha,\beta}$ defined on a product of two homogeneous groups, where only the right-hand side weight is of product type. As far as we know the derived results for $I_{\alpha,\beta}$ are new, even in the case of Euclidean spaces. The proofs of the main results are based on Sawyer (see [3]) type duality theorem which is also true for homogeneous groups (see Propositions C and E below) and Hardy-type two-weight inequalities in homogeneous groups. Analogous results for multiple potential operators defined on $\mathbb{R}_+^n$ with respect to the cone of non-negative

decreasing functions on $\mathbb{R}^n_+$ were studied in [4, 5]. It should be emphasized that the two-weight problem for a multiple Hardy operator for the cone of decreasing functions on $\mathbb{R}^n_+$ was investigated by Barza, Heinig and Persson [6] under the restriction that both weights are of product type.

Historically, the one-weight inequality for the classical Hardy operator on decreasing functions was characterize by Arino and Muckenhoupt [7] under the so called B_p condition. The same problem for multiple Hardy transform was studied by Arcozzi, Barza, Garcia-Domingo and Soria [8]. This problem in the two-weight setting was solved by Sawyer [3]. Some sufficient conditions guaranteeing the two-weight inequality for the Riesz potential I_α on $\mathbb{R}^n$ were given by Rakotondratsimba [9]. In particular, the author showed that I_α is bounded from $L^p_{\mathrm{dec},r}(w, \mathbb{R}^n)$ to $L^q(v, \mathbb{R}^n)$ if the weighted Hardy operators $(\mathcal{H}f)(x) = \frac{1}{|x|^{n-\alpha}} \int_{|y|<|x|} f(y)\,dy$ and $(\mathcal{H}'f)(x) = \int_{|y|>|x|} \frac{f(y)}{|y|^{n-\alpha}}\,dy$ are bounded from $L^p(w, \mathbb{R}^n)$ to $L^q(v, \mathbb{R}^n)$. In fact, the author studied the problem on the cone of monotone decreasing functions.

Now we give some comments regarding the notation: in the sequel under the symbol $A \approx B$ we mean that there are positive constants c_1 and c_2 (depending on appropriate parameters) such that $c_1 A \le B \le c_2 A$; $A \ll B$ means that there is a positive constant c such that $A \le cB$; integral over a product set $E_1 \times E_2$ from g will be denoted by $\iint_{E_1 \times E_2} g(x,y)\,dx\,dy$ or $\int_{E_1} \int_{E_2} g(x,y)\,dx\,dy$; for a weight functions w and w_i on G, by the symbols $W(t)$ and $W_i(t)$ will be denoted the integrals $\int_{B(e,t)} w(x)\,dx$ and $\int_{B(e_i,t)} w_i(x)\,dx$ respectively; for a weight w on $G_1 \times G_2$, we denote $W(t,\tau) := \int_{B(e_1,t) \times B(e_2,\tau)} w(x,y)\,dx\,dy$, where e_1 and e_1 are neutral elements in G_1 and G_2, respectively. Finally, we mention that constants (often different constants in one and the same line of inequalities) will be denoted by c or C. The symbol p' stands for the conjugate number of p: $p' = p/(p-1)$, where $1 < p < \infty$.

2 Preliminaries

We begin this section with the statements regarding polar coordinates in G (see *e.g.*, [1, p.14]).

Proposition A *Let G be a homogeneous group and let $S = \{x \in G : r(x) = 1\}$. There is a (unique) Radon measure σ on S such that for all $u \in L^1(G)$,*

$$\int_G u(x)\,dx = \int_0^\infty \int_S u(\delta_t \bar{y}) t^{Q-1}\,d\sigma(\bar{y})\,dt.$$

Let a be a positive number. The two-weight inequality for the Hardy-type transforms

$$\left(H^a f\right)(x) = \int_{B(e,ar(x))} f(y)\,dy, \quad x \in G, \qquad \left(\widetilde{H}^a f\right)(x) = \int_{G \backslash B(e,ar(x))} f(y)\,dy, \quad x \in G,$$

reads as follows (see [10], Chapter 1 for more general case, in particular, for quasi-metric measure spaces):

Theorem A *Let $1 < p \le q < \infty$ and let a be a positive number. Then*
(i) *The operator H^a is bounded from $L^p(u_1, G)$ to $L^q(u_2, G)$ if and only if*

$$\sup_{t>0} \left(\int_{G \backslash B(e,t)} u_2(x)\,dx\right)^{1/q} \left(\int_{B(e,at)} u_1^{1-p'}(x)\,dx\right)^{1/p'} < \infty.$$

(ii) *The operator $\widetilde{H}^a$ is bounded from $L^p(u_1, G)$ to $L^q(u_2, G)$ if and only if*

$$\sup_{t>0}\left(\int_{B(e,t)} u_2(x)\,dx\right)^{1/q}\left(\int_{G\setminus B(e,at)} u_1^{1-p'}(x)\,dx\right)^{1/p'} < \infty.$$

We refer also to [11] for the Hardy inequality written for balls with center at the origin. In the sequel we denote H^1 by H.

The following statement for Euclidean spaces was derived by Barza, Johansson and Persson [12].

Proposition B *Let w be a weight function on G and let $1 < p < \infty$. If $f \in \mathcal{DR}(G)$ and $g \geq 0$, then*

$$\sup_{f\downarrow r}\frac{\int_G f(x)g(x)\,dx}{(\int_G f(x)^p w(x)\,dx)^{1/p}} \approx \|w\|_{L^1(G)}^{-1/p}\|g\|_{L^1(G)} + \left(\int_G H^{p'}\big(r(x)\big)W^{-p'}\big(r(x)\big)w(x)\,dx\right)^{1/p'},$$

where $H(t) = \int_{B(e,t)} g(x)\,dx$, $W(t) = \int_{B(e,t)} w(x)\,dx$.

The proof of Proposition B repeats the arguments (for $\mathbb{R}^n$) used in the proof of Theorem 3.1 of [12] taking Proposition A and the following lemma into account.

Lemma A *Let $1 < p < \infty$. For a weight function w, the inequality*

$$\int_G w(x)\left(\int_{G\setminus B(e,r(x))} f(y)\,dy\right)^p dx \leq p\int_G f^p(x)W^p\big(r(x)\big)w^{1-p}(x)\,dx, \quad f \geq 0,$$

holds.

Proof The proof of this lemma is based on Theorem A (part (ii)) taking $a = 1$, $p = q$, $u_2(x) = v(x)$, $u_1 = w^{1-p}(x)W^p(r(x))$ there. Details are omitted. □

Corollary A *Let the conditions of Proposition B be satisfied and let $\int_G w(x)\,dx = \infty$. Then the following relation holds:*

$$\sup_{f\downarrow r}\frac{\int_G f(x)g(x)\,dx}{(\int_G f^p(x)w(x)\,dx)^{1/p}} \approx \left(\int_G H^{p'}\big(r(x)\big)W\big(r(x)\big)w(x)\,dx\right)^{1/p}.$$

Corollary A implies the following duality result, which follows in the standard way (see [3, 12] for details).

Proposition C *Let $1 < p, q < \infty$ and let v, w be weight functions on G with $\int_G w(x)\,dx = \infty$. Then the integral operator T defined on functions on G is bounded from $L^p_{\mathrm{dec},r}(w, G)$ to $L^q(v, G)$ if and only if*

$$\left(\int_G\left(\int_{B(e,r(x))}(T^*g)(y)\,dy\right)^{p'} W^{-p'}\big(r(x)\big)w(x)\,dx\right)^{1/p'} \leq C\left(\int_G g^{q'}(x)v^{1-q'}(x)\,dx\right)^{1/q'}$$

(2.1)

holds for every positive measurable g on G.

The next statement yields the criteria for the two-weight boundedness of the operator H on the cone $\mathcal{DR}(G)$. In particular the following statement is true.

Theorem B *Let $1 < p \le q < \infty$ and let v and w be weights on G such that $\|w\|_{L^1(G)} = \infty$. Then H is bounded from $L^p_{\mathrm{dec},r}(w, G)$ to $L^q_v(v, G)$ if and only if*

(i)

$$\sup_{t>0} \left(\int_{B(e,t)} w(x)\, dx \right)^{-1/p} \left(\int_{B(e,t)} v(x) r^{Qq}(x)\, dx \right)^{1/q} < \infty;$$

(ii)

$$\sup_{t>0} \left(\int_{B(e,t)} r^{Qp'}(x) W^{-p'}\big(r(x)\big) w(x)\, dx \right)^{1/p'} \left(\int_{G \setminus B(e,t)} v(x)\, dx \right)^{1/q} < \infty.$$

Proof The proof of this statement follows by the standard way applying Proposition C (see *e.g.* [3, 12]). □

Definition 2.1 Let ρ be a locally integrable a.e. positive function on G. We say that ρ satisfies the doubling condition at e ($\rho \in DC(G)$) if there is a positive constant $b > 1$ such that for all $t > 0$ the following inequality holds:

$$\int_{B(e,2t)} \rho(x)\, dx \le b \int_{B(e,t)} \rho(x)\, dx.$$

Further, we say that $w \in DC^{\gamma,p}(G)$, where $1 < p < \infty$, $0 < \gamma < Q/p$, if there is a positive constant b such that for all $t > 0$

$$\int_{G \setminus B(e,t)} r^{\gamma p'}(x) W^{-p'}\big(r(x)\big) w(x)\, dx \le b \int_{G \setminus B(e,2t)} r^{\gamma p'}(x) W^{-p'}\big(r(x)\big) w(x)\, dx.$$

Remark 2.1 It can be checked that if a weight w satisfies the doubling condition et e in the strong sense, *i.e.*, $w \in DC(G)$ and $\int_{B(e,2t)} w(x)\, dx \le c \int_{B(e,2t) \setminus B(e,t)} w(x)\, dx$ with a constant c independent of t, then $w \in DC^{\gamma,p}(G)$.

Definition 2.2 We say that a locally integrable a.e. positive function ρ on $G_1 \times G_2$ satisfies the doubling condition with respect to the second variable ($\rho \in DC(y)$) uniformly to the first one if there is a positive constant c such that for all $t > 0$ and almost every $x \in G_1$ the following inequality holds:

$$\int_{B(e_2,2t)} \rho(x,y)\, dy \le c \int_{B(e_2,t)} \rho(x,y)\, dy.$$

Analogously is defined the class of weights $DC(x)$.

3 Riesz potentials on G

The main result of this section reads as follows.

Theorem 3.1 *Let $1 < p \le q < \infty$ and let v and w be weights such that either $w \in DC^{\alpha,p}(G)$ or $v \in DC(G)$; let $\|w\|_{L^1(G)} = \infty$. Then the operator I_α is bounded from $L^p_{\mathrm{dec},r}(w, G)$ to $L^q(v, G)$*

if and only if

(i)

$$\sup_{t>0}\left(\int_{B(e,t)} w(x)\,dx\right)^{-1/p}\left(\int_{B(e,t)} r^{\alpha q}(x)v(x)\,dx\right)^{1/q} < \infty; \tag{3.1}$$

(ii)

$$\sup_{t>0}\left(\int_{B(e,t)} r^{p'Q}(x)W^{-p'}\big(r(x)\big)w(x)\,dx\right)^{1/p'}\left(\int_{G\setminus B(e,t)} r^{(\alpha-Q)q}(x)v(x)\,dx\right)^{1/q} < \infty; \tag{3.2}$$

(iii)

$$\sup_{t>0}\left(\int_{B(e,t)} v(x)\,dx\right)^{1/q}\left(\int_{G\setminus B(e,t)} r^{\alpha p'}(x)W^{-p'}\big(r(x)\big)w(x)\,dx\right)^{1/p'} < \infty. \tag{3.3}$$

To prove this result we need to prove some auxiliary statements.

Lemma 3.1 *Let $0 < \alpha < Q$ and let c_0 be the constant from the triangle inequality of r. Then there is a positive constant c depending only on Q, α, and c_0 such that for all $s \in B(e, r(x)/2)$,*

$$I(x,y) := \int_{B(e,r(x))\setminus B(e,2c_0 r(y))} r\big(ty^{-1}\big)^{\alpha-Q}\,dt \le cr\big(xy^{-1}\big)^{\alpha}. \tag{3.4}$$

Proof We have

$$I(x,y) = \int_0^{\infty}\Big|\big\{t \in G : r\big(ty^{-1}\big)^{\alpha-Q} > \lambda\big\} \cap B\big(e,r(x)\big)\setminus B\big(e,2c_0 r(y)\big)\Big|\,d\lambda$$

$$= \int_0^{r(xy^{-1})^{\alpha-Q}}(\cdots) + \int_{r(xy^{-1})^{\alpha-Q}}^{\infty}(\cdots) =: I^{(1)}(x,y) + I^{(2)}(x,y).$$

Observe that, by the triangle inequality for r, we have $r^Q(x) \le c_0^Q 2^{Q-1}(r^Q(xy^{-1}) + r^Q(y))$. This implies that $r^Q(x) - (2c_0)^Q r^Q(y) \le c_0^Q 2^{Q-1} r^Q(xy^{-1})$. Hence,

$$I^{(1)}(x,y) \le r\big(xy^{-1}\big)^{\alpha-Q}\Big|B\big(e,r(x)\big)\setminus B\big(e,2c_0 r(y)\big)\Big|$$

$$= r\big(xy^{-1}\big)^{\alpha-Q}\big(r^Q(x) - (2c_0)^Q r^Q(y)\big) \le cr\big(xy^{-1}\big)^{\alpha}.$$

Further, it is easy to see that

$$I^{(2)}(x,y) \le cr\big(xy^{-1}\big)^{\alpha}.$$

Finally we have (3.4). □

Let us introduce the following potential operators:

$$(J_\alpha f)(x) = \int_{B(e,2c_0 r(x))} f(y)r^{\alpha-Q}\big(xy^{-1}\big)\,dy, \qquad (S_\alpha f)(x) = \int_{G\setminus B(e,2c_0 r(x))} f(y)r^{\alpha-Q}\big(xy^{-1}\big)\,dy,$$

$$x \in G, 0 < \alpha < Q.$$

It is easy to see that

$$I_\alpha f = J_\alpha f + S_\alpha f. \tag{3.5}$$

We need also to introduce the following weighted Hardy operator:

$$(H_\alpha f)(x) = r(x)^{\alpha - Q}(Hf)(x).$$

Proposition 3.1 *The following relation holds for all $f \in \mathcal{DR}(G)$:*

$$J_\alpha f \approx H_\alpha f. \tag{3.6}$$

Proof We have

$$(J_\alpha f)(x) = \int_{B(e,r(x)/2c_0)} f(y) r^{\alpha - Q}\left(xy^{-1}\right) dy + \int_{B(e,2c_0 r(x)) \setminus B(e,r(x)/(2c_0))} f(y) r^{\alpha - Q}\left(xy^{-1}\right) dy$$

$$=: \left(J_\alpha^{(1)} f\right)(x) + \left(J_\alpha^{(2)} f\right)(x).$$

If $y \in B(e, r(x)/2c_0)$, then $r(x) \le c_0(r(xy^{-1}) + r(y)) \le c_0 r(xy^{-1}) + r(x)/2$. Hence $r(x) \le 2c_0 r(xy^{-1})$. Consequently,

$$\left(J_\alpha^{(1)} f\right)(x) \le c(H_\alpha f)(x).$$

Applying now the fact that $f \in DR(G)$ we see that

$$\left(J_\alpha^{(2)} f\right)(x) \le f\left(r(x)/2c_0\right) \int_{B(e,r(x)/2c_0) \setminus B(e,2c_0 r(x))} r^{\alpha - Q}\left(xy^{-1}\right) dy$$

$$\le cf\left(r(x)/2c_0\right) r(x)^\alpha \le c(H_\alpha f)(x). \qquad \square$$

Lemma 3.2 *Let $1 < p \le q < \infty$ and let v and w be weights on G such that $\|w\|_{L^1(G)} = \infty$. Then the operator S_α is bounded from $L^p_{\mathrm{dec},r}(w,G)$ to $L^q(v,G)$ if*

$$\sup_{t>0} \left(\int_{G \setminus B(e,t)} r^{\alpha p'}(x) W^{-p'}\left(r(x)\right) w(x)\, dx \right)^{1/p'} \left(\int_{B(e,t/(2c_0))} v(x)\, dx \right)^{1/q} < \infty.$$

Conversely, if S_α is bounded from $L^p_{\mathrm{dec},r}(w,G)$ to $L^q(v,G)$, then the condition

$$\sup_{t>0} \left(\int_{G \setminus B(e,t)} r^{\alpha p'}(x) W^{-p'}(x) w(x)\, dx \right)^{1/q} \left(\int_{B(e,t/(4c_0))} v(x)\, dx \right)^{1/p'} < \infty$$

is satisfied. Furthermore, if either $w \in DC^{\alpha,p}(G)$ or $v \in DC(G)$, then the operator S_α is bounded from $L^p_{\mathrm{dec},r}(w,G)$ to $L^q(v,G)$ if and only if

$$\sup_{t>0} \left(\int_{G \setminus B(e,t)} r^{\alpha p'}(x) W^{-p'}\left(r(x)\right) w(x)\, dx \right)^{1/q} \left(\int_{B(e,t)} v(x)\, dx \right)^{1/p'} < \infty.$$

Proof Applying Proposition C, S_α is bounded from $L^p_{\mathrm{dec},r}(w,G)$ to $L^q(v,G)$ if and only if

$$\left(\int_G \left(\int_{B(e,r(x))} (S^*_\alpha f)(y)\,dy\right)^{p'} W^{-p'}\big(r(x)\big) w(x)\,dx\right)^{1/p'} \le c \left(\int_G g^{q'}(x) v^{1-q'}(x)\,dx\right)^{1/q'},$$

where

$$(S^*_\alpha f)(x) = \int_{B(e,r(x)/(2c_0))} f(y) r^{\alpha-Q}\big(xy^{-1}\big)\,dy.$$

Now we show that

$$c_1 r^\alpha(x) \int_{B(e,r(x)/(4c_0))} g(s)\,ds \le \int_{B(e,r(x))} (S^*_\alpha g)(y)\,dy$$

$$\le c_2 r^\alpha(x) \int_{B(e,r(x)/(2c_0))} g(s)\,ds, \quad g \ge 0. \tag{3.7}$$

To prove the right-hand side estimate in (3.7) observe that by Tonelli's theorem and Lemma 3.1 we have

$$\int_{B(e,r(x))} (S^*_\alpha g)(y)\,dy = \int_{B(e,r(x)/(2c_0))} f(s) \left(\int_{B(e,r(x))\setminus B(e,2c_0 r(s))} r^{\alpha-Q}\big(sy^{-1}\big)\,dy\right) ds$$

$$\le c_2 r(x)^\alpha \int_{B(e,r(x)/(2c_0))} f(s)\,ds.$$

On the other hand,

$$\int_{B(e,r(x))} (S^*_\alpha g)(y)\,dy \ge c r^{\alpha-Q}(x) \left(\int_{B(e,r(x))\setminus B(e,r(x)/2)} \left(\int_{B(e,r(y)/(2c_0))} f(s)\,ds\right) dy\right)$$

$$\ge c_1 r^\alpha(x) \left(\int_{B(e,r(x)/(4c_0))} f(s)\,ds\right).$$

Thus, Theorem A completes the proof. □

Proof of Theorem 3.1 By (3.5) it is enough to estimate the terms with $J_\alpha f$ and $S_\alpha f$. By applying Proposition 3.1 and Theorem B we find that J_α is bounded from $L^p_{\mathrm{dec},r}(w,G)$ to $L^q(v,G)$ if and only if the conditions (ii) and (iii) are satisfied. Now by Lemma 3.2 and the equality (which is a consequence of Proposition A)

$$\left(\int_{G\setminus B(e,t)} W\big(r(x)\big) w(x)\,dx\right)^{1/p'} = \left(\int_{B(e,t)} w(x)\,dx\right)^{-1/p}$$

we see that S_α is bounded from $L^p_{\mathrm{dec},r}(w,G)$ to $L^q(v,G)$ if and only if (i) is satisfied. □

4 Multiple potentials on $G_1 \times G_2$

Let us now investigate the two-weight problem for the operator I_{α_1,α_2} on the cone $\mathcal{DR}(G_1 \times G_2)$. In the sequel without loss of generality we denote the triangle inequality constants for G_1 and G_2 by one and the same symbol c_0.

The following statement can be derived just in the same way as Theorem 3.1 was obtained in [6]. The proof is omitted to avoid repeating those arguments.

Proposition D *Let* $1 < p < \infty$ *and let* $w(x,y) = w_1(x)w_2(y)$ *be a product weight on* $G_1 \times G_2$. *Then the relation*

$$\sup_{0 \le f \downarrow r} \frac{\iint_{G_1 \times G_2} f(x,y)g(x,y)\,dx\,dy}{(\iint_{G_1 \times G_2} f^p(x,y)w(x,y))^{1/p}} \approx \sum_{i=1}^{4} I_k$$

holds for a non-negative measurable function g, where

$$I_1 := \|w\|_{L^1(G_1 \times G_2)}^{-1/p} \|g\|_{L^1(G_1 \times G_2)},$$

$$I_2 := \|w_2\|_{L^1(G_1)}^{-1/p} \left(\int_{G_1} \left(\int_{B(e_1,r_1(x))} \|g(t,\cdot)\|_{L^1(G_2)}\,dt \right)^{p'} W_1^{-p'}(r_1(x))w_1(x)\,dx \right)^{1/p'},$$

$$I_3 := \|w_1\|_{L^1(G_1)}^{-1/p} \left(\int_{G_2} \left(\int_{B(e_2,r_2(y))} \|g(\cdot,\tau)\|_{L^1(G_1)}\,d\tau \right)^{p'} W_2^{-p'}(r_2(y))w_2(y)\,dy \right)^{1/p'},$$

$$I_4 := \left(\int_{G_1 \times G_2} \left(\int_{G_1 \times G_2} g(t,\tau)\,dt\,d\tau \right)^{p'} W^{-p'}(r_1(x),r_2(y))w(x,y)\,dx\,dy \right)^{1/p'}.$$

Applying Proposition D together with the duality arguments we can get the following statement (*cf.* [6]).

Proposition E *Let* $1 < p < \infty$ *and let* v *and* w *be weights on* $G_1 \times G_2$ *such that* $w(x,y) = w_1(x)w_2(y)$, $\|w\|_{L^1(G_1 \times G_2)} = \infty$. *Then an integral operator* T *defined for functions from* $\mathcal{DR}(G_1 \times G_2)$ *is bounded from* $L_{\mathrm{dec},r}^p(w, G_1 \times G_2)$ *to* $L^p(v, G_1 \times G_2)$ *if and only if for all non-negative measurable g on* $G_1 \times G_2$,

$$\left(\iint_{G_1 \times G_2} \left(\iint_{B(e_1,r_1(x)) \times B(e_2,r_2(y))} (T^*g)(t,\tau)\,dt\,d\tau \right)^{p'} W^{-p'}(x,y)w(x,y)\,dx\,dy \right)^{1/p'}$$

$$\le C \left(\iint_{G_1 \times G_2} g^{q'}(x,y)v^{1-q'}(x,y)\,dx\,dy \right)^{1/q'}.$$

The next statements deal with the double Hardy-type operators defined on $G_1 \times G_2$:

$$(H^{a,b}f)(x,y) = \int_{B(e_1,ar_1(x))} \int_{B(e_2,br_2(x))} f(t,\tau)\,dt\,d\tau, \quad (x,y) \in G_1 \times G_2,$$

$$(\tilde{H}^{a,b}f)(x,y) = \int_{G_1 \setminus B(e_1,ar_1(x))} \int_{G_2 \setminus B(e_2,br_2(x))} f(t,\tau)\,dt\,d\tau, \quad (x,y) \in G_1 \times G_2,$$

$$(H_1^{a,b}f)(x,y) = \int_{B(e_1,ar_1(x))} \int_{G_2 \setminus B(e_2,br_2(y))} f(t,\tau)\,dt\,d\tau, \quad (x,y) \in G_1 \times G_2,$$

$$(H_2^{a,b}f)(x,y) = \int_{G_1 \setminus B(e_1,ar_1(x))} \int_{B(e_2,br_2(y))} f(t,\tau)\,dt\,d\tau, \quad (x,y) \in G_1 \times G_2.$$

Proposition 4.1 *Let* $1 < p \le q < \infty$. *Suppose that* v *and* w *be weights on* $G_1 \times G_2$ *such that either* $w(x,y) = w_1(x)w_2(y)$ *or* $v(x,y) = v_1(x)v_2(y)$. *Then:*

(i) *The operator $H^{a,b}$ is bounded from $L^p(w, G_1 \times G_2)$ to $L^q(v, G_1 \times G_2)$ if and only if*

$$
A := \sup_{t>0, \tau>0} \left(\int_{G_1 \backslash B(e_1,t)} \int_{G_2 \backslash B(e_2,\tau)} v(x,y)\, dx\, dy \right)^{1/q}
$$
$$
\times \left(\int_{B(e_1,at)} \int_{B(e_2,b\tau)} w^{1-p'}(x,y)\, dx\, dy \right)^{1/p'} < \infty.
$$

(ii) *The operator $\tilde{H}^{a,b}$ is bounded from $L^p(w, G_1 \times G_2)$ to $L^q(v, G_1 \times G_2)$ if and only if*

$$
\sup_{t>0, \tau>0} \left(\int_{B(e_1,t)} \int_{B(e_2,\tau)} v(x,y)\, dx\, dy \right)^{1/q} \left(\int_{G_1 \backslash B(e_1,at)} \int_{G_2 \backslash B(e_2,b\tau)} w^{1-p'}(x,y)\, dx\, dy \right)^{1/p'} < \infty.
$$

(iii) *The operator $H_1^{a,b}$ is bounded from $L^p(w, G_1 \times G_2)$ to $L^q(v, G_1 \times G_2)$ if and only if*

$$
\sup_{t>0, \tau>0} \left(\int_{G_1 \backslash B(e_1,t)} \int_{B(e_2,\tau)} v(x,y)\, dx\, dy \right)^{1/q} \left(\int_{B(e_1,at)} \int_{G_2 \backslash B(e_2,b\tau)} w^{1-p'}(x,y)\, dx\, dy \right)^{1/p'} < \infty.
$$

(iv) *The operator $H_2^{a,b}$ is bounded from $L^p(w, G_1 \times G_2)$ to $L^q(v, G_1 \times G_2)$ if and only if*

$$
\sup_{t>0, \tau>0} \left(\int_{B(e_1,t)} \int_{G_2 \backslash B(e_2,\tau)} v(x,y)\, dx\, dy \right)^{1/q} \left(\int_{G_1 \backslash B(e_1,at)} \int_{B(e_2,b\tau)} w^{1-p'}(x,y)\, dx\, dy \right)^{1/p'} < \infty.
$$

Proof Let $w(x,y) = w_1(x)w_2(y)$. Then the proposition follows in the same way as the appropriate statements regarding the Hardy operators defined on $\mathbb{R}_+^2$ in [13, 14] (see also Theorem 1.1.6 of [15]). If v is a product weight, *i.e.* $v(x,y) = v_1(x)v_2(y)$, then the result follows from the duality arguments. We give the proof, for example, for $H^{a,b}$ in the case when $w(x,y) = w_1(x)w_2(y)$.

First suppose that $S := \int_{G_2} w_2^{1-p'}(y)\, dy = \infty$. Let $\{x_k\}_{k=-\infty}^{+\infty}$ be a sequence of positive numbers for which the equality

$$
2^k = \int_{B(e_2,bx_k)} w_2^{1-p'}(y)\, dy \tag{4.1}
$$

holds for all $k \in \mathbb{Z}$. This equality follows because of the continuity in t of the integral over the ball $B(e_2, bt)$. It is clear that $\{x_k\}$ is increasing and $\mathbb{R}_+ = \bigcup_{k \in \mathbb{Z}} [x_k, x_{k+1})$. Moreover, it is easy to verify that

$$
2^k = \int_{B(e_2,bx_{k+1}) \backslash B(e_2,bx_k)} w_2^{1-p'}(y)\, dy.
$$

Let $f \geq 0$. We have

$$
\left\| H^{a,b}f \right\|_{L_v^q(G_1 \times G_2)}^q = \iint_{G_1 \times G_2} v(x,y) \left(H^{a,b}f \right)^q (x,y)\, dx\, dy \leq \sum_{k \in \mathbb{Z}} \int_{G_1} \int_{B(e_2,x_{k+1}) \backslash B(e_2,x_k)} v(x,y)
$$
$$
\times \left(\iint_{B(e_1,ar_1(x)) \times B(e_2,br_2(x))} f(t,\tau)\, dt\, d\tau \right)^q dx\, dy
$$
$$
\leq \sum_{k \in \mathbb{Z}} \int_{G_1} \left(\int_{B(e_2,x_{k+1}) \backslash B(e_2,x_k)} v(x,y)\, dy \right)
$$

$$\times \left(\int_{B(e_1,ar_1(x))} \left(\int_{B(e_2,bx_{k+1})} f(t,\tau)\,d\tau \right) dt \right)^q dx$$

$$= \sum_{k\in\mathbb{Z}} \int_{G_1} V_k(x) \left(\int_{B(e_1,ar_1(x))} F_k(t)\,dt \right)^q dx,$$

where

$$V_k(x) := \int_{B(e_2,x_{k+1})\setminus B(e_2,x_k)} v(x,y)\,dy; \qquad F_k(t) := \int_{B(e_2,bx_{k+1})} f(t,\tau)\,d\tau.$$

It is obvious that

$$A^q \geq \sup_{\substack{a>0 \\ j\in\mathbb{Z}}} \left(\int_{G_1\setminus B(e_1,t)} v_j(y)\,dy \right) \left(\iint_{B(e_1,at)\times B(e_2,bx_j)} w^{1-p'}(x,y)\,dx\,dy \right)^{q/p'}.$$

Hence, by Theorem A

$$\|H^{a,b}f\|^q_{L^q_v(G_1\times G_2)} \leq cA^q \sum_{j\in\mathbb{Z}} \left[\int_{G_1} w_1(x) \left(\int_{B(e_2,bx_j)} w_2^{1-p'}(y)\,dy \right)^{1-p} \left(F_k(x) \right)^p dx \right]^{q/p}$$

$$\leq cA^q \left[\int_{G_1} w_1(x) \sum_{j\in Z} \left(\int_{B(e_2,bx_j)} w_2^{1-p'}(y)\,dy \right)^{1-p} \right.$$

$$\times \left. \left(\sum_{k=-\infty}^{j} \int_{B(e_2,bx_{k+1})\setminus B(e_2,bx_k)} f(x,\tau)\,d\tau \right)^p dx \right]^{q/p}.$$

On the other hand, (4.1) yields

$$\sum_{k=n}^{+\infty} \left(\int_{B(e_2,bx_k)} w_2^{1-p'}(y)\,dy \right)^{1-p} \left(\sum_{k=-\infty}^{n} \int_{B(e_2,bx_{k+1})\setminus B(e_2,bx_k)} w_2^{1-p'}(y)\,dy \right)^{p-1}$$

$$= \sum_{k=n}^{+\infty} \left(\int_{B(e_2,bx_k)} w_2^{1-p'}(y)\,dy \right)^{1-p} \left(\int_{B(e_2,bx_{n+1})} w_2^{1-p'}(y)\,dy \right)^{p-1}$$

$$= \left(\sum_{k=n}^{+\infty} 2^{k(1-p)} \right) 2^{(n+1)(p-1)} \leq c$$

for all $n\in\mathbb{Z}$. Hence by the discrete Hardy inequality (see *e.g.* [16]) and Hölder's inequality we have

$$\|H^{a,b}f\|^q_{L^q_v(G_1\times G_2)} \leq cA^q \left[\int_{G_1} w_1(x) \sum_{j\in\mathbb{Z}} \left(\int_{B(e_2,bx_{j+1})\setminus B(e_2,bx_j)} w_2^{1-p'}(y)\,dy \right)^{1-p} \right.$$

$$\times \left. \left(\int_{B(e_2,bx_{j+1})\setminus B(e_2,bx_j)} f(x,\tau)\,d\tau \right)^p dx \right]^{q/p}$$

$$\leq cA^q \left[\int_{G_1} w_1(x) \sum_{j\in\mathbb{Z}} \left(\int_{B(e_2,bx_{j+1})\setminus B(e_2,bx_j)} w_2(\tau) f^p(x,\tau)\,d\tau \right) dx \right]^{q/p}$$

$$= cA^q \|f\|^q_{L^p_w(G_1\times G_2)}.$$

If $S < \infty$, then without loss of generality we can assume that $S = 1$. In this case we choose the sequence $\{x_k\}_{k=-\infty}^{0}$ for which (4.1) holds for all $k \in \mathbb{Z}_-$. Arguing as in the case $S = \infty$ and using slight modification of the discrete Hardy inequality (see also [15], Chapter 1 for similar arguments), we finally obtain the desired result.

Finally we notice that the part (i) can also be proved if we first establish the boundedness of the operator $(\mathcal{H}^{a,b}\varphi)(t, \tau) = \int_0^{at} \int_0^{b\tau} \varphi(s, r) \, ds \, dr$ in the spirit of Theorem 1.1.6 in [15] and then pass to the case of $G_1 \times G_2$ by Proposition A. □

The next statement will be useful for us.

Proposition 4.2 *Let* $1 < p \leq q < \infty$. *Assume that* v *and* w *are weights on* $G_1 \times G_2$. *Suppose that* $w(x, y) = w_1(x)w_2(y)$ *and that* $W_i(\infty) = \infty$, $i = 1, 2$. *Then the operator* $H^{1,1}$ *is bounded from* $L_{\mathrm{dec},r}^p(w, G_1 \times G_2)$ *to* $L^q(v, G_1 \times G_2)$ *if and only if the following four conditions are satisfied*:

(i)

$$
\sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} \int_{B(e_2, a_2)} w(x, y) \, dx \, dy \right)^{-1/p}
$$
$$
\times \left(\int_{B(e_1, a_1)} \int_{B(e_2, a_2)} r_1^{Q_1 q}(x) r_2(y)^{Q_2 q} v(x, y) \, dx \, dy \right)^{1/q} < \infty;
$$

(ii)

$$
\sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} \int_{B(e_2, a_2)} r_1^{Q_1 p'}(x) r_2(y)^{Q_2 p'} W^{-p'}(r_1(x), r_2(y)) w(x, y) \, dx \, dy \right)^{1/p'}
$$
$$
\times \left(\int_{G_1 \setminus B(e_1, a_1)} \int_{G_2 \setminus B(e_2, a_2)} v(x, y) \, dx \, dy \right)^{1/q} < \infty;
$$

(iii)

$$
\sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} w_1(r_1(x)) \, dx \right)^{-1/p} \left(\int_{B(e_2, a_2)} r_2(y)^{Q_2 p'} W_2^{-p'}(r_2(y)) w_2(y) \, dy \right)^{1/p'}
$$
$$
\times \left(\int_{B(e_1, a_1)} \int_{G_2 \setminus B(e_2, a_2)} r_1(x)^{Q_1 q} v(x, y) \, dx \, dy \right)^{1/q} < \infty;
$$

(iv)

$$
\sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} r_1(x)^{Q_1 p'} W_1^{-p'}(r_1(x)) w_1(x) \, dt_1 \right)^{1/p'} \left(\int_{B(e_2, a_2)} w_2(y) \, dy \right)^{-1/p}
$$
$$
\times \left(\int_{G_1 \setminus B(e_1, a_1)} \int_{B(e_2, a_2)} r_2(y)^{Q_2 q} v(x, y) \, dx \, dy \right)^{1/q} < \infty.
$$

Proof We follow the proof of Theorem 5.3 in [6]. First of all observe that by Proposition E, if w is a product weight, *i.e.*, $w(x_1, x_2) = w_1(x_1)w_2(x_2)$, such that $W_i(\infty) = \infty$, $i = 1, 2$, and v is any weight on $G_1 \times G_2$, then $H^{1,1}$ is bounded from $L_{\mathrm{dec},r}^p(w, G_1)$ to $L^q(v, G_2)$ if and only

if

$$
\left(\iint_{G_1 \times G_2} \left(\int_{B(e_1, r_1(x))} \int_{B(e_2, r_2(x))} \left[\int_{G_1 \setminus B(e_1, r_1(t))} \int_{G_1 \setminus B(e_2, r_2(\tau))} g(s, \varepsilon) \, ds \, d\varepsilon \right] dt \, d\tau \right)^{p'} \right.
$$

$$
\left. \times \, W^{-p'} \big(r_1(x), r_2(y) \big) w(x, y) \, dx \, dy \right)^{1/p'}
$$

$$
\leq c \left(\iint_{G_1 \times G_2} g^{q'}(x, y) v^{1-q'}(x, y) \, dx \, dy \right)^{1/q'}, \quad g \geq 0. \tag{4.2}
$$

Further, we have

$$
\iint_{B(e_1, r_1(x)) \times B(e_2, r_2(x))} \left(\int_{G_1 \setminus B(e_1, r_1(t))} \int_{G_2 \setminus B(e_2, r_2(t))} g(s, \varepsilon) \, ds \, d\varepsilon \right) dt \, d\tau
$$

$$
= \int_{B(e_1, r_1(x))} \int_{B(e_2, r_2(x))} r_1^{Q_1}(t) r_2^{Q_2}(\tau) g(t, \tau) \, dt \, d\tau
$$

$$
+ r_1^{Q_1}(x) \int_{G_1 \setminus B(e_1, r_1(x))} \int_{B(e_2, r_2(y))} r_2^{Q_2}(\tau) g(t, \tau) \, dt \, d\tau
$$

$$
+ r_2^{Q_2}(y) \int_{B(e_1, r_1(x))} \int_{G_2 \setminus B(e_2, r_2(y))} r_1^{Q_1}(t) g(t, \tau) \, dt \, d\tau
$$

$$
+ r_1^{Q_1}(x) r_2^{Q_2}(y) \int_{G_1 \setminus B(e_1, r_1(x))} \int_{G_2 \setminus B(e_2, r_2(y))} g(t, \tau) \, dt \, d\tau
$$

$$
=: I^{(1)}(x, y) + I^{(2)}(x, y) + I^{(3)}(x, y) + I^{(4)}(x, y).
$$

It is obvious that (4.2) holds if and only if

$$
\left(\iint_{G_1 \times G_2} \big(I^{(j)} \big)^{p'}(x, y) W^{-p'} \big(r_1(x), r_2(y) \big) w(x, y) \, dx \, dy \right)^{1/p'}
$$

$$
\leq c \left(\iint_{G_1 \times G_2} g^{q'}(x, y) v^{1-q'}(x, y) \, dx \, dy \right)^{1/q'} \tag{4.3}
$$

for $j = 1, 2, 3, 4$. By using Proposition 4.1 (part (i)) we find that

$$
\left(\iint_{G_1 \times G_2} \big(I^{(1)} \big)^{p'}(x, y) W^{-p'} \big(r_1(x), r_2(y) \big) w(x, y) \, dx \, dy \right)^{1/p'}
$$

$$
\leq c \left(\iint_{G_1 \times G_2} g^{q'}(x, y) v^{1-q'}(x, y) \, dx \, dy \right)^{1/q'}
$$

if and only if

$$
\left(\int_{G_1 \setminus B(e_1, t)} \int_{G_2 \setminus B(e_2, \tau)} W^{-p'} \big(r_1(x), r_2(y) \big) w(x, y) \, dx \, dy \right)^{1/p'}
$$

$$
\times \left(\iint_{B(e_1, t) \times B(e_2, \tau)} \left(\frac{v^{1-q'}(x, y)}{r_1^{Q_1 q'}(x) r_2^{Q_2 q'}(y)} \right)^{1-q} dx \, dy \right)^{1/q}
$$

$$
= c_p \left(\iint_{B(e_1, t) \times B(e_2, \tau)} w(x, y) \, dx \, dy \right)^{-1/p}
$$

$$\times \left(\iint_{B(e_1,t) \times B(e_2,\tau)} v(x,y) r_1^{Q_1 q}(x) r_2^{Q_2 q}(y)\, dx\, dy \right)^{1/q}$$

$$\leq C.$$

In the latter equality we used the equality

$$\left(\int_{G_i \setminus B(e_i,t)} W_i^{-p'}(r_i(x)) w_i(x)\, dx \right)^{1/p'} = \left(\int_{B(e_i,t)} w_i(x)\, dx \right)^{-1/p}, \quad i = 1, 2,$$

which is a direct consequence of integration by parts and Proposition A. Taking now Proposition 4.1 (part (ii)) into account we find that (4.3) holds for $j = 4$ if and only if condition (ii) is satisfied, while Proposition 4.1 (parts (iii) and (iv)) and the following observation:

$$\sup_{a_1,a_2>0} \left(\int_{G_1 \setminus B(e_1,a_1)} w_1(x) W_1^{-p'}(r_1(x))\, dx \right)^{1/p'} \left(\int_{B(e_2,a_2)} r_2^{p'Q_2}(y) W_2^{-p'}(r_2(y)) w_2(y)\, dy \right)^{1/p'}$$

$$\times \left(\int_{B(e_1,a_1)} \int_{G_2 \setminus B(e_2,a_2)} r_1^{Q_1 q}(x) v(x,y)\, dx\, dy \right)^{1/q}$$

$$= c_p \sup_{a_1,a_2>0} \left(\int_{B(e_1,a_1)} w_1(x)\, dx \right)^{-1/p} \left(\int_{B(e_2,a_2)} r_2^{Q_2 p'}(y) W_2^{-p'}(r_2(y)) w_2(y)\, dy \right)^{1/p'}$$

$$\times \left(\int_{B(e_1,a_1)} \int_{G_2 \setminus B(e_2,a_2)} r_1^{Qq}(x) v(x,y)\, dx\, dy \right)^{1/q} < \infty;$$

$$\sup_{a_1,a_2>0} \left(\int_{B(e_1,a_1)} r_1^{Q_1 p'}(x) W_1^{-p'}(r_1(x)) w_1(x)\, dx \right)^{1/p'} \left(\int_{G_2 \setminus B(e_2,a_2)} w_2(y) W_2^{-p'}(r_2(y))\, dy \right)^{1/p'}$$

$$\times \left(\int_{G_1 \setminus B(e_1,a_1)} \int_{B(e_2,a_2)} r_2^{Q_2 q}(y) v(x,y)\, dx\, dy \right)^{1/q}$$

$$= c_p \sup_{a_1,a_2>0} \left(\int_{B(e_1,a_1)} r_1^{Q_1 p'}(x) W_1^{-p'}(r_1(x)) w_1(x)\, dx \right)^{1/p'} \left(\int_{B(e_2,a_2)} w_2(t_2)\, dt_2 \right)^{-1/p}$$

$$\times \left(\int_{G_1 \setminus B(e_1,a_1)} \int_{B(e_2,a_2)} r_2^{Q_2 q}(y) v(x,y)\, dx\, dy \right)^{1/q} < \infty$$

yield (4.3) for $j = 2, 3$. □

Let

$$(J_{\alpha_1,\alpha_2} f)(x,y) = \int_{B(e_1,2c_0 r_1(x))} \int_{B(e_2,2c_0 r_2(y))} f(t,\tau) r_1(xt^{-1})^{\alpha_1-Q_1} r_2(y\tau^{-1})^{\alpha_2-Q_2}\, dt\, d\tau,$$

$$(J_{\alpha_1} S_{\alpha_2} f)(x,y) = \int_{B(e_1,2c_0 r_1(x))} \int_{G_2 \setminus B(e_2,2c_0 r_2(y))} f(t,\tau) r_1(xt^{-1})^{\alpha_1-Q_1} r_2(y\tau^{-1})^{\alpha_2-Q_2}\, dt\, d\tau,$$

$$(S_{\alpha_1} J_{\alpha_2} f)(x,y) = \int_{G_1 \setminus B(e_1,2c_0 r_1(x))} \int_{B(e_2,2c_0 r_2(y))} f(t,\tau) r_1(xt^{-1})^{\alpha_1-Q_1} r_2(y\tau^{-1})^{\alpha_2-Q_2}\, dt\, d\tau,$$

$$(S_{\alpha_1,\alpha_2} f)(x,y) = \int_{G_1 \setminus B(e_1,2c_0 r_1(x))} \int_{G_2 \setminus B(e_2,2c_0 r_2(y))} f(t,\tau) r_1(xt^{-1})^{\alpha_1-Q_1} r_2(y\tau^{-1})^{\alpha_2-Q_2}\, dt\, d\tau,$$

where c_0 is the constant from the triangle inequality for the homogeneous norms r_1 and r_2.

It is obvious that

$$I_{\alpha_1,\alpha_2} f = J_{\alpha_1,\alpha_2} f + J_{\alpha_1} S_{\alpha_2} f + S_{\alpha_1} J_{\alpha_2} f + S_{\alpha_1,\alpha_2} f. \tag{4.4}$$

Now we formulate the main result of this section.

Theorem 4.1 *Let* $1 < p \le q < \infty$. *Assume that* v *and* w *are weights on* $G_1 \times G_2$ *such that* $w(x,y) = w_1(x)w_2(y)$. *Suppose that either* $w_i \in DC^{\alpha_i,p}(G)$, $i = 1,2$, *or* $v \in DC(x) \cap DC(y)$. *Then the operator* I_{α_1,α_2} *is bounded from* $L^p_{\mathrm{dec},r}(w, G_1 \times G_2)$ *to* $L^q(v, G_1 \times G_2)$ *if and only if the following conditions are satisfied*:

(i)

$$A_1 := \sup_{a_1,a_2>0} \left(\int_{B(e_1,a_1)} \int_{B(e_2,a_2)} w(x,y)\,dx\,dy \right)^{-1/p}$$
$$\times \left(\int_{B(e_1,a_1)} \int_{B(e_2,a_2)} \left(r_1^{\alpha_1}(x) r_2^{\alpha_2}(y) \right)^q v(x,y)\,dx\,dy \right)^{1/q} < \infty;$$

(ii)

$$A_2 := \sup_{a_1,a_2>0} \left(\int_{B(e_1,a_1)} \int_{B(e_2,a_2)} r_1^{Q_1 p'}(x) r_2^{Q_2 p'}(y) W^{-p'}(r_1(x), r_2(y)) w(x,y)\,dx\,dy \right)^{1/p'}$$
$$\times \left(\int_{G_1 \backslash B(e_1,a_1)} \int_{G_2 \backslash B(e_2,a_2)} \left(r_1^{\alpha_1 - Q_1}(x) r_2^{\alpha_2 - Q_2}(y) \right)^q v(x,y)\,dx\,dy \right)^{1/q} < \infty;$$

(iii)

$$A_3 := \sup_{a_1,a_2>0} \left(\int_{B(e_1,a_1)} w_1(x)\,dx \right)^{-1/p} \left(\int_{B(e_2,a_2)} r_2^{Q_2 p'}(y) W_2^{-p'}(r_2(y)) w_2(y)\,dy \right)^{1/p'}$$
$$\times \left(\int_{B(e_1,a_1)} \int_{G_2 \backslash B(e_2,a_2)} r_1^{\alpha_1 q}(x) r_2^{q(\alpha_2 - Q_2)}(y) v(x,y)\,dx\,dy \right)^{1/q} < \infty;$$

(iv)

$$A_4 := \sup_{a_1,a_2>0} \left(\int_{B(e_1,a_1)} r_1^{Q_1 p'}(x) W_1^{-p'}(r_1(x)) w_1(x)\,dx \right)^{1/p'} \left(\int_{B(e_2,a_2)} w_2(y)\,dy \right)^{-1/p}$$
$$\times \left(\int_{G_1 \backslash B(e_1,a_1)} \int_{B(e_2,a_2)} r_1^{q(\alpha_1 - Q_1)}(x) r_2^{q\alpha_2}(y) v(x,y)\,dx\,dy \right)^{1/q} < \infty;$$

(v)

$$A_5 := \sup_{a_1,a_2>0} \left(\int_{G_1 \backslash B(e_1,a_1)} \int_{G_2 \backslash B(e_2,a_2)} r_1^{\alpha_1 p'}(x) r_2^{\alpha_2 p'}(y) W^{-p'}(r_1(x), r_2(y)) w(x,y)\,dx\,dy \right)^{1/p'}$$
$$\times \left(\int_{B(e_1,a_1)} \int_{B(e_2,a_2)} v(x,y)\,dx\,dy \right)^{1/q} < \infty;$$

(vi)

$$A_6 := \sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} w_1(x) \, dx \right)^{-1/p} \left(\int_{G_2 \backslash B(e_2, a_2)} r_2^{\alpha_2 p'}(y) W_2^{-p'}(r_2(y)) w_2(y) \, dy \right)^{1/p'}$$

$$\times \left(\int_{B(e_1, a_1)} \int_{B(e_2, a_2)} r_1^{\alpha_1 q}(x) v(x, y) \, dx \, dy \right)^{1/q} < \infty;$$

(vii)

$$A_7 := \sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} r_1^{Q_1 p'}(x) W_1^{-p'}(r_1(x)) w_1(x) \, dx \right)^{1/p'}$$

$$\times \left(\int_{G_2 \backslash B(e_2, a_2)} r_2^{\alpha_2 p'}(y) W_2^{-p'}(r_2(y)) w_2(y) \, dy \right)^{1/p'}$$

$$\times \left(\int_{G_1 \backslash B(e_1, a_1)} \int_{B(e_2, a_2)} r_1^{(\alpha_1 - Q_1)q}(x) v(x, y) \, dx \, dy \right)^{1/q} < \infty;$$

(viii)

$$A_8 := \sup_{a_1, a_2 > 0} \left(\int_{G_2 \backslash B(e_1, a_1)} r_1^{\alpha_1 p'}(x) W_1^{-p'}(r_1(x)) w_1(x) \, dx \right)^{-1/p} \left(\int_{B(e_2, a_2)} w_2(y) \, dy \right)^{1/p'}$$

$$\times \left(\int_{B(e_1, a_1)} \int_{B(e_2, a_2)} r_2^{\alpha_2 q}(x) v(x, y) \, dx \, dy \right)^{1/q} < \infty;$$

(ix)

$$A_9 := \sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} r_2^{Q_2 p'}(y) W_2^{-p'}(r_2(y)) w_2(y) \, dy \right)^{1/p'}$$

$$\times \left(\int_{G_1 \backslash B(e_1, a_1)} r_1^{\alpha_1 p'}(x) W_1^{-p'}(r_1(x)) w_1(x) \, dx \right)^{1/p'}$$

$$\times \left(\int_{B(e_1, a_1)} \int_{G_2 \backslash B(e_2, a_2)} r_2^{(\alpha_2 - Q_2)q}(y) v(x, y) \, dx \, dy \right)^{1/q} < \infty.$$

Proof Let us assume that $v \in DC(x) \cap DC(y)$. The case when $w_i \in DC^{\alpha_i, p}(G_i)$, $i = 1, 2$, follows analogously. By using representation (4.4) we have to investigate the boundedness of the operators $J_{\alpha_1, \alpha_2} f$, $J_{\alpha_1} S_{\alpha_2} f$, $S_{\alpha_1} J_{\alpha_2} f$, $S_{\alpha_1, \alpha_2} f$ separately.

Since $f \in \mathcal{DR}(G_1 \times G_2)$ by using the arguments of the proof of Proposition 3.1 it can be checked that

$$(J_{\alpha_1, \alpha_2} f)(x, y) \approx r_1^{\alpha_1 - Q_1}(x) r_2^{\alpha_2 - Q_2}(y) \iint_{B(e_1, r_1(x)) \times B(e_2, r_2(y))} f(t, \tau) \, dt \, d\tau$$

(see also [4] for a similar estimate in the case of the multiple one-sided potentials on $\mathbb{R}_+^2$). Hence, by Proposition 4.2 we find that J_{α_1, α_2} is bounded from $L^p_{\text{dec}, r}(w, G_1 \times G_2)$ to $L^q(v, G_1 \times G_2)$ if and only if conditions (i)-(iv) hold.

Observe that the dual to S_{α_1,α_2} is given by

$$\left(S^*_{\alpha_1,\alpha_2}g\right)(x,y) = \iint_{B(e_1,r_1(x)/(2c_0))\times B(e_2,r_2(y)/(2c_0))} g(t,\tau)r_1^{\alpha_1-Q_1}\left(xt^{-1}\right)r_2^{\alpha_2-Q_2}\left(y\tau^{-1}\right)dt\,d\tau.$$

Further, Tonelli's theorem together with Lemma 3.1 for both variables implies that there are positive constants c_1 and c_2 such that for all $(x,y)\in G_1\times G_2$ for the dual (see also the proof of Lemma 3.2),

$$r_1^{\alpha_1}(x)r_2^{\alpha_2}(y)\iint_{B(e_1,r_1(x)/(4c_0))\times B(e_2,r_2(y)/(4c_0))} g(t,\tau)\,dt\,d\tau$$

$$\leq c_1 \iint_{B(e_1,r_1(x))\times B(e_2,r_2(y))} \left(S^*_{\alpha_1,\alpha_2}g\right)(t,\tau)\,dt\,d\tau$$

$$\leq c_2 r_1^{\alpha_1}(x)r_2^{\alpha_2}(y)\iint_{B(e_1,r_1(x)/(2c_0))\times B(e_2,r_2(y)/(2c_0))} g(t,\tau)\,dt\,d\tau.$$

Applying Propositions 4.1 and 4.2 with the condition that $v\in DC(G_1\times G_2)$ we find that the operator S_{α_1,α_2} is bounded from $L^p_{\mathrm{dec},r}(w,G_1\times G_2)$ to $L^q(v,G_1\times G_2)$ if and only if condition (v) is satisfied.

Further, observe that due to the fact that f is radially decreasing with respect to the first variable we have

$$(J_{\alpha_1}S_{\alpha_2}f)(x,y)\approx(\mathcal{H}_{\alpha_1}S_{\alpha_2}f)(x,y),$$

where

$$(\mathcal{H}_{\alpha_1}S_{\alpha_2}f)(x,y) = r_1^{\alpha_1-Q_1}(x)\int_{B(e_1,2c_0r_1(x))}\int_{G_2\setminus B(e_2,2c_0r_2(y))} f(t,\tau)r_2\left(y\tau^{-1}\right)^{\alpha_2-Q_2}dt\,d\tau.$$

Dual of $\mathcal{H}_{\alpha_1}S_{\alpha_2}$ is given by

$$\left(\mathcal{H}^*_{\alpha_1}S^*_{\alpha_2}g\right)(t,\tau) = \int_{G_1\setminus B(e_1,r(t))}\int_{B(e_2,r(\tau)/2c_0)} r_1^{\alpha_1-Q_1}(s)r_2^{\alpha_2-Q_2}\left(\varepsilon\tau^{-1}\right)f(s,\varepsilon)\,ds\,d\varepsilon.$$

Further, we have

$$T(x,y) := \iint_{B(e_1,r_1(x))\times B(e_2,r_2(y))}\left(\mathcal{H}^*_{\alpha_1}S^*_{\alpha_2}g\right)(t,\tau)\,dt\,d\tau$$

$$= \iint_{B(e_1,r_1(x))\times B(e_2,r_2(y))}\left(\int_{B(e_1,r_1(x))\setminus B(e_1,r(t))}\int_{B(e_2,r(\tau)/2c_0)} r_1^{\alpha_1-Q_1}(s)\right.$$

$$\times r_2^{\alpha_2-Q_2}\left(\tau\varepsilon^{-1}\right)f(s,\varepsilon)\,ds\,d\varepsilon\Bigg)\,dt\,d\tau$$

$$+ \iint_{B(e_1,r_1(x))\times B(e_2,r_2(y))}\left(\int_{G_1\setminus B(e_1,r_1(x))}\int_{B(e_2,r(\tau)/(2c_0))} r_1^{\alpha_1-Q_1}(s)\right.$$

$$\times r_2^{\alpha_2-Q_2}\left(\tau\varepsilon^{-1}\right)f(s,\varepsilon)\,ds\,d\varepsilon\Bigg)\,dt\,d\tau$$

$$=: T_1(x,y) + T_2(x,y).$$

Tonelli's theorem for G_1, the inequality $r_2^{\alpha_2-Q_2}(\tau\varepsilon^{-1}) \geq cr_2^{\alpha_2-Q_2}(y)$ for $\tau \in B(e_2, r(y))$, $\varepsilon \in B(e_2, r(\tau)/(2c_0))$, and the fact that the integral $\int_{B(e_1,\tau)} f(s,\varepsilon)\,ds$ is decreasing in τ uniformly to ε yield

$$
T_1(x,y) \geq cr_2^{\alpha_2-Q_2}(y)
$$

$$
\times \int_{B(e_1,r_1(x))} \int_{B(e_2,r_2(y))\backslash B(e_2,r_2(y)/2)} \left(\int_{B(e_1,r_1(x))\backslash B(e_1,r(t))} \int_{B(e_2,r_2(y)/(4c_0))} r_1^{\alpha_1-Q_1}(s) \right.
$$

$$
\left. \times f(s,\varepsilon)\,ds\,d\varepsilon \right) dt\,d\tau
$$

$$
= cr_2^{\alpha_2}(y) \int_{B(e_1,r_1(x))} \left(\int_{B(e_1,r_1(x))\backslash B(e_1,r(t))} r_1^{\alpha_1-Q_1}(s) \right.
$$

$$
\times \left(\int_{B(e_2,r_2(y)/(4c_0))} f(s,\varepsilon)\,d\varepsilon \right) ds \right) dt
$$

$$
= cr_2^{\alpha_2}(y) \int_{B(e_1,r_1(x))} \left(\int_{B(e_1,r_1(x))\backslash B(e_1,r(t))} F(s,y)\,ds \right) dt
$$

$$
= cr_2^{\alpha_2}(y) \int_{B(e_1,r_1(x))} F(s,y) \left(\int_{B(e_1,r(s))} dt \right) ds
$$

$$
= cr_2^{\alpha_2}(y) \int_{B(e_1,r_1(x))} \int_{B(e_2,r_2(y)/(4c_0))} r_1^{\alpha_1}(s) f(t,\tau)\,d\varepsilon\,ds.
$$

Here we used the notation

$$
F(s,y) := \int_{B(e_2,r_2(y)/(4c_0))} f(s,\varepsilon)\,d\varepsilon.
$$

Taking into account that the function $\int_{B(e_2,2c_0\lambda)} f(s,\varepsilon)\,d\varepsilon$ is decreasing in λ uniformly to s, the inequality $r_2(\tau\varepsilon^{-1}) \leq cr_2(y)$ for $\tau \in B(e_2, r(y))$, $\varepsilon \in B(e_2, r(\tau)/(2c_0))$, and Tonelli's theorem for G_1 we find that

$$
T_2(x,y) \geq cr_1^{Q_1}(x) r_2^{\alpha_2}(y) \int_{G_1\backslash B(e_1,r_1(x))} \int_{B(e_2,r_2(y)/(4c_0))} r_1^{\alpha_1-Q_1}(s) f(t,\tau)\,d\varepsilon\,ds.
$$

To get the upper estimate, observe that Tonelli's theorem for $G_1 \times G_2$ and Lemma 3.1 for r_2 yield

$$
T_1(x,y) \leq \int_{B(e_1,r_1(x))} \int_{B(e_2,r_2(y)/(2c_0))} r_1^{\alpha_1-Q_1}(s) f(s,\varepsilon)
$$

$$
\times \left(\int_{B(e_1,r_1(s))} \int_{B(e_2,r_2(y))\backslash B(e_2,2c_0r_2(\varepsilon))} r_2^{\alpha_2-Q_2}(\tau\varepsilon^{-1})\,dt\,d\tau \right) ds\,d\varepsilon
$$

$$
\leq cr_2^{\alpha_2}(y) \iint_{B(e_1,r_1(x))\times B(e_2,r_2(y)/(2c_0))} r_1^{\alpha_1}(s) f(s,\varepsilon)\,ds\,d\varepsilon.
$$

Similarly,

$$
T_2(x,y) \leq cr_1^{Q_1}(x) r_2^{\alpha_2}(y) \iint_{G_1\backslash B(e_1,r_1(x))\times B(e_2,r_2(y)/(2c_0))} r_1^{\alpha_1-Q_1}(s) f(s,\varepsilon)\,ds\,d\varepsilon.
$$

Summarizing these estimates we see that there are positive constants c_1 and c_2 depending only on α_1, α_2, Q_1, and Q_2 such that

$$r_2^{\alpha_2}(y) \iint_{B(e_1,r_1(x))\times B(e_2,r_2(y)/(4c_0))} r_1^{\alpha_1}(s)f(s,\varepsilon)\,ds\,d\varepsilon$$

$$+ r_1^{Q_1}(x)r_2^{\alpha_2}(y) \iint_{G_1\setminus B(e_1,r_1(x))\times B(e_2,r_1(y)/(4c_0))} r_1^{\alpha_1-Q_1}(s)f(s,\varepsilon)\,ds\,d\varepsilon.$$

$$\leq c_1 T(x,y) \leq r_2^{\alpha_2}(y) \iint_{B(e_1,r_1(x))\times B(e_2,r2(y)/(2c_0))} r_1^{\alpha_1}(s)f(s,\varepsilon)\,ds\,d\varepsilon$$

$$+ r_1^{Q_1}(x)r_2^{\alpha_2}(y) \iint_{G_1\setminus B(e_1,r_1(x))\times B(e_2,r_1(y)/(2c_0))} r_1^{\alpha_1-Q_1}(s)f(s,\varepsilon)\,ds\,d\varepsilon.$$

Taking Propositions 4.1 and E into account together with the doubling condition for v with respect to the second variable we see that the operator $J_{\alpha_1}S_{\alpha_2}$ is bounded from $L^p_{\mathrm{dec},r}(w,G_1)$ to $L^q(v,G_2)$ if and only if the conditions (vi) and (vii) are satisfied.

In a similar manner (changing the roles of the first and second variables) we can see that $S_{\alpha_1}J_{\alpha_2}$ is bounded from $L^p_{\mathrm{dec},r}(w,G_1)$ to $L^q(v,G_2)$ if and only if the conditions (viii) and (ix) are satisfied. □

Theorem 4.1 and Remark 2.1 imply criteria for the trace inequality for I_{α_1,α_2}. Namely the following statement holds.

Theorem 4.2 *Let* $1 < p \leq q < \infty$ *and let* $0 < \alpha_i < Q_i/p$, $i = 1, 2$. *Then* I_{α_1,α_2} *is bounded from* $L^p_{\mathrm{dec},r}(G_1 \times G_2)$ *to* $L^q(v, G_1 \times G_2)$ *if and only if the following condition holds*:

$$B := \sup_{a_1,a_2>0} \left(\int_{B(e_1,a_1)} \int_{B(e_2,a_2)} v(x,y)\,dx\,dy\right)^{1/q} a_1^{\alpha_1-Q_1/p} a_2^{\alpha_2-Q_2/p} < \infty.$$

Proof Sufficiency is a consequence of the inequality $\max\{A_1,\ldots,A_9\} \leq cB$, while necessity follows immediately by taking the test function $f_{a_1,a_2}(x,y) = \chi_{B(e_1,a_1)}(x)\chi_{B(e_2,a_2)}(y)$, $a_1, a_2 > 0$. □

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