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Weighted boundedness of a multilinear operator associated to a singular integral operator with general kernels

Qiufen Feng*

*Correspondence:
fengqiufen@126.com
Changsha Commerce and Tourism
College, Changsha, 410116,
P.R. China

Abstract

In this paper, we establish the weighted sharp maximal function inequalities for a multilinear operator associated to a singular integral operator with general kernels. As an application, we obtain the boundedness of the operator on weighted Lebesgue and Morrey spaces.

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1 Introduction and preliminaries

As the development of singular integral operators (see [1–3]), their commutators and multilinear operators have been well studied. In [4–6], the authors prove that the commutators generated by singular integral operators and *BMO* functions are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chanillo (see [7]) proves a similar result when singular integral operators are replaced by fractional integral operators. In [8, 9], the boundedness for the commutators generated by singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(\mathbb{R}^n)$ ($1 < p < \infty$) spaces is obtained. In [10, 11], the boundedness for the commutators generated by singular integral operators and weighted *BMO* and Lipschitz functions on $L^p(\mathbb{R}^n)$ ($1 < p < \infty$) spaces is obtained. In [12], some singular integral operators with general kernel are introduced, and the boundedness for the operators and their commutators generated by *BMO* and Lipschitz functions is obtained (see [8, 12, 13]). Motivated by these, in this paper, we study the multilinear operator generated by the singular integral operator with general kernel and the weighted Lipschitz and *BMO* functions.

First, let us introduce some notations. Throughout this paper, Q denotes a cube of \mathbb{R}^n with sides parallel to the axes. For any locally integrable function f , the sharp maximal function of f is defined by

$$M^\#(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well known that (see [1, 2])

$$M^\#(f)(x) \approx \sup_{Q \ni x} \inf_{c \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy.$$

For $\eta > 0$, let $M_\eta^\#(f)(x) = M^\#(|f|^\eta)^{1/\eta}(x)$ and $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < n$, $1 \leq p < \infty$ and the non-negative weight function w , set

$$M_{\eta,p,w}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{w(Q)^{1-p\eta/n}} \int_Q |f(y)|^p w(y) \, dy \right)^{1/p}.$$

We write $M_{\eta,p,w}(f) = M_{p,w}(f)$ if $\eta = 0$.

The A_p weight is defined by (see [1]), for $1 < p < \infty$,

$$A_p = \left\{ w \in L^1_{\text{loc}}(\mathbb{R}^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x) \, dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} \, dx \right)^{p-1} < \infty \right\}$$

and

$$A_1 = \{ w \in L^p_{\text{loc}}(\mathbb{R}^n) : M(w)(x) \leq Cw(x), \text{ a.e.} \}.$$

Given a non-negative weight function w , for $1 \leq p < \infty$, the weighted Lebesgue space $L^p(\mathbb{R}^n, w)$ is the space of functions f such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty.$$

For $0 < \beta < 1$ and the non-negative weight function w , the weighted Lipschitz space $Lip_\beta(w)$ is the space of functions b such that

$$\|b\|_{Lip_\beta(w)} = \sup_Q \frac{1}{w(Q)^{\beta/n}} \left(\frac{1}{w(Q)} \int_Q |b(y) - b_Q|^p w(x)^{1-p} \, dy \right)^{1/p} < \infty,$$

and the weighted BMO space $BMO(w)$ is the space of functions b such that

$$\|b\|_{BMO(w)} = \sup_Q \left(\frac{1}{w(Q)} \int_Q |b(y) - b_Q|^p w(x)^{1-p} \, dy \right)^{1/p} < \infty.$$

Remark (1) It has been known that (see [11, 14]), for $b \in Lip_\beta(w)$, $w \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^k Q}| \leq Ck \|b\|_{Lip_\beta(w)} w(x) w(2^k Q)^{\beta/n}.$$

(2) It has been known that (see [2, 14]), for $b \in BMO(w)$, $w \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^k Q}| \leq Ck \|b\|_{BMO(w)} w(x).$$

(3) Let $b \in Lip_\beta(w)$ or $b \in BMO(w)$ and $w \in A_1$. By [14], we know that spaces $Lip_\beta(w)$ or $BMO(w)$ coincide and the norms $\|b\|_{Lip_\beta(w)}$ or $\|b\|_{BMO(w)}$ are equivalent with respect to different values $1 \leq p < \infty$.

Definition 1 Let φ be a positive, increasing function on R^+ , and there exists a constant $D > 0$ such that

$$\varphi(2t) \leq D\varphi(t) \quad \text{for } t \geq 0.$$

Let w be a non-negative weight function on R^n and f be a locally integrable function on R^n . Set, for $1 \leq p < \infty$,

$$\|f\|_{L^{p,\varphi}(w)} = \sup_{x \in R^n, d > 0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} |f(y)|^p w(y) dy \right)^{1/p},$$

where $Q(x, d) = \{y \in R^n : |x - y| < d\}$. The generalized weighted Morrey space is defined by

$$L^{p,\varphi}(R^n, w) = \{f \in L^1_{\text{loc}}(R^n) : \|f\|_{L^{p,\varphi}(w)} < \infty\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n, w) = L^{p,\delta}(R^n, w)$, which is the classical Morrey space (see [15, 16]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n, w) = L^p(R^n, w)$, which is the weighted Lebesgue space (see [1]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [17–20]).

In this paper, we study some singular integral operators as follows (see [12]).

Definition 2 Let $T : S \rightarrow S'$ be a linear operator such that T is bounded on $L^2(R^n)$ and has a kernel K , that is, there exists a locally integrable function $K(x, y)$ on $R^n \times R^n \setminus \{(x, y) \in R^n \times R^n : x = y\}$ such that

$$T(f)(x) = \int_{R^n} K(x, y)f(y) dy$$

for every bounded and compactly supported function f , where K satisfies

$$|K(x, y)| \leq C|x - y|^{-n},$$

$$\int_{2|y-z| < |x-y|} (|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)|) dx \leq C,$$

and there is a sequence of positive constant numbers $\{C_k\}$ such that for any $k \geq 1$,

$$\left(\int_{2^k|z-y| \leq |x-y| < 2^{k+1}|z-y|} (|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)|)^q dy \right)^{1/q}$$

$$\leq C_k(2^k|z - y|)^{-n/q'},$$

where $1 < q' < 2$ and $1/q + 1/q' = 1$. Moreover, let m be a positive integer and b be a function on R^n . Set

$$R_{m+1}(b; x, y) = b(x) - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha b(y)(x - y)^\alpha.$$

The multilinear operator related to the operator T is defined by

$$T^b(f)(x) = \int_{\mathbb{R}^n} \frac{R_{m+1}(b; x, y)}{|x - y|^m} K(x, y) f(y) dy.$$

Note that the classical Calderón-Zygmund singular integral operator satisfies Definition 1 (see [1–3, 5, 6]) and that the commutator $[b, T](f) = bT(f) - T(bf)$ is a particular operator of the multilinear operator T^b if $m = 0$. The multilinear operator T^b is a non-trivial generalization of the commutator. It is well known that commutators and multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [21–23]). The main purpose of this paper is to prove the sharp maximal inequalities for the multilinear operator T^b . As application, we obtain the weighted L^p -norm inequality and Morrey space boundedness for the multilinear operator T^b .

2 Theorems and lemmas

We shall prove the following theorems.

Theorem 1 *Let T be a singular integral operator as in Definition 2, the sequence $\{kC_k\} \in l^1$, $w \in A_1$, $0 < \eta < 1$, $q' < r < \infty$ and $D^\alpha b \in BMO(w)$ for all α with $|\alpha| = m$. Then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(\mathbb{R}^n)$ and $\tilde{x} \in \mathbb{R}^n$,*

$$M_\eta^\#(T^b(f))(\tilde{x}) \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).$$

Theorem 2 *Let T be a singular integral operator as in Definition 2, the sequence $\{kC_k\} \in l^1$, $w \in A_1$, $0 < \eta < 1$, $q' < r < \infty$, $0 < \beta < 1$ and $D^\alpha b \in Lip_\beta(w)$ for all α with $|\alpha| = m$. Then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(\mathbb{R}^n)$ and $\tilde{x} \in \mathbb{R}^n$,*

$$M_\eta^\#(T^b(f))(\tilde{x}) \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,r,w}(f)(\tilde{x}).$$

Theorem 3 *Let T be a singular integral operator as in Definition 2, the sequence $\{kC_k\} \in l^1$, $w \in A_1$, $q' < u < \infty$ and $D^\alpha b \in BMO(w)$ for all α with $|\alpha| = m$. Then T^b is bounded from $L^u(\mathbb{R}^n, w)$ to $L^u(\mathbb{R}^n, w^{1-u})$.*

Theorem 4 *Let T be a singular integral operator as in Definition 2, the sequence $\{kC_k\} \in l^1$, $w \in A_1$, $q' < u < \infty$, $0 < D < 2^n$ and $D^\alpha b \in BMO(w)$ for all α with $|\alpha| = m$. Then T^b is bounded from $L^{u,\varphi}(\mathbb{R}^n, w)$ to $L^{u,\varphi}(\mathbb{R}^n, w^{1-u})$.*

Theorem 5 *Let T be a singular integral operator as in Definition 2, the sequence $\{kC_k\} \in l^1$, $w \in A_1$, $0 < \beta < 1$, $q' < u < n/\beta$, $1/v = 1/u - \beta/n$ and $D^\alpha b \in Lip_\beta(w)$ for all α with $|\alpha| = m$. Then T^b is bounded from $L^u(\mathbb{R}^n, w)$ to $L^v(\mathbb{R}^n, w^{1-v})$.*

Theorem 6 *Let T be a singular integral operator as in Definition 2, the sequence $\{kC_k\} \in l^1$, $w \in A_1$, $0 < \beta < 1$, $0 < D < 2^n$, $q' < u < n/\beta$, $1/v = 1/u - \beta/n$ and $D^\alpha b \in Lip_\beta(w)$ for all α with $|\alpha| = m$. Then T^b is bounded from $L^{u,\varphi}(\mathbb{R}^n, w)$ to $L^{v,\varphi}(\mathbb{R}^n, w^{1-v})$.*

To prove the theorems, we need the following lemmas.

Lemma 1 (see [1, p.485]) *Let $0 < u < v < \infty$ and for any function $f \geq 0$, we define that, for $1/r = 1/u - 1/v$,*

$$\|f\|_{WL^v} = \sup_{\lambda > 0} \lambda \left| \{x \in R^n : f(x) > \lambda\} \right|^{1/v}, \quad N_{u,v}(f) = \sup_Q \|f \chi_Q\|_{L^u} / \|\chi_Q\|_{L^r},$$

where the sup is taken for all measurable sets Q with $0 < |Q| < \infty$. Then

$$\|f\|_{WL^v} \leq N_{u,v}(f) \leq (v/(v-u))^{1/u} \|f\|_{WL^v}.$$

Lemma 2 (see [12]) *Let T be a singular integral operator as in Definition 2, the sequence $\{C_k\} \in l^1$. Then T is bounded on $L^p(R^n, w)$ for $w \in A_p$ with $1 < p < \infty$, and weak (L^1, L^1) bounded.*

Lemma 3 (see [1, 7]) *Let $0 \leq \eta < n, 1 \leq s < u < n/\eta, 1/v = 1/u - \eta/n$ and $w \in A_1$. Then*

$$\|M_{\eta,s,w}(f)\|_{L^v(w)} \leq C \|f\|_{L^u(w)}.$$

Lemma 4 (see [1]) *Let $0 < p, \eta < \infty$ and $w \in \bigcup_{1 \leq r < \infty} A_r$. Then, for any smooth function f for which the left-hand side is finite,*

$$\int_{R^n} M_{\eta}(f)(x)^p w(x) dx \leq C \int_{R^n} M_{\eta}^{\#}(f)(x)^p w(x) dx.$$

Lemma 5 (see [17, 20]) *Let $0 < p < \infty, 0 < \eta < \infty, 0 < D < 2^n$ and $w \in A_1$. Then, for any smooth function f for which the left-hand side is finite,*

$$\|M_{\eta}(f)\|_{L^{p,\varphi}(w)} \leq C \|M_{\eta}^{\#}(f)\|_{L^{p,\varphi}(w)}.$$

Lemma 6 (see [17, 20]) *Let $0 \leq \eta < n, 0 < D < 2^n, 1 \leq s < u < n/\eta, 1/v = 1/u - \eta/n$ and $w \in A_1$. Then*

$$\|M_{\eta,s,w}(f)\|_{L^v,\varphi(w)} \leq C \|f\|_{L^u,\varphi(w)}.$$

Lemma 7 (see [22]) *Let b be a function on R^n and $D^{\alpha} A \in L^u(R^n)$ for all α with $|\alpha| = m$ and any $u > n$. Then*

$$|R_m(b; x, y)| \leq C |x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x, y)} |D^{\alpha} b(z)|^u dz \right)^{1/u},$$

where \tilde{Q} is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

3 Proofs of theorems

Proof of Theorem 1 It suffices to prove for $f \in C_0^{\infty}(R^n)$ and some constant C_0 that the following inequality holds:

$$\left(\frac{1}{|Q|} \int_Q |T^b(f)(x) - C_0|^n dx \right)^{1/n} \leq C \sum_{|\alpha|=m} \|D^{\alpha} b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}(x) = b(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha b)_{\tilde{Q}} x^\alpha$, then $R_m(b; x, y) = R_m(\tilde{b}; x, y)$ and $D^\alpha \tilde{b} = D^\alpha b - (D^\alpha b)_{\tilde{Q}}$ for $|\alpha| = m$. We write, for $f_1 = f \chi_{\tilde{Q}}$ and $f_2 = f \chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned} T^{\tilde{b}}(f)(x) &= \int_{R^n} \frac{R_m(\tilde{b}; x, y)}{|x - y|^m} K(x, y) f_1(y) dy \\ &\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{(x - y)^\alpha D^\alpha \tilde{b}(y)}{|x - y|^m} K(x, y) f_1(y) dy \\ &\quad + \int_{R^n} \frac{R_{m+1}(\tilde{b}; x, y)}{|x - y|^m} K(x, y) f_2(y) dy \\ &= T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x - \cdot|^m} f_1\right) - T\left(\sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x - \cdot)^\alpha D^\alpha \tilde{b}}{|x - \cdot|^m} f_1\right) + T^{\tilde{b}}(f_2)(x), \end{aligned}$$

then

$$\begin{aligned} &\left(\frac{1}{|Q|} \int_Q |T^{\tilde{b}}(f)(x) - T^{\tilde{b}}(f_2)(x_0)|^\eta dx\right)^{1/\eta} \\ &\leq C \left(\frac{1}{|Q|} \int_Q \left|T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x - \cdot|^m} f_1\right)\right|^\eta dx\right)^{1/\eta} \\ &\quad + C \left(\frac{1}{|Q|} \int_Q \left|T\left(\sum_{|\alpha|=m} \frac{(x - \cdot)^\alpha D^\alpha \tilde{b}}{|x - \cdot|^m} f_1\right)\right|^\eta dx\right)^{1/\eta} \\ &\quad + C \left(\frac{1}{|Q|} \int_Q |T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0)|^\eta dx\right)^{1/\eta} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

For I_1 , noting that $w \in A_1$, w satisfies the reverse of Hölder's inequality,

$$\left(\frac{1}{|Q|} \int_Q w(x)^{p_0} dx\right)^{1/p_0} \leq \frac{C}{|Q|} \int_Q w(x) dx$$

for all cube Q and some $1 < p_0 < \infty$ (see [1]). We take $u = rp_0/(r + p_0 - 1)$ in Lemma 7 and have $1 < u < r$ and $p_0 = u(r - 1)/(r - u)$. Then by Lemma 7 and Hölder's inequality, we get

$$\begin{aligned} &|R_m(\tilde{b}; x, y)| \\ &\leq C|x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x, y)} |D^\alpha \tilde{b}(z)|^u dz\right)^{1/u} \\ &\leq C|x - y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/u} \left(\int_{\tilde{Q}(x, y)} |D^\alpha \tilde{b}(z)|^u w(z)^{u(1-r)/r} w(z)^{u(r-1)/r} dz\right)^{1/u} \\ &\leq C|x - y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/u} \left(\int_{\tilde{Q}(x, y)} |D^\alpha \tilde{b}(z)|^r w(z)^{1-r} dz\right)^{1/r} \\ &\quad \times \left(\int_{\tilde{Q}(x, y)} w(z)^{u(r-1)/(r-u)} dz\right)^{(r-u)/ru} \end{aligned}$$

$$\begin{aligned}
 &\leq C|x-y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/u} \|D^\alpha b\|_{BMO(w)} w(\tilde{Q})^{1/r} |\tilde{Q}|^{(r-u)/ru} \\
 &\quad \times \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} w(z)^{p_0} dz \right)^{(r-u)/ru} \\
 &\leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} |\tilde{Q}|^{-1/u} w(\tilde{Q})^{1/r} |\tilde{Q}|^{1/u-1/r} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} w(z) dz \right)^{(r-1)/r} \\
 &\leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} |\tilde{Q}|^{-1/u} w(\tilde{Q})^{1/r} |\tilde{Q}|^{1/u-1/r} w(\tilde{Q})^{1-1/r} |\tilde{Q}|^{1/r-1} \\
 &\leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} \frac{w(\tilde{Q})}{|\tilde{Q}|} \\
 &\leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}).
 \end{aligned}$$

Thus, by the L^s -boundedness of T (see Lemma 2) for $1 < s < r$ and $w \in A_1 \subseteq A_{r/s}$, we obtain

$$\begin{aligned}
 I_1 &\leq \frac{C}{|Q|} \int_Q \left| T \left(\frac{R_m(\tilde{b}; x, \cdot)}{|x - \cdot|^m} f_1 \right) \right| dx \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \left(\frac{1}{|Q|} \int_{R^n} |T(f_1)(x)|^s dx \right)^{1/s} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) |Q|^{-1/s} \left(\int_{R^n} |f_1(x)|^s dx \right)^{1/s} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) |Q|^{-1/s} \left(\int_Q |f(x)|^s w(x)^{s/r} w(x)^{-s/r} dx \right)^{1/s} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) |Q|^{-1/s} \left(\int_Q |f(x)|^r w(x) dx \right)^{1/r} \left(\int_Q w(x)^{-s/(r-s)} dx \right)^{(r-s)/rs} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) |Q|^{-1/s} w(\tilde{Q})^{1/r} \left(\frac{1}{w(\tilde{Q})} \int_{\tilde{Q}} |f(x)|^r w(x) dx \right)^{1/r} \\
 &\quad \times \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x)^{-s/(r-s)} dx \right)^{(r-s)/rs} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x) dx \right)^{1/r} |\tilde{Q}|^{1/s} w(\tilde{Q})^{-1/r} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).
 \end{aligned}$$

For I_2 , by the weak (L^1, L^1) boundedness of T (see Lemma 2) and Kolmogoro's inequality (see Lemma 1), we obtain

$$\begin{aligned}
 I_2 &\leq C \sum_{|\alpha|=m} \left(\frac{1}{|Q|} \int_Q |T(D^\alpha \tilde{b} f_1)(x)|^\eta dx \right)^{1/\eta} \\
 &\leq C \sum_{|\alpha|=m} \frac{|Q|^{1/\eta-1} \|T(D^\alpha \tilde{b} f_1)\chi_Q\|_{L^\eta}}{\| \chi_Q \|_{L^\eta/(1-\eta)}} \\
 &\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \|T(D^\alpha \tilde{b} f_1)\|_{WL^1}
 \end{aligned}$$

$$\begin{aligned}
 &\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \int_{R^n} |D^\alpha \tilde{b}(x) f_1(x)| \, dx \\
 &\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \int_{\tilde{Q}} |D^\alpha b(x) - (D^\alpha b)_{\tilde{Q}}| w(x)^{-1/r} |f(x)| w(x)^{1/r} \, dx \\
 &\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \left(\int_{\tilde{Q}} |(D^\alpha b(x) - (D^\alpha b)_{\tilde{Q}})|^{r'} w(x)^{1-r'} \, dx \right)^{1/r'} \left(\int_{\tilde{Q}} |f(x)|^r w(x) \, dx \right)^{1/r} \\
 &\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \|D^\alpha b\|_{BMO(w)} w(\tilde{Q})^{1/r'} w(\tilde{Q})^{1/r} \left(\frac{1}{w(\tilde{Q})} \int_{\tilde{Q}} |f(x)|^r w(x) \, dx \right)^{1/r} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} \frac{w(\tilde{Q})}{|\tilde{Q}|} M_{r,w}(f)(\tilde{x}) \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).
 \end{aligned}$$

For I_3 , noting that $|x - y| \approx |x_0 - y|$ for $x \in Q$ and $y \in R^n \setminus Q$, we write

$$\begin{aligned}
 &|T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0)| \\
 &\leq \int_{R^n} |R_m(\tilde{b}; x, y) - R_m(\tilde{b}; x_0, y)| \frac{|K(x, y)|}{|x - y|^m} |f_2(y)| \, dy \\
 &\quad + \int_{R^n} \left| \frac{K(x, y)}{|x - y|^m} - \frac{K(x_0, y)}{|x_0 - y|^m} \right| |R_m(\tilde{b}; x_0, y)| |f_2(y)| \, dy \\
 &\quad + \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \left| \frac{K(x, y)(x - y)^\alpha}{|x - y|^m} - \frac{K(x_0, y)(x_0 - y)^\alpha}{|x_0 - y|^m} \right| |D^\alpha \tilde{b}(y)| |f_2(y)| \, dy \\
 &= I_3^{(1)}(x) + I_3^{(2)}(x) + I_3^{(3)}(x).
 \end{aligned}$$

For $I_3^{(1)}(x)$, by the formula (see [22])

$$R_m(\tilde{b}; x, y) - R_m(\tilde{b}; x_0, y) = \sum_{|\gamma| < m} \frac{1}{\gamma!} R_{m-|\gamma|}(D^\gamma \tilde{b}; x, x_0)(x - y)^\gamma$$

and Lemma 7, we have, similar to the proof of I_1 ,

$$|R_m(\tilde{b}; x, y) - R_m(\tilde{b}; x_0, y)| \leq C \sum_{|\gamma| < m} \sum_{|\alpha|=m} |x - x_0|^{m-|\gamma|} |x - y|^{|\gamma|} \|D^\alpha b\|_{BMO(w)} w(\tilde{x})$$

and

$$|R_m(\tilde{b}; x_0, y)| \leq C \sum_{|\alpha|=m} |x - x_0|^m \|D^\alpha b\|_{BMO(w)} w(\tilde{x}).$$

Thus, by $w \in A_1 \subseteq A_r$, we get

$$\begin{aligned}
 I_3^{(1)}(x) &\leq \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |R_m(\tilde{b}; x, y) - R_m(\tilde{b}; x_0, y)| \frac{|K(x, y)|}{|x - y|^m} |f(y)| \, dy \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} \frac{|x - x_0|}{|x_0 - y|^{m+1}} |f(y)| \, dy
 \end{aligned}$$

$$\begin{aligned}
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \sum_{k=1}^{\infty} \frac{d}{(2^k d)^{n+1}} \int_{2^k \tilde{Q}} |f(y)| w(y)^{1/r} w(y)^{-1/r} dy \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \sum_{k=1}^{\infty} \frac{d}{(2^k d)^{n+1}} \left(\int_{2^k \tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \\
 &\quad \times \left(\int_{2^k \tilde{Q}} w(y)^{-1/(r-1)} dy \right)^{(r-1)/r} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \sum_{k=1}^{\infty} \frac{d}{(2^k d)^{n+1}} w(2^k \tilde{Q})^{1/r} \\
 &\quad \times \left(\frac{1}{w(2^k \tilde{Q})} \int_{2^k \tilde{Q}} |f(y)|^r w(y) dx \right)^{1/r} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y)^{-1/(r-1)} dy \right)^{(r-1)/r} \\
 &\quad \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y) dy \right)^{1/r} |2^k \tilde{Q}| w(2^k \tilde{Q})^{-1/r} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}) \sum_{k=1}^{\infty} 2^{-k} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).
 \end{aligned}$$

For $I_3^{(2)}(x)$, we take $1 < p < \infty$ such that $1/p + 1/q + 1/r = 1$. Recalling $r > q'$ and $w \in A_1 \subseteq A_{r/p+1}$, we get

$$\begin{aligned}
 I_3^{(2)}(x) &\leq \sum_{k=0}^{\infty} \int_{2^{k+1} \tilde{Q} \setminus 2^k \tilde{Q}} |K(x, y) - K(x_0, y)| \frac{|R_m(\tilde{b}; x_0, y)|}{|x - y|^m} |f(y)| w(y)^{1/r} w(y)^{-1/r} dy \\
 &\quad + \int_{2^{k+1} \tilde{Q} \setminus 2^k \tilde{Q}} \left| \frac{1}{|x - y|^m} - \frac{1}{|x_0 - y|^m} \right| |K(x_0, y)| |R_m(\tilde{b}; x_0, y)| |f(y)| dy \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \sum_{k=0}^{\infty} \left(\int_{2^{k+1} \tilde{Q} \setminus 2^k \tilde{Q}} |K(x, y) - K(x_0, y)|^q dy \right)^{1/q} \\
 &\quad \times \left(\int_{2^{k+1} \tilde{Q} \setminus 2^k \tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \left(\int_{2^{k+1} \tilde{Q} \setminus 2^k \tilde{Q}} w(y)^{-p/r} dy \right)^{1/p} \\
 &\quad + C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \sum_{k=0}^{\infty} \int_{2^{k+1} \tilde{Q} \setminus 2^k \tilde{Q}} \frac{|x - x_0|}{|x_0 - y|^{n+1}} |f(y)| w(y)^{1/r} w(y)^{-1/r} dy \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \sum_{k=1}^{\infty} C_k |2^k Q|^{-1/q'} w(2^k \tilde{Q})^{1/r} \\
 &\quad \times \left(\frac{1}{w(2^k \tilde{Q})} \int_{2^k \tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y)^{-p/r} dy \right)^{1/p} \\
 &\quad \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y) dy \right)^{1/r} |2^k \tilde{Q}|^{1/r+1/p} w(2^k \tilde{Q})^{-1/r} \\
 &\quad + C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \sum_{k=1}^{\infty} \frac{d}{(2^k d)^{n+1}} \\
 &\quad \times \left(\int_{2^k \tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \left(\int_{2^k \tilde{Q}} w(y)^{-1/(r-1)} dy \right)^{(r-1)/r}
 \end{aligned}$$

$$\begin{aligned} &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \sum_{k=1}^{\infty} (C_k + 2^{-k}) \left(\frac{1}{w(2^k \tilde{Q})} \int_{2^k \tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}). \end{aligned}$$

Similarly, we have, for $r < p'_1 < \infty$, $1 < s_1, s_2 < \infty$ with $1/p_1 + 1/q + 1/r + 1/s_1 = 1$ and $1/q + 1/r + 1/s_2 = 1$,

$$\begin{aligned} I_3^{(3)}(x) &\leq \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x, y) - K(x_0, y)| \frac{|(x-y)^\alpha|}{|x-y|^m} \\ &\quad \times |D^\alpha b(y) - (D^\alpha b)_{2^{k+1}\tilde{Q}}| w(y)^{(1-p_1)/p_1} |f(y)| w(y)^{1/r} w(y)^{(p_1-1)/p_1-1/r} dy \\ &\quad + \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x, y) - K(x_0, y)| \frac{|(x-y)^\alpha|}{|x-y|^m} \\ &\quad \times |(D^\alpha b)_{2^{k+1}\tilde{Q}} - (D^\alpha b)_{\tilde{Q}}| |f(y)| w(y)^{1/r} w(y)^{-1/r} dy \\ &\quad + \sum_{|\alpha|=m} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} \left| \frac{(x-y)^\alpha}{|x-y|^m} - \frac{(x_0-y)^\alpha}{|x_0-y|^m} \right| |K(x_0, y)| |f(y)| |D^\alpha \tilde{b}(y)| dy \\ &\leq C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x, y) - K(x_0, y)|^q dy \right)^{1/q} \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \\ &\quad \times \left(\int_{2^{k+1}\tilde{Q}} |D^\alpha b(y) - (D^\alpha b)_{2^{k+1}\tilde{Q}}|^{p_1} w(y)^{1-p_1} dy \right)^{1/p_1} \\ &\quad \times \left(\int_{2^{k+1}\tilde{Q}} w(y)^{-(1/r-1/p'_1)s_1} dy \right)^{1/s_1} \\ &\quad + C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x, y) - K(x_0, y)|^q dy \right)^{1/q} \\ &\quad \times \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \\ &\quad \times |(D^\alpha b)_{2^{k+1}\tilde{Q}} - (D^\alpha b)_{\tilde{Q}}| \left(\int_{2^{k+1}\tilde{Q}} w(y)^{-s_2/r} dy \right)^{1/s_2} \\ &\quad + C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} \frac{|x-x_0|}{|x_0-y|^{n+1}} \\ &\quad \times |f(y)| |D^\alpha b(y) - (D^\alpha b)_{2^{k+1}\tilde{Q}}| w(y)^{1/r} w(y)^{-1/r} dy \\ &\quad + C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} \frac{|x-x_0|}{|x_0-y|^{n+1}} \\ &\quad \times |f(y)| |(D^\alpha b)_{2^{k+1}\tilde{Q}} - (D^\alpha b)_{\tilde{Q}}| w(y)^{1/r} w(y)^{-1/r} dy \\ &\leq C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} C_k |2^k Q|^{-1/q'} \|D^\alpha b\|_{BMO(w)} w(2^k \tilde{Q})^{1/p_1} w(2^k \tilde{Q})^{1/r} \\ &\quad \times \left(\frac{1}{w(2^k \tilde{Q})} \int_{2^k \tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \end{aligned}$$

$$\begin{aligned}
 & \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y) dy \right)^{1/r-1/p'_1} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y)^{-(1/r-1/p'_1)s_1} dy \right)^{1/s_1} \\
 & \times |2^k \tilde{Q}|^{1/s_1+1/r-1/p'_1} w(2^k \tilde{Q})^{-1/r+1/p'_1} \\
 & + C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} C_k |2^k Q|^{-1/q'} k \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) w(2^k \tilde{Q})^{1/r} \\
 & \times \left(\frac{1}{w(2^k \tilde{Q})} \int_{2^k \tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y) dy \right)^{1/r} \\
 & \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y)^{-s_2/r} dy \right)^{1/s_2} |2^k \tilde{Q}|^{1/s_2+1/r} w(2^k \tilde{Q})^{-1/r} \\
 & + C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} \frac{d}{(2^k d)^{n+1}} \left(\int_{2^k \tilde{Q}} |D^\alpha b(y) - (D^\alpha b)_{2^k \tilde{Q}}|^{r'} w(y)^{1-r'} dy \right)^{1/r'} \\
 & \times \left(\int_{2^{k+1} \tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \\
 & + C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} \frac{d}{(2^k d)^{n+1}} k \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \left(\int_{2^k \tilde{Q}} w(y)^{-r'/r} dy \right)^{1/r'} \\
 & \times \left(\int_{2^{k+1} \tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \\
 \leq & \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} \sum_{k=1}^{\infty} C_k \frac{w(2^k \tilde{Q})}{|2^k \tilde{Q}|} \left(\frac{1}{w(2^k \tilde{Q})} \int_{2^k \tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \\
 & + \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \sum_{k=1}^{\infty} k C_k \left(\frac{1}{w(2^k \tilde{Q})} \int_{2^k \tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \\
 & + C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} \sum_{k=1}^{\infty} 2^{-k} \frac{w(2^k \tilde{Q})}{|2^k \tilde{Q}|} \left(\frac{1}{w(2^k \tilde{Q})} \int_{2^k \tilde{Q}} |f(y)|^r w(y) dx \right)^{1/r} \\
 & + C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \sum_{k=1}^{\infty} k 2^{-k} \left(\frac{1}{w(2^k \tilde{Q})} \int_{2^k \tilde{Q}} |f(y)|^r w(y) dx \right)^{1/r} \\
 \leq & C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).
 \end{aligned}$$

Thus

$$I_3 \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).$$

These complete the proof of Theorem 1. □

Proof of Theorem 2 It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0 that the following inequality holds:

$$\left(\frac{1}{|Q|} \int_Q |T^b(f)(x) - C_0|^\eta dx \right)^{1/\eta} \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,r,w}(f)(\tilde{x}).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Similar to the proof of Theorem 1, we have, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned} & \left(\frac{1}{|Q|} \int_Q |T^{\tilde{b}}(f)(x) - T^{\tilde{b}}(f_2)(x_0)|^\eta dx \right)^{1/\eta} \\ & \leq C \left(\frac{1}{|Q|} \int_Q \left| T \left(\frac{R_m(\tilde{b}; x, \cdot)}{|x - \cdot|^m} f_1 \right) \right|^\eta dx \right)^{1/\eta} \\ & \quad + C \left(\frac{1}{|Q|} \int_Q \left| T \left(\sum_{|\alpha|=m} \frac{(x - \cdot)^\alpha D^\alpha \tilde{b}}{|x - \cdot|^m} f_1 \right) \right|^\eta dx \right)^{1/\eta} \\ & \quad + C \left(\frac{1}{|Q|} \int_Q |T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0)|^\eta dx \right)^{1/\eta} \\ & = J_1 + J_2 + J_3. \end{aligned}$$

For J_1 and J_2 , by using the same argument as in the proof of Theorem 1, we get

$$\begin{aligned} & |R_m(\tilde{b}; x, y)| \\ & \leq C|x - y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/q} \left(\int_{\tilde{Q}(x,y)} |D^\alpha \tilde{b}(z)|^q w(z)^{q(1-r)/r} w(z)^{q(r-1)/r} dz \right)^{1/q} \\ & \leq C|x - y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/q} \left(\int_{\tilde{Q}(x,y)} |D^\alpha \tilde{b}(z)|^r w(z)^{1-r} dz \right)^{1/r} \\ & \quad \times \left(\int_{\tilde{Q}(x,y)} w(z)^{q(r-1)/(r-q)} dz \right)^{(r-q)/rq} \\ & \leq C|x - y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/q} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{Q})^{\beta/n+1/r} |\tilde{Q}|^{(r-q)/rq} \\ & \quad \times \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} w(z)^{p_0} dz \right)^{(r-q)/rq} \\ & \leq C|x - y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} |\tilde{Q}|^{-1/q} w(\tilde{Q})^{\beta/n+1/r} |\tilde{Q}|^{1/q-1/r} \\ & \quad \times \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} w(z) dz \right)^{(r-1)/r} \\ & \leq C|x - y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} |\tilde{Q}|^{-1/q} w(\tilde{Q})^{\beta/n+1/r} |\tilde{Q}|^{1/q-1/r} w(\tilde{Q})^{1-1/r} |\tilde{Q}|^{1/r-1} \\ & \leq C|x - y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \frac{w(\tilde{Q})^{\beta/n+1}}{|\tilde{Q}|} \\ & \leq C|x - y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{Q})^{\beta/n} w(\tilde{x}). \end{aligned}$$

Thus

$$\begin{aligned} J_1 & \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{Q})^{\beta/n} w(\tilde{x}) |Q|^{-1/s} \left(\int_{R^n} |f_1(x)|^s dx \right)^{1/s} \\ & \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{Q})^{\beta/n} w(\tilde{x}) |Q|^{-1/s} \end{aligned}$$

$$\begin{aligned}
 & \times \left(\int_{\tilde{Q}} |f(x)|^r w(x) dx \right)^{1/r} \left(\int_{\tilde{Q}} w(x)^{-s/(r-s)} dx \right)^{(r-s)/rs} \\
 & \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x}) |\tilde{Q}|^{-1/s} w(\tilde{Q})^{1/r} \left(\frac{1}{w(\tilde{Q})^{1-r\beta/n}} \int_{\tilde{Q}} |f(x)|^r w(x) dx \right)^{1/r} \\
 & \quad \times \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x)^{-s/(r-s)} dx \right)^{(r-s)/rs} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x) dx \right)^{1/r} |\tilde{Q}|^{1/s} w(\tilde{Q})^{-1/r} \\
 & \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,r,w}(f)(\tilde{x}), \\
 J_2 & \leq C \sum_{|\alpha|=m} \frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^\alpha b(x) - (D^\alpha b)_{\tilde{Q}}| w(x)^{-1/r} |f(x)| w(x)^{1/r} dx \\
 & \leq C \sum_{|\alpha|=m} \frac{1}{|\tilde{Q}|} \left(\int_{\tilde{Q}} |(D^\alpha b(x) - (D^\alpha b)_{\tilde{Q}})|^{r'} w(x)^{1-r'} dx \right)^{1/r'} \left(\int_{\tilde{Q}} |f(x)|^r w(x) dx \right)^{1/r} \\
 & \leq C \sum_{|\alpha|=m} \frac{1}{|\tilde{Q}|} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{Q})^{\beta/n+1/r'} w(\tilde{Q})^{1/r-\beta/n} \left(\frac{1}{w(\tilde{Q})^{1-r\beta/n}} \int_{\tilde{Q}} |f(x)|^r w(x) dx \right)^{1/r} \\
 & \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \frac{w(\tilde{Q})}{|\tilde{Q}|} M_{\beta,r,w}(f)(\tilde{x}) \\
 & \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,r,w}(f)(\tilde{x}).
 \end{aligned}$$

For J_3 , we have

$$\begin{aligned}
 & |R_m(\tilde{b}; x, y) - R_m(\tilde{b}; x_0, y)| \\
 & \leq C \sum_{|\gamma|<m} \sum_{|\alpha|=m} |x - x_0|^{m-|\gamma|} |x - y|^{|\gamma|} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x}) w(2^k \tilde{Q})
 \end{aligned}$$

and

$$|R_m(\tilde{b}; x, y)| \leq C \sum_{|\alpha|=m} |x - x_0|^m \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x}) w(2^k \tilde{Q}).$$

Thus, for $1 < p < \infty$ with $1/p + 1/q + 1/r = 1$ and $r < p'_1 < \infty$, $1 < s_1, s_2 < \infty$ with $1/p_1 + 1/q + 1/r + 1/s_1 = 1$ and $1/q + 1/r + 1/s_2 = 1$, we obtain

$$\begin{aligned}
 & |T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0)| \\
 & \leq \int_{R^n} |R_m(\tilde{b}; x, y) - R_m(\tilde{b}; x_0, y)| \frac{|K(x, y)|}{|x - y|^m} |f_2(y)| dy \\
 & \quad + \int_{R^n} \left| \frac{K(x, y)}{|x - y|^m} - \frac{K(x_0, y)}{|x_0 - y|^m} \right| |R_m(\tilde{b}; x_0, y)| |f_2(y)| dy \\
 & \quad + \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \left| \frac{K(x, y)(x - y)^\alpha}{|x - y|^m} - \frac{K(x_0, y)(x_0 - y)^\alpha}{|x_0 - y|^m} \right| |D^\alpha \tilde{b}(y)| |f_2(y)| dy \\
 & \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x}) \sum_{k=0}^{\infty} w(2^{k+1} \tilde{Q})^{\beta/n} \int_{2^{k+1} \tilde{Q} \setminus 2^k \tilde{Q}} \frac{|x - x_0|}{|x_0 - y|^{n+1}} |f(y)| dy
 \end{aligned}$$

$$\begin{aligned}
 & + C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x}) \sum_{k=0}^{\infty} w(2^{k+1}\tilde{Q})^{\beta/n} \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x, y) - K(x_0, y)|^q dy \right)^{1/q} \\
 & \times \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} w(y)^{-p/r} dy \right)^{1/p} \\
 & + C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} \frac{|x - x_0|}{|x_0 - y|^{n+1}} |D^\alpha b(y) - (D^\alpha b)_{2^{k+1}\tilde{Q}}| |f(y)| dy \\
 & + C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} \frac{|x - x_0|}{|x_0 - y|^{n+1}} |(D^\alpha b)_{2^{k+1}\tilde{Q}} - (D^\alpha b)_{\tilde{Q}}| |f(y)| dy \\
 & + C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x, y) - K(x_0, y)|^q dy \right)^{1/q} \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \\
 & \times \left(\int_{2^{k+1}\tilde{Q}} |D^\alpha b(y) - (D^\alpha b)_{2^{k+1}\tilde{Q}}|^{p_1} w(y)^{1-p_1} dy \right)^{1/p_1} \left(\int_{2^{k+1}\tilde{Q}} w(y)^{-(1/r-1/p'_1)s_1} dy \right)^{1/s_1} \\
 & + C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x, y) - K(x_0, y)|^q dy \right)^{1/q} \left(\int_{2^{k+1}\tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \\
 & \times |(D^\alpha b)_{2^{k+1}\tilde{Q}} - (D^\alpha b)_{\tilde{Q}}| \left(\int_{2^{k+1}\tilde{Q}} w(y)^{-s_2/r} dy \right)^{1/s_2} \\
 \leq & C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x}) \sum_{k=1}^{\infty} k \frac{d}{(2^k d)^{n+1}} w(2^k \tilde{Q})^{\beta/n} \left(\int_{2^k \tilde{Q}} |f(y)|^r w(y) dx \right)^{1/r} \\
 & \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y)^{-1/(r-1)} dy \right)^{(r-1)/r} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y) dy \right)^{1/r} |2^k \tilde{Q}| w(2^k \tilde{Q})^{-1/r} \\
 & + C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x}) \sum_{k=1}^{\infty} C_k |2^k Q|^{-1/q'} w(2^k \tilde{Q})^{\beta/n} \left(\int_{2^k \tilde{Q}} |f(y)|^r w(y) dx \right)^{1/r} \\
 & \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y)^{-p/r} dy \right)^{1/p} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y) dy \right)^{1/r} |2^k \tilde{Q}|^{1/p+1/r} w(2^k \tilde{Q})^{-1/r} \\
 & + C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} \frac{d}{(2^k d)^{n+1}} \left(\int_{2^k \tilde{Q}} |D^\alpha b(y) - (D^\alpha b)_{2^k \tilde{Q}}|^{r'} w(y)^{1-r'} dy \right)^{1/r'} \\
 & \times \left(\int_{2^k \tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \\
 & + C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x}) \sum_{k=1}^{\infty} C_k |2^k Q|^{-1/q'} w(2^k \tilde{Q})^{\beta/n} \left(\int_{2^k \tilde{Q}} |f(y)|^r w(y) dx \right)^{1/r} \\
 & \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y)^{-(1/r-1/p'_1)s_1} dy \right)^{1/s_1} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y) dy \right)^{1/r-1/p'_1} \\
 & \times |2^k \tilde{Q}|^{1/s_1+1/r-1/p'_1} w(2^k \tilde{Q})^{-1/r+1/p'_1} \\
 & + C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x}) \sum_{k=1}^{\infty} k C_k |2^k Q|^{-1/q'} w(2^k \tilde{Q})^{\beta/n} \left(\int_{2^k \tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \\
 & \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y) dy \right)^{1/r} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y)^{-s_2/r} dy \right)^{1/s_2} |2^k \tilde{Q}|^{1/s_2+1/r} w(2^k \tilde{Q})^{-1/r}
 \end{aligned}$$

$$\begin{aligned}
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x}) \sum_{k=1}^{\infty} k(C_k + 2^{-k}) \\
 &\quad \times \left(\frac{1}{w(2^k \tilde{Q})^{1-r\beta/n}} \int_{2^k \tilde{Q}} |f(y)|^r w(y) dx \right)^{1/r} \\
 &\quad + C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \sum_{k=1}^{\infty} k(C_k + 2^{-k}) \frac{w(2^k \tilde{Q})}{|2^k \tilde{Q}|} \\
 &\quad \times \left(\frac{1}{w(2^k \tilde{Q})^{1-r\beta/n}} \int_{2^k \tilde{Q}} |f(y)|^r w(y) dx \right)^{1/r} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,r,w}(f)(\tilde{x}).
 \end{aligned}$$

This completes the proof of Theorem 2. □

Proof of Theorem 3 Choose $1 < r < u$ in Theorem 1 and notice $w^{1-u} \in A_1$, then we have, by Lemmas 3 and 4,

$$\begin{aligned}
 \|T^b(f)\|_{L^u(w^{1-u})} &\leq \|M_\eta(T^b(f))\|_{L^u(w^{1-u})} \leq C \|M_\eta^\#(T^b(f))\|_{L^u(w^{1-u})} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} \|wM_{r,w}(f)\|_{L^u(w^{1-u})} \\
 &= C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} \|M_{r,w}(f)\|_{L^u(w)} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} \|f\|_{L^u(w)}.
 \end{aligned}$$

This completes the proof of Theorem 3. □

Proof of Theorem 4 Choose $1 < r < u$ in Theorem 1 and notice $w^{1-u} \in A_1$, then we have, by Lemmas 5 and 6,

$$\begin{aligned}
 \|T^b(f)\|_{L^{u,\varphi}(w^{1-u})} &\leq \|M_\eta(T^b(f))\|_{L^{u,\varphi}(w^{1-u})} \leq C \|M_\eta^\#(T^b(f))\|_{L^{u,\varphi}(w^{1-u})} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} \|wM_{r,w}(f)\|_{L^{u,\varphi}(w^{1-u})} \\
 &= C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} \|M_{r,w}(f)\|_{L^{u,\varphi}(w)} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} \|f\|_{L^{u,\varphi}(w)}.
 \end{aligned}$$

This completes the proof of Theorem 4. □

Proof of Theorem 5 Choose $1 < r < u$ in Theorem 2 and notice $w^{1-v} \in A_1$, then we have, by Lemmas 3 and 4,

$$\begin{aligned}
 \|T^b(f)\|_{L^v(w^{1-v})} &\leq \|M_\eta(T^b(f))\|_{L^v(w^{1-v})} \leq C \|M_\eta^\#(T^b(f))\|_{L^v(w^{1-v})} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \|wM_{\beta,r,w}(f)\|_{L^v(w^{1-v})}
 \end{aligned}$$

$$\begin{aligned}
 &= C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \|M_{\beta,r,w}(f)\|_{L^v(w)} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \|f\|_{L^u(w)}.
 \end{aligned}$$

This completes the proof of Theorem 5. □

Proof of Theorem 6 Choose $1 < r < u$ in Theorem 2 and notice $w^{1-v} \in A_1$, then we have, by Lemmas 5 and 6,

$$\begin{aligned}
 \|T^b(f)\|_{L^{v,\varphi}(w^{1-v})} &\leq \|M_\eta(T^b(f))\|_{L^{v,\varphi}(w^{1-v})} \leq C \|M_\eta^\#(T^b(f))\|_{L^{v,\varphi}(w^{1-v})} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \|wM_{\beta,r,w}(f)\|_{L^{v,\varphi}(w^{1-v})} \\
 &= C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \|M_{\beta,r,w}(f)\|_{L^{v,\varphi}(w)} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \|f\|_{L^{u,\varphi}(w)}.
 \end{aligned}$$

This completes the proof of Theorem 6. □

Competing interests

The author declares that they have no competing interests.

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References

- García-Cuerva, J, Rubio de Francia, JL: Weighted Norm Inequalities and Related Topics. North-Holland Mathematics Studies, vol. 16. North-Holland, Amsterdam (1985)
- Stein, EM: Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals. Princeton University Press, Princeton (1993)
- Torchinsky, A: Real Variable Methods in Harmonic Analysis. Pure and Applied Math., vol. 123. Academic Press, New York (1986)
- Coifman, RR, Rochberg, R, Weiss, G: Factorization theorems for Hardy spaces in several variables. *Ann. Math.* **103**, 611-635 (1976)
- Pérez, C: Endpoint estimate for commutators of singular integral operators. *J. Funct. Anal.* **128**, 163-185 (1995)
- Pérez, C, Trujillo-Gonzalez, R: Sharp weighted estimates for multilinear commutators. *J. Lond. Math. Soc.* **65**, 672-692 (2002)
- Chanillo, S: A note on commutators. *Indiana Univ. Math. J.* **31**, 7-16 (1982)
- Lin, Y: Sharp maximal function estimates for Calderón-Zygmund type operators and commutators. *Acta Math. Sci., Ser. A* **31**, 206-215 (2011)
- Paluszynski, M: Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss. *Indiana Univ. Math. J.* **44**, 1-17 (1995)
- Bloom, S: A commutator theorem and weighted BMO. *Trans. Am. Math. Soc.* **292**, 103-122 (1985)
- Hu, B, Gu, J: Necessary and sufficient conditions for boundedness of some commutators with weighted Lipschitz spaces. *J. Math. Anal. Appl.* **340**, 598-605 (2008)
- Chang, DC, Li, JF, Xiao, J: Weighted scale estimates for Calderón-Zygmund type operators. *Contemp. Math.* **446**, 61-70 (2007)
- Liu, LZ: Sharp maximal function estimates and boundedness for commutators associated with general integral operator. *Filomat* **25**(4), 137-151 (2011)
- García-Cuerva, J: Weighted H^p spaces. *Diss. Math.* **162**, 1-63 (1979)
- Peetre, J: On convolution operators leaving $L^{p,\lambda}$ -spaces invariant. *Ann. Mat. Pura Appl.* **72**, 295-304 (1966)
- Peetre, J: On the theory of $L^{p,\lambda}$ -spaces. *J. Funct. Anal.* **4**, 71-87 (1969)
- Di Fazio, G, Ragusa, MA: Commutators and Morrey spaces. *Boll. Unione Mat. Ital., A* **5**, 323-332 (1991)
- Di Fazio, G, Ragusa, MA: Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients. *J. Funct. Anal.* **112**, 241-256 (1993)
- Liu, LZ: Interior estimates in Morrey spaces for solutions of elliptic equations and weighted boundedness for commutators of singular integral operators. *Acta Math. Sci., Ser. B* **25**, 89-94 (2005)
- Mizuhara, T: Boundedness of some classical operators on generalized Morrey spaces. In: *Harmonic Analysis: Proceedings of a Conference Held in Sendai, Japan*, pp. 183-189 (1990)

21. Cohen, J, Gosselin, J: On multilinear singular integral operators on R^n . *Stud. Math.* **72**, 199-223 (1982)
22. Cohen, J, Gosselin, J: A BMO estimate for multilinear singular integral operators. III. *J. Math.* **30**, 445-465 (1986)
23. Ding, Y, Lu, SZ: Weighted boundedness for a class rough multilinear operators. *Acta Math. Sin.* **17**, 517-526 (2001)

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