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Robust finite-time $\mathcal{H}_\infty$ filtering for uncertain systems subject to missing measurements

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Abstract

In this paper, the robust finite-time $\mathcal{H}_\infty$ filter design problem for uncertain systems subject to missing measurements is investigated. It is assumed that the system is subject to the norm-bounded uncertainties and the measurements of the output are intermittent. For the model of the missing measurements, the Bernoulli process is adopted. A full-order filter is proposed to estimate the signal which can track the signal to be estimated. By augmenting the system vector, a stochastic augmented system is obtained. Based on the analysis of the robust stochastic finite-time stability and the $\mathcal{H}_\infty$ performance, the filter design method is obtained. The filter parameters can be calculated by solving a sequence of linear matrix inequalities. Finally, a numerical example is used to show the design procedure and the effectiveness of the proposed design approach.

Keywords: finite-time stability; robust filtering; $\mathcal{H}_\infty$ filtering; linear matrix inequalities

1 Introduction

In the modern control, a filter plays an important role since the filter can be used to estimate the unavailable state and filter the external noise. Therefore, the filter design has been a hot research topic since the original development of the modern control. It is well known that the Kalman filter is an effective way to estimate state. However, the Kalman filter requires the preliminary knowledge of the spectrum of the noise and the precise system model. However, in many practical cases, these requirements cannot be satisfied. In these cases, the $\mathcal{H}_\infty$ filter is a great alternative. The $\mathcal{H}_\infty$ filter, which was originally proposed in the late 1980s [1], has attracted a lot of attention due to the fact that the filter can be easily utilized to deal with the uncertainties and the attenuation effect from the external input to the estimated signal [2–5].

In the state-space model, it is always assumed that system matrices are precise. However, in the real world, these matrices are unavoidable to contain uncertainties which can result from the modeling error or variations of the system parameters. During the past 20 years, the norm-bounded uncertainties have been widely adopted in the system modeling for practical plants, such as the works in [6–10]. In [11], the norm-bounded uncertainties were used in the time-delay linear systems. While in [9], the norm-bounded uncertainties were used in the neutral systems.

In the literature, most of the works on the $\mathcal{H}_\infty$ filtering were based on the Lyapunov asymptotic stability. However, in many practical applications, the asymptotic stability is

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not enough if large values of the state are not acceptable, see [4, 12–27] and the references therein. Although the finite-time stability was early proposed in 1960s [12], it was not a hot research topic in the following 40 years. Recently, as the development and the application of the linear matrix inequalities [28, 29], the finite-time stability has been devoted considerable efforts.

The missing measurements have been attracting a great number of attention due to the fact that the measurements are missing when sensors temporally fail [30–34]. If the phenomenon of missing measurements is not considered during the filter design, the actual missing measurements may deteriorate the designed filters. Although, there are many results on the $\mathcal{H}_\infty$ filtering, uncertain systems, and finite-time stability, there are few results on the $\mathcal{H}_\infty$ filtering for uncertain systems subject to missing measurements. This fact motivates me to do the research. In this paper, the contributions can be summarized as follows. The missing measurements are considered the finite-time framework. Due to the existence of the stochastic variable in the augmented system, the robust stochastic finite-time boundedness is studied for the uncertain stochastic system. Moreover, the $\mathcal{H}_\infty$ filtering with the robust stochastic finite-time stability is investigated.

2 Problem formulation

In this paper, the following uncertain discrete-time linear system is considered:

$$\begin{cases} x_{k+1} = (A + \Delta A)x_k + (B + \Delta B)\omega_k, \\ y_k = (C + \Delta C)x_k + (D + \Delta D)\omega_k, \\ z_k = Ex_k, \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ denotes the state vector, $y_k \in \mathbb{R}^m$ is the system output, $z_k \in \mathbb{R}^p$ is the signal to be estimated, and $\omega_k \in \mathbb{R}^r$ is the time-varying disturbance which satisfies

$$\sum_{k=1}^{\infty} \omega_k^{\mathrm{T}} \omega_k \leq d^2 \quad (k \in \mathbb{N}_0), \tag{2}$$

where $d > 0$ is a given scalar.

The matrices A, B, C, D, and E are constant matrices with appropriate dimensions. ΔA, ΔB, ΔC, and ΔD are real time-varying matrix functions representing the time-varying parameter uncertainties. It is assumed that the uncertainties are norm-bounded and admissible, which can be modeled as

$$\begin{bmatrix} \Delta A & \Delta B \\ \Delta C & \Delta D \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} G_k \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}, \tag{3}$$

where H_1, H_2, M_1, M_2, M_3, and M_4 are known real constant matrices and G_k is an unknown time-varying matrix function satisfying

$$\|G_k\| \leq I, \quad \forall k \in \mathbb{N}_0. \tag{4}$$

If the sampling of the output is perfect, the input of the filter is equal to the output of the system. However, if considering the intermittent sensor failures, the phenomenon of missing measurements occurs. This phenomenon was firstly proposed in [35]. Since the reliability of the system becomes more and more important, the filter and control design problem for systems subject to missing measurements has been a hot topic in recent years [31, 34, 36]. Inspired by the work in [35], the model of the missing measurement in this paper is expressed as follows:

$$\hat{y}_k = \begin{cases} (C + \Delta C)x_k + (D + \Delta D)\omega_k, & \text{the measurement is perfect,} \\ (D + \Delta D)\omega_k, & \text{the measurement is missing and only the noise is left,} \end{cases} \tag{5}$$

where $\hat{y}_k$ is the input of the filter to be designed. If a Bernoulli process is used to describe the phenomenon, the measured output is expressed as

$$\hat{y}_k = \alpha_k(C + \Delta C)x_k + (D + \Delta D)\omega_k, \tag{6}$$

where the stochastic variable r_k is a Bernoulli distributed white sequence taking values in the set $\{0, 1\}$.

The main objective of this paper is to design a full-order filter for the system (1) in the following form:

$$\begin{cases} \hat{x}_{k+1} = A_f\hat{x}_k + B_f\hat{y}_k, \\ \hat{z}_k = E_f\hat{x}_k, \end{cases} \tag{7}$$

where $\hat{x}_k$ is the state of the filter, $\hat{z}_k$ is an estimation of z_k, and A_f, B_f and E_f are filter parameters to be designed later.

Suppose that β is the probability of the available measuring. Defining the filtering error e_k as $e_k = z_k - \hat{z}_k$, the following augmented system can be obtained:

$$\begin{cases} \xi_{k+1} = (\bar{A}_1 + \Delta A_1)\xi_k + (\alpha_k - \beta)(\bar{A}_2 + \Delta A_2)\xi_k + (\bar{B} + \Delta B)\omega_k, \\ e_k = \bar{E}\xi_k, \end{cases} \tag{8}$$

where

$$\xi_k = \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}, \qquad \bar{A}_1 = \begin{bmatrix} A & 0 \\ \beta B_f C & A_f \end{bmatrix}, \qquad \Delta\bar{A}_1 = \begin{bmatrix} \Delta A & 0 \\ \beta B_f \Delta C & 0 \end{bmatrix},$$

$$\bar{A}_2 = \begin{bmatrix} 0 & 0 \\ B_f C & 0 \end{bmatrix}, \qquad \Delta\bar{A}_2 = \begin{bmatrix} 0 & 0 \\ B_f \Delta C & 0 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} B \\ B_f D \end{bmatrix}, \qquad \Delta\bar{B} = \begin{bmatrix} \Delta B \\ B_f \Delta D \end{bmatrix}, \quad \text{and} \quad \bar{E} = [E \quad -E_f].$$

Note that there is a stochastic variable α_k and some norm-bounded uncertainties in the augmented system in (8). Therefore, the challenge now is how to design the filter such that the augmented system in (8) is robustly stochastically finite-time bounded and the effect of the disturbance input to the signal to be estimated is constrained to a prescribed level.

Before proceeding, the following definitions are introduced.

Definition 1 (Finite-time stable (FTS) [13]) For a class of discrete-time linear systems,

$$\xi_{k+1} = A\xi_k, \quad k \in \mathbb{N}_0, \tag{9}$$

is said to be FTS with respect to (c_1, c_2, R, N), where R is a positive definite matrix, $0 < c_1 < c_2$ and $N \in \mathbb{N}_0$, if $\xi_0^{\mathrm{T}} R\xi_0 \le c_1^2$, then $\xi_k^{\mathrm{T}} R\xi_k \le c_2^2$ for all $k \in \{1, 2, \ldots, N\}$.

Definition 2 (Robustly stochastically finite-time stable (RSFTS)) For a class of discrete-time linear uncertain systems,

$$\xi_{k+1} = \bar{A}(\Delta A, \alpha_k)\xi_k, \quad k \in \mathbb{N}_0, \tag{10}$$

is said to be RSFTS with respect to (c_1, c_2, R, N), where the system matrix $\bar{A}(\Delta A, \alpha_k)$ has the uncertainty and the stochastic variable, R is a positive definite matrix, $0 < c_1 < c_2$ and $N \in \mathbb{N}_0$, if for all admissible uncertainties ΔA, stochastic variable α_k, $\xi_0^{\mathrm{T}} R\xi_0 \le c_1^2$, then $\mathbb{E}\{\xi_k^{\mathrm{T}} R\xi_k\} \le c_2^2$ for all $k \in \{1, 2, \ldots, N\}$.

Definition 3 (Robustly stochastically finite-time bounded (RSFTB)) For a class of discrete-time linear uncertain systems,

$$\xi_{k+1} = \bar{A}(\Delta A, \alpha_k)\xi_k + \bar{B}(\Delta B)\omega_k, \quad k \in \mathbb{N}_0, \tag{11}$$

is said to be RSFTB with respect to (c_1, c_2, d, R, N), where the system matrix $\bar{A}(\Delta A, \alpha_k)$ has the uncertainty and the stochastic variable, the input matrix contains the norm-bounded uncertainty, R is a positive definite matrix, $0 < c_1 < c_2$ and $N \in \mathbb{N}_0$, if for all admissible uncertainties ΔA and ΔB, stochastic variable α_k, $\xi_0^{\mathrm{T}} R\xi_0 \le c_1^2$, then $\mathbb{E}\{\xi_k^{\mathrm{T}} R\xi_k\} \le c_2^2$ for all $k \in \{1, 2, \ldots, N\}$.

With the above definitions, the main objectives in this paper can be summarized as follows. For the uncertainty in 1, design the full-order filter (7) such that for all the admissible uncertainties and the missing measurements,

- the augmented system (8) is RSFTS;
- under the zero-initial condition, the signal to be estimated z_k satisfies

$$\mathbb{E}\left\{\sum_{i=1}^{N} z_k^{\mathrm{T}} z_k\right\} < \gamma^2 \sum_{i=1}^{N} \omega_k^{\mathrm{T}} \omega_k \tag{12}$$

  for all l_2-bounded ω_k, where the prescribed value γ is the $\mathcal{H}_\infty$ attenuation level.

In addition, some useful lemmas are also needed.

Lemma 1 (Schur complement [6]) *Given a symmetric matrix* $\Phi = \left[\begin{smallmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{smallmatrix}\right]$, *the following three conditions are equivalent to each other*:

- $\Phi < 0$;
- $\Phi_{11} < 0$, $\Phi_{22} - \Phi_{12}^{\mathrm{T}} \Phi_{11}^{-1} \Phi_{12} < 0$;
- $\Phi_{22} < 0$, $\Phi_{11} - \Phi_{12} \Phi_{22}^{-1} \Phi_{12}^{\mathrm{T}} < 0$.

Lemma 2 [37, 38] *Let* $\Theta = \Theta^{\mathrm{T}}$, $\bar{H}$ *and* $\bar{M}$ *be real matrices with compatible dimensions, and let* G_k *be time-varying and satisfy* (4). *Then it can be concluded that the following condition*:

$$\Theta + \bar{H}G_k\bar{M} + (\bar{H}G_k\bar{M})^{\mathrm{T}} < 0 \tag{13}$$

holds if and only if there exists a positive scaler $\varepsilon > 0$ *such that*

$$\begin{bmatrix} \Theta & \bar{H} & \varepsilon\bar{M}^{\mathrm{T}} \\ * & -\varepsilon I & 0 \\ * & * & -\varepsilon I \end{bmatrix} < 0 \tag{14}$$

is satisfied.

3 Main results

3.1 Finite-time stability and $\mathcal{H}_\infty$ performance analysis

In this section, the finite-time stability, robust finite-time stability, and robust stochastic finite-time stability will be analyzed by assuming the parameters of the filter to be designed are given.

Theorem 1 *The augmented system in* (8) *is RSFTB with respect to* (c_1, c_2, d, R, N) *if there exist positive-definite matrices* $P_1 = P_1^{\mathrm{T}}$, $P_2 = P_2^{\mathrm{T}}$ *and two scalars* $\theta \geq 1$ *and* $\varepsilon > 0$ *such that the following conditions hold*:

$$\begin{bmatrix} -P_1 & 0 & hP_1\bar{A}_2 & 0 & hP_1\bar{H}_1 & 0 & 0 & 0 \\ * & -P_1 & P_1\bar{A}_1 & P_1\bar{B} & 0 & P_1\bar{H}_2 & 0 & 0 \\ * & * & -\theta P_1 & 0 & 0 & 0 & \varepsilon\bar{M}_1^{\mathrm{T}} & \varepsilon\bar{M}_1^{\mathrm{T}} \\ * & * & * & -\theta P_2 & 0 & 0 & 0 & \varepsilon M_3^{\mathrm{T}} \\ * & * & * & * & -\varepsilon I & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon I & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon I & 0 \\ * & * & * & * & * & * & * & -\varepsilon I \end{bmatrix} < 0, \tag{15}$$

and

$$\lambda_{\max}(\tilde{P}_1)c_1^2 + \lambda_{\max}(P_2)d^2 < \frac{c_2^2\lambda_{\min}(\tilde{P}_1)}{\theta^N}, \tag{16}$$

where

$$\tilde{P}_1 = R^{-1/2}P_1R^{-1/2}, \qquad h = \sqrt{\beta(1-\beta)},$$

$$\bar{H}_1 = \begin{bmatrix} 0 & 0 \\ B_fH_2 & 0 \end{bmatrix}, \qquad \bar{M}_1 = \begin{bmatrix} 0 & 0 \\ M_3 & 0 \end{bmatrix},$$

$$\bar{H}_2 = \begin{bmatrix} H_1 & 0 \\ B_fH_2 & 0 \end{bmatrix}, \qquad \bar{M}_2 = \begin{bmatrix} M_1 & 0 \\ \beta M_3 & 0 \end{bmatrix}, \qquad \bar{M}_3 = \begin{bmatrix} M_2 \\ M_4 \end{bmatrix}.$$

Proof Consider the following Lyapunov function:

$$V(k) = \xi_k^{\mathrm{T}} P_1 \xi_k, \tag{17}$$

where P_1 is a symmetric positive-definite matrix. For the augmented system in (8), the expectation of one step advance of the Lyapunov function can be derived as

$$\begin{aligned}\mathbb{E}\{V(k+1)|\xi_k\} &= \xi_k^{\mathrm{T}}\big((\bar{A}_1 + \Delta\bar{A}_1) + h(\bar{A}_2 + \Delta\bar{A}_2)\big)^{\mathrm{T}} P_1\big((\bar{A}_1 + \Delta\bar{A}_1) + h(\bar{A}_2 + \Delta\bar{A}_2)\big)\xi_k \\ &\quad + 2\xi_k^{\mathrm{T}}(\bar{A}_1 + \Delta\bar{A}_1)^{\mathrm{T}} P_1(\bar{B} + \Delta\bar{B})\omega_k + \omega_k^{\mathrm{T}}(\bar{B} + \Delta\bar{B})^{\mathrm{T}} P_1(\bar{B} + \Delta\bar{B})\omega_k \\ &= \begin{bmatrix}\xi_k \\ \omega_k\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}\Omega_{11} & \Omega_{12} \\ * & \Omega_{22}\end{bmatrix} \begin{bmatrix}\xi_k \\ \omega_k\end{bmatrix} = \begin{bmatrix}\xi_k \\ \omega_k\end{bmatrix}^{\mathrm{T}} \Omega \begin{bmatrix}\xi_k \\ \omega_k\end{bmatrix},\end{aligned} \tag{18}$$

where

$$\begin{aligned}\Omega_{11} &= \big((\bar{A}_1 + \Delta\bar{A}_1) + h(\bar{A}_2 + \Delta\bar{A}_2)\big)^{\mathrm{T}} P_1\big((\bar{A}_1 + \Delta\bar{A}_1) + h(\bar{A}_2 + \Delta\bar{A}_2)\big), \\ \Omega_{12} &= (\bar{A}_1 + \Delta\bar{A}_1)^{\mathrm{T}} P_1(\bar{B} + \Delta\bar{B}), \\ \Omega_{22} &= (\bar{B} + \Delta\bar{B})^{\mathrm{T}} P_1(\bar{B} + \Delta\bar{B}).\end{aligned}$$

Note that

$$\Delta\bar{A}_1 = \bar{H}_2\bar{G}_k\bar{M}_2, \qquad \Delta\bar{A}_2 = \bar{H}_1\bar{G}_k\bar{M}_1, \qquad \Delta\bar{B} = \bar{H}_2\bar{G}_k\bar{M}_3, \tag{19}$$

where

$$\bar{G}_k = \begin{bmatrix} G_k & 0 \\ 0 & G_k \end{bmatrix}. \tag{20}$$

By using the Schur complement, the condition in (15) implies that

$$\Omega < \begin{bmatrix} \theta P_1 & 0 \\ 0 & \theta P_2 \end{bmatrix}, \tag{21}$$

since the condition in (15) can be rewritten as

$$\Theta + \bar{H}G_k\bar{M} + (\bar{H}G_k\bar{M})^{\mathrm{T}} < 0, \tag{22}$$

where

$$\Theta = \begin{bmatrix} -P_1 & 0 & hP_1\bar{A}_2 & 0 \\ * & -P_1 & P_1\bar{A}_1 & P_1\bar{B} \\ * & * & -\theta P_1 & 0 \\ * & * & * & -\theta P_2 \end{bmatrix},$$

$$\bar{H} = \begin{bmatrix} hP_1\bar{H}_1 & 0 \\ 0 & P_1\bar{H}_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \bar{M} = \begin{bmatrix} 0 & 0 & \bar{M}_1 & 0 \\ 0 & 0 & \bar{M}_2 & \bar{M}_3 \end{bmatrix}.$$

With the condition (21), the following inequality can be obtained:

$$\mathbb{E}\{V(k+1)|\xi_k\} < \theta V(k) + \theta\omega_k^{\mathrm{T}} P_2 \omega_k. \tag{23}$$

Taking the iterative operation with respect to the time instant k, the following inequality is derived:

$$\mathbb{E}\{V(k)|\xi_0\} < \theta^k V(0) + \sum_{i=1}^{k} \theta^{k-i+1} \omega_{j-1}^{\mathrm{T}} P_2 \omega_{j-1} < \theta^N (\lambda_{\max}(\tilde{P}_1) c_1^2 + \lambda_{\max}(P_2) d^2). \tag{24}$$

It follows from the Lyapunov function that

$$\mathbb{E}\{V(k)|\xi_0\} > \lambda_{\min}(\tilde{P}_1) \xi_k^{\mathrm{T}} R \xi_k. \tag{25}$$

Combing (24) and (25), one gets

$$\mathbb{E}\{\xi_k^{\mathrm{T}} R \xi_k\} < \frac{\theta^N}{\lambda_{\min}(\tilde{P}_1)} (\lambda_{\max}(\tilde{P}_1) c_1^2 + \lambda_{\max}(P_2) d^2). \tag{26}$$

It is inferred from the conditions (16) and (26) that

$$\mathbb{E}\{\xi_k^{\mathrm{T}} R \xi_k\} < c_2^2. \tag{27}$$

Therefore, if the conditions (15) and (16) hold, the augmented system (8) is RSFTB. The proof is completed. □

It is noticed that there is a positive-definite matrix P_2 in Theorem 2. The matrix P_2 can be randomly chosen. For considering the $\mathcal{H}_\infty$ performance, other sufficient conditions are provided in the following theorem.

Theorem 2 *The augmented system in* (8) *is RSFTB with respect to* (c_1, c_2, d, R, N) *if there exist positive-definite matrix* $P = P^{\mathrm{T}}$ *and three scalars* $\theta \geq 1$, $\varepsilon > 0$, *and* $\gamma > 0$ *such that the following conditions hold*:

$$\begin{bmatrix} -P & 0 & hP\bar{A}_2 & 0 & hP\bar{H}_1 & 0 & 0 & 0 \\ * & -P & P\bar{A}_1 & P\bar{B} & 0 & P\bar{H}_2 & 0 & 0 \\ * & * & -\theta P & 0 & 0 & 0 & \varepsilon\bar{M}_1^{\mathrm{T}} & \varepsilon\bar{M}_2^{\mathrm{T}} \\ * & * & * & -\gamma^2 I & 0 & 0 & 0 & \varepsilon M_3^{\mathrm{T}} \\ * & * & * & * & -\varepsilon I & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon I & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon I & 0 \\ * & * & * & * & * & * & * & -\varepsilon I \end{bmatrix} < 0, \tag{28}$$

and

$$\lambda_{\max}(\tilde{P}) c_1^2 + \gamma^2 d^2 < \frac{c_2^2 \lambda_{\min}(\tilde{P})}{\theta^N}, \tag{29}$$

where $\tilde{P} = R^{-1/2} P R^{-1/2}$ *and* $h = \sqrt{\beta(1-\beta)}$.

Proof To prove the theorem, P_1 and P_2 in Theorem 1 can be replaced with P and $\gamma^2 I/\theta$, respectively. The proof is completed. □

The robust stochastic finite-time stability and the robust stochastic finite-time boundedness of the augmented system (8) have been offered. Now, we are going to consider the $\mathcal{H}_\infty$ performance.

Theorem 3 *The augmented system in* (8) *is RSFTB with respect to* $(0, c_2, d, R, N)$ *and with an* $\mathcal{H}_\infty$ *attenuation level* γ *if there exist positive-definite matrix* $P = P^{\mathrm{T}}$ *and three scalars* $\theta \geq 1$, $\varepsilon > 0$, *and* $\gamma > 0$ *such that the following conditions hold*:

$$\begin{bmatrix} -P & 0 & 0 & hP\bar{A}_2 & 0 & hP\bar{H}_1 & 0 & 0 & 0 \\ * & -P & 0 & P\bar{A}_1 & P\bar{B} & 0 & P\bar{H}_2 & 0 & 0 \\ * & * & -I & \bar{E} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\theta P & 0 & 0 & 0 & \varepsilon\bar{M}_1^{\mathrm{T}} & \varepsilon\bar{M}_2^{\mathrm{T}} \\ * & * & * & * & -\gamma^2 I & 0 & 0 & 0 & \varepsilon M_3^{\mathrm{T}} \\ * & * & * & * & * & -\varepsilon I & 0 & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon I & 0 & 0 \\ * & * & * & * & * & * & * & -\varepsilon I & 0 \\ * & * & * & * & * & * & * & * & -\varepsilon I \end{bmatrix} < 0, \tag{30}$$

and

$$\gamma^2 d^2 < \frac{c_2^2 \lambda_{\min}(\tilde{P})}{\theta^N}, \tag{31}$$

where $\tilde{P} = R^{-1/2} P R^{-1/2}$ *and* $h = \sqrt{\beta(1-\beta)}$.

Proof In the proof of $\mathcal{H}_\infty$ performance, it is required that the initial value of the state is zero. Therefore, c_1 in Theorem 2 is set to be zero. Under the zero-initial condition, consider the following cost function:

$$J = \mathbb{E}\{V(k+1)|\xi_k\} + \mathbb{E}\{z_k^{\mathrm{T}} z_k\} - \gamma^2 I. \tag{32}$$

The cost function can be revaluated with similar lines in Theorem 1. □

3.2 Filter design

The robust stochastic finite-time stability and the $\mathcal{H}_\infty$ performance have been investigated in the above subsection. In this subsection, the filter design method will be proposed.

Theorem 4 *Given a positive constant* γ *and two scalars* σ *and* ρ, *the closed-loop system in* (8) *is RSFTB with respect to* $(0, c_2, d, R, N)$ *and with a prescribed* $\mathcal{H}_\infty$ *attenuation level* γ *if there exists a positive-definite matrices* $P = P^{\mathrm{T}} = \left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & \sigma P_{12} \end{smallmatrix}\right]$, *matrices* $\left[\begin{smallmatrix} A_F & B_F \\ E_f & 0 \end{smallmatrix}\right]$, *two scalars*

$\theta \geq 1$, *and* $\varepsilon > 0$ *such that the following conditions hold*:

$$\begin{bmatrix} -P & 0 & 0 & h\Omega_1 & 0 & h\Omega_3 & 0 & 0 & 0 \\ * & -P & 0 & \Omega_2 & \Omega_4 & 0 & \Omega_5 & 0 & 0 \\ * & * & -I & \bar{E} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\theta P & 0 & 0 & 0 & \varepsilon \bar{M}_1^{\mathrm{T}} & \varepsilon \bar{M}_2^{\mathrm{T}} \\ * & * & * & * & -\gamma^2 I & 0 & 0 & 0 & \varepsilon M_3^{\mathrm{T}} \\ * & * & * & * & * & -\varepsilon I & 0 & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon I & 0 & 0 \\ * & * & * & * & * & * & * & -\varepsilon I & 0 \\ * & * & * & * & * & * & * & * & -\varepsilon I \end{bmatrix} < 0, \tag{33}$$

$$\gamma^2 d^2 < \frac{c_2^2 \rho}{\theta^N}, \tag{34}$$

and

$$\rho I < R^{-1/2} P R^{-1/2}, \tag{35}$$

where

$$\Omega_1 = \begin{bmatrix} B_F C & 0 \\ \sigma B_F C & 0 \end{bmatrix}, \qquad \Omega_2 = \begin{bmatrix} P_{11} A + \beta B_F C & A_F \\ P_{12}^{\mathrm{T}} A + \beta \sigma B_F C & \sigma A_F \end{bmatrix}, \qquad \Omega_3 = \begin{bmatrix} B_F H_2 & 0 \\ \sigma B_F H_2 & 0 \end{bmatrix},$$

$$\Omega_4 = \begin{bmatrix} P_{11} B + B_F D \\ P_{12}^{\mathrm{T}} B + \sigma B_F D \end{bmatrix}, \qquad \Omega_5 = \begin{bmatrix} P_{11} H_1 + B_F H_2 & 0 \\ P_{12}^{\mathrm{T}} H_1 + \sigma B_F H_2 & 0 \end{bmatrix}.$$

Moreover, the filter parameters can be calculated as $A_f = P_{12}^{-1} A_F$ *and* $B_f = P_{12}^{-1} B_F$.

Proof It is assumed that the Lyapunov weighting matrix has the following structure:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & \sigma P_{12} \end{bmatrix}, \tag{36}$$

where σ is a prescribed scalar. With this assumption, the coupled terms in Theorem 3 can be evaluated as follows:

$$\begin{aligned} P\bar{A}_1 &= \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & \sigma P_{12} \end{bmatrix} \begin{bmatrix} A & 0 \\ \beta B_f C & A_f \end{bmatrix} = \begin{bmatrix} P_{11} A + \beta P_{12} B_f C & P_{12} A_f \\ P_{12}^{\mathrm{T}} A + \beta \sigma P_{12} B_f C & \sigma P_{12} A_f \end{bmatrix}, \\ P\bar{A}_2 &= \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & \sigma P_{12} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \beta B_f C & 0 \end{bmatrix} = \begin{bmatrix} P_{12} B_f C & 0 \\ \sigma P_{12} B_f C & 0 \end{bmatrix}, \\ P\bar{B} &= \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & \sigma P_{12} \end{bmatrix} \begin{bmatrix} B \\ B_f D \end{bmatrix} = \begin{bmatrix} P_{11} B + P_{12} B_f D \\ P_{12}^{\mathrm{T}} B + \sigma P_{12} B_f D \end{bmatrix}, \\ P\bar{H}_1 &= \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & \sigma P_{12} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ B_f H_2 & 0 \end{bmatrix} = \begin{bmatrix} P_{12} B_f H_2 & 0 \\ \sigma P_{12} B_f H_2 & 0 \end{bmatrix}, \\ P\bar{H}_2 &= \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & \sigma P_{12} \end{bmatrix} \begin{bmatrix} H_1 & 0 \\ B_f H_2 & 0 \end{bmatrix} = \begin{bmatrix} P_{11} H_1 + P_{12} B_f H_2 & 0 \\ P_{12}^{\mathrm{T}} H_1 + \sigma P_{12} B_f H_2 & 0 \end{bmatrix}. \end{aligned} \tag{37}$$

Defining new variables as $A_F = P_{12}A_f$ and $B_F = P_{12}B_f$, the condition in (30) is equivalent to (33). Supposing that

$$\lambda_{\min}(\tilde{P}) \geq \rho, \tag{38}$$

the conditions (34) and (35) can guarantee that the condition (31) is satisfied. □

The $\mathcal{H}_\infty$ performance γ refers to the attenuation level from the external noise to the signal to be estimated. Therefore, it is desired that the performance γ should be as small as possible. For fixed θ and c_2, the optimal γ can be obtained by

$$\begin{cases} \min \gamma^2, \\ \text{s.t. (33), (34) and (35).} \end{cases} \tag{39}$$

4 Numerical example

Consider the system in (1) with the following matrix:

$$A = \begin{bmatrix} 0.2 & 0.1 \\ -0.2 & 0.3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.1 \\ -0.2 \end{bmatrix},$$

$$C = [1 \quad 0], \qquad D = 0.01, \qquad E = [0 \quad 1],$$

$$H_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \qquad H_2 = 0.05, \qquad M_1 = [1 \quad -1],$$

$$M_2 = 0.1, \qquad M_3 = [1 \quad 0], \qquad M_4 = 0.2.$$

In this example, the following values are chosen for the finite-time stability:

$$R = I, \qquad N = 5, \qquad c_2 = 5, \qquad d = 0.1, \qquad \theta = 1.2.$$

It is assumed that the probability of the available measurements is 0.95, that is, 5% of the output is randomly missing. With the proposed filter design problem in (39), the achieved minimum $\mathcal{H}_\infty$ performance index is $\gamma = 0.5474$ and the corresponding optimal filter is

$$\begin{cases} \hat{x}_{k+1} = \begin{bmatrix} -0.2415 & -0.0454 \\ 0.8518 & 0.4759 \end{bmatrix} \hat{x}_k + \begin{bmatrix} -0.7680 \\ 1.9164 \end{bmatrix} \hat{y}_k, \\ \hat{z}_k = \begin{bmatrix} 0.3451 & -0.5608 \end{bmatrix} \hat{x}_k. \end{cases}$$

In the simulation, assume that the external disturbance satisfies

$$\omega_k = \frac{0.1}{k+1},$$

and the time-varying parameter satisfies

$$G_k = \sin(k).$$

It is easy to check that the 2-norm of the external disturbance is less than d which is 0.1 and the time-varying parameter $\|G_k\| \leq 1$. It can be seen from Figure 1 that the estimated

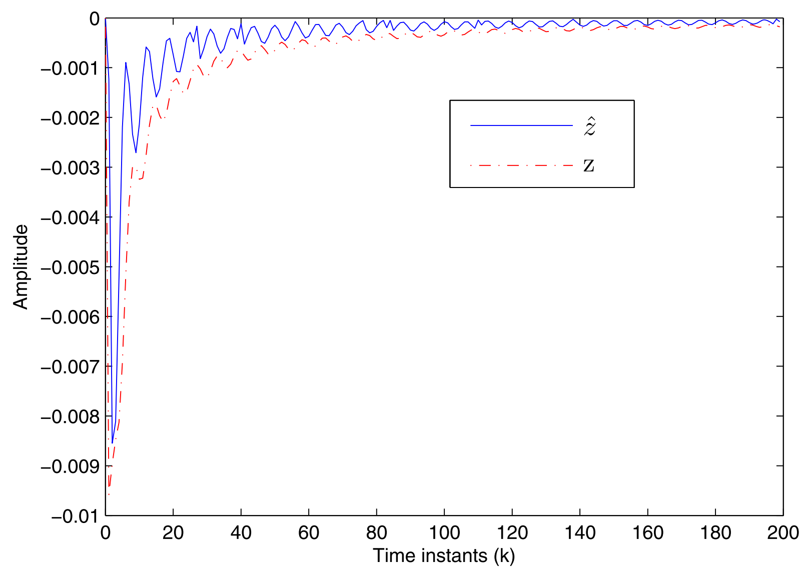

Figure 1 Trajectories of the signal to be estimated and the estimated signal.

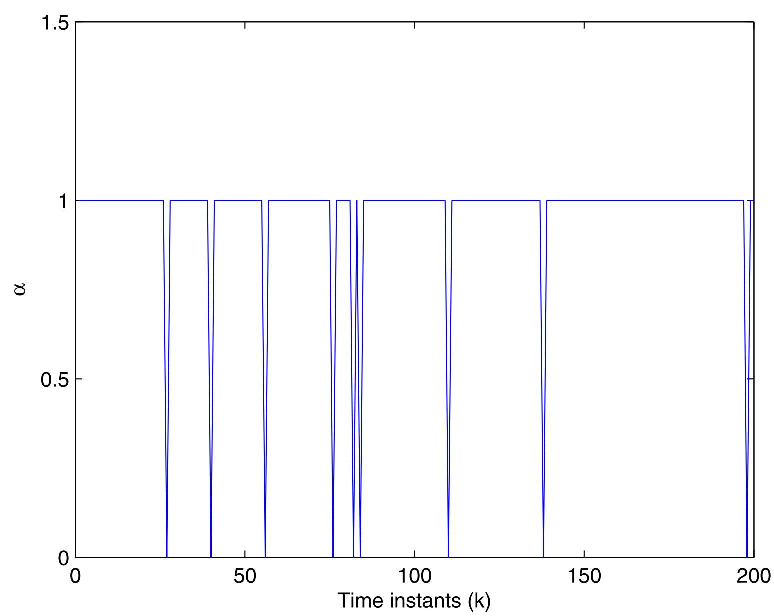

Figure 2 Stochastic values of the intermittent measurements in the simulation.

signal $\hat{z}_k$ can track the signal to be estimated well. The intermittent measurements in the random simulation are shown in Figure 2.

5 Conclusion

In this paper, the robust finite-time $\mathcal{H}_\infty$ filter design problem of discrete-time systems subject to missing measurements has been investigated. The uncertainties in the system matrices are assumed to be norm-bounded. The measurements of the system output are intermittent and a Bernoulli process is used to model the intermittent measurements.

Based on the results of the robust stochastic finite-time stability and the $\mathcal{H}_\infty$ performance, the filter design approach was proposed. Finally, an illustrative example was used to show the design procedure and the effectiveness of the proposed design approach.

Competing interests
The author declares that they have no competing interests.

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