Guo et al. *Journal of Inequalities and Applications* 2013, **2013**:67
http://www.journalofinequalitiesandapplications.com/content/2013/1/67

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH | Open Access

Some exact constants for the approximation of the quantity in the Wallis' formula

Senlin Guo[1*], Jian-Guo Xu[1] and Feng Qi[2]

*Correspondence:
sguo@hotmail.com
[1]Department of Mathematics, Zhongyuan University of Technology, Zhengzhou, Henan 450007, China
Full list of author information is available at the end of the article

Abstract

In this article, a sharp two-sided bounding inequality and some best constants for the approximation of the quantity associated with the Wallis' formula are presented.

MSC: Primary 41A44; secondary 26D20; 33B15

Keywords: approximation; gamma function; Stirling's formula; Wallis' formula

1 Introduction and main result

Throughout the paper, $\mathbb{Z}$ denotes the set of all integers, $\mathbb{N}$ denotes the set of all positive integers,

$$\begin{aligned} &\mathbb{N}_0 := \mathbb{N} \cup \{0\}, \\ &n!! := \prod_{i=0}^{[(n-1)/2]} (n-2i), \end{aligned} \tag{1}$$

and

$$W_n := \frac{(2n-1)!!}{(2n)!!}. \tag{2}$$

Here in (1), the floor function $[t]$ denotes the integer which is less than or equal to the number t.

The Euler gamma function is defined and denoted for $\operatorname{Re} z > 0$ by

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t}\, dt. \tag{3}$$

One of the elementary properties of the gamma function is that

$$\Gamma(x+1) = x\Gamma(x). \tag{4}$$

In particular,

$$\Gamma(n+1) = n!, \quad n \in \mathbb{N}_0. \tag{5}$$

Also, note that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{6}$$

For the approximation of $n!$, a well-known result is the following Stirling's formula:

$$n! \sim \sqrt{2\pi n} n^n e^{-n}, \quad n \to \infty, \tag{7}$$

which is an important tool in analytical probability theory, statistical physics and physical chemistry.

Consider the quantity W_n, defined by (2). This quantity is important in the probability theory - for example, the three events, (a) a return to the origin takes place at time $2n$, (b) no return occurs up to and including time $2n$, and (c) the path is non-negative between 0 and $2n$, have the common probability W_n. Also, the probability that in the time interval from 0 to $2n$ the particle spends $2k$ time units on the positive side and $2n - 2k$ time units on the negative side is $W_k W_{n-k}$. For details of these interesting results, one may see [1, Chapter III].

W_n is closely related to the Wallis' formula.

The Wallis' formula

$$\frac{2}{\pi} = \prod_{n=1}^{\infty} \frac{(2n-1)(2n+1)}{(2n)^2} \tag{8}$$

can be obtained by taking

$$x = \frac{\pi}{2}$$

in the infinite product representation of $\sin x$ (see [2, p.10], [3, p.211])

$$\sin x = x \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right), \quad x \in \mathbb{R}. \tag{9}$$

Since

$$\prod_{n=1}^{\infty} \frac{(2n-1)(2n+1)}{(2n)^2} = \lim_{n\to\infty} (2n+1) W_n^2, \tag{10}$$

another important form of Wallis' formula is (see [4, pp.181-184])

$$\lim_{n\to\infty} (2n+1) W_n^2 = \frac{2}{\pi}. \tag{11}$$

The following generalization of Wallis' formula was given in [5].

$$\frac{\pi}{t\sin(\pi/t)} = \frac{1}{t-1} \prod_{i=1}^{\infty} \frac{(it)^2}{(it+t-1)(it-t+1)}, \quad t > 1. \tag{12}$$

In fact, by letting

$$x = (1 - 1/t)\pi, \quad t \neq 0$$

in (9), we have

$$\sin\frac{\pi}{t} = \frac{\pi}{t}(t-1)\prod_{i=1}^{\infty}\frac{(it+t-1)(it-t+1)}{(it)^2}, \quad t \neq 0. \tag{13}$$

From (13), we get

$$\frac{\pi}{t\sin(\pi/t)} = \frac{1}{t-1}\prod_{i=1}^{\infty}\frac{(it)^2}{(it+t-1)(it-t+1)} \tag{14}$$

for

$$t \neq 0, \qquad t \neq \frac{1}{k}, \quad k \in \mathbb{Z}.$$

(12) is a special case of (14). The proof of (12) in [5] involves integrating powers of a generalized sine function.

There is a close relationship between Stirling's formula and Wallis' formula. The determination of the constant $\sqrt{2\pi}$ in the usual proof of Stirling's formula (7) or Stirling's asymptotic formula

$$\Gamma(x) \sim \sqrt{2\pi}x^{x-1/2}e^{-x}, \quad x \to \infty, \tag{15}$$

relies on Wallis' formula (see [2, pp.18-20], [3, pp.213-215], [4, pp.181-184]).

Also, note that

$$W_n = \left[(2n+1)\int_0^{\pi/2}\sin^{2n+1}x\,dx\right]^{-1} \tag{16}$$

$$= \left[(2n+1)\int_0^{\pi/2}\cos^{2n+1}x\,dx\right]^{-1} \tag{17}$$

and Wallis' sine (cosine) formula (see [6, p.258])

$$W_n = \frac{2}{\pi}\int_0^{\pi/2}\sin^{2n}x\,dx \tag{18}$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\cos^{2n}x\,dx. \tag{19}$$

Some inequalities involving W_n were given in [7–12].

In this article, we give a sharp two-sided bounding inequality and some exact constants for the approximation of W_n, defined by (2). The main result of the paper is as follows.

Theorem 1 *For all* $n \in \mathbb{N}$, $n \geq 2$,

$$\sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n\frac{\sqrt{n-1}}{n} < W_n \leq \frac{4}{3}\left(1-\frac{1}{2n}\right)^n\frac{\sqrt{n-1}}{n}. \tag{20}$$

The constants $\sqrt{e/\pi}$ *and* $4/3$ *in* (20) *are best possible.*

Moreover,

$$W_n \sim \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n}, \quad n \to \infty. \tag{21}$$

Remark 1 By saying that the constants $\sqrt{e/\pi}$ and 4/3 in (20) are best possible, we mean that the constant $\sqrt{e/\pi}$ in (20) cannot be replaced by a number which is greater than $\sqrt{e/\pi}$ and the constant 4/3 in (20) cannot be replaced by a number which is less than 4/3.

2 Lemmas

We need the following lemmas to prove our result.

Lemma 1 ([13, Theorem 1.1]) *The function*

$$f(x) := \frac{x^{x+\frac{1}{2}}}{e^x\Gamma(x+1)} \tag{22}$$

is strictly logarithmically concave and strictly increasing from $(0,\infty)$ *onto* $(0, \frac{1}{\sqrt{2\pi}})$.

Lemma 2 ([13, Theorem 1.3]) *The function*

$$h(x) := \frac{e^x\sqrt{x-1}\Gamma(x+1)}{x^{x+1}} \tag{23}$$

is strictly logarithmically concave and strictly increasing from $(1,\infty)$ *onto* $(0,\sqrt{2\pi})$.

Lemma 3 ([6, p.258]) *For all* $n \in \mathbb{N}$,

$$\Gamma\left(n+\frac{1}{2}\right) = \sqrt{\pi}\, n!\, W_n, \tag{24}$$

where W_n *is defined by* (2).

Remark 2 Some functions associated with the functions $f(x)$ and $h(x)$, defined by (22) and (23) respectively, were proved to be logarithmically completely monotonic in [14–16]. For more recent work on (logarithmically) completely monotonic functions, please see, for example, [17–43].

3 Proof of the main result

Proof of Theorem 1 By Lemma 1, we have

$$\frac{3}{e\sqrt{e\pi}} = f\left(\frac{3}{2}\right) \le f\left(n-\frac{1}{2}\right) = \frac{(n-\frac{1}{2})^n}{e^{n-1/2}\Gamma(n+1/2)} < \frac{1}{\sqrt{2\pi}}, \quad n \ge 2, \tag{25}$$

and

$$\lim_{n\to\infty} \frac{(n-\frac{1}{2})^n}{e^{n-1/2}\Gamma(n+1/2)} = \frac{1}{\sqrt{2\pi}}. \tag{26}$$

The lower and upper bounds in (25) are best possible.

By Lemma 3, (25) and (26) can be rewritten respectively as

$$\frac{3}{e^2} \le \frac{(n-\frac{1}{2})^n}{W_n e^n n!} < \frac{1}{\sqrt{2e}}, \quad n \ge 2, \tag{27}$$

and

$$\lim_{n\to\infty} \frac{(n-\frac{1}{2})^n}{W_n e^n n!} = \frac{1}{\sqrt{2e}}. \tag{28}$$

The constants $3/e^2$ and $1/\sqrt{2e}$ in (27) are best possible.

By Lemma 2, we get

$$\left(\frac{e}{2}\right)^2 = h(2) \le h(n) = \frac{e^n n! \sqrt{n-1}}{n^{n+1}} < \sqrt{2\pi}, \quad n \ge 2, \tag{29}$$

and

$$\lim_{n\to\infty} \frac{e^n n! \sqrt{n-1}}{n^{n+1}} = \sqrt{2\pi}. \tag{30}$$

The lower bound $(e/2)^2$ and the upper bound $\sqrt{2\pi}$ in (29) are best possible.

From (27) and (29), we obtain that for all $n \ge 2$,

$$\frac{3}{4} \le \frac{\sqrt{n-1}(n-\frac{1}{2})^n}{W_n n^{n+1}} < \sqrt{\frac{\pi}{e}}. \tag{31}$$

The constants $3/4$ and $\sqrt{\pi/e}$ in (31) are best possible. From (31) we get that for all $n \ge 2$,

$$\sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n} < W_n \le \frac{4}{3}\left(1-\frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n}. \tag{32}$$

The constants $\sqrt{e/\pi}$ and $4/3$ in (32) are best possible.

From (28) and (30), we see that

$$\lim_{n\to\infty} \frac{\sqrt{n-1}(n-\frac{1}{2})^n}{W_n n^{n+1}} = \sqrt{\frac{\pi}{e}}, \tag{33}$$

which is equivalent to (21).

The proof is thus completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed to the writing of the present article. They also read and approved the final manuscript.

Author details
[1]Department of Mathematics, Zhongyuan University of Technology, Zhengzhou, Henan 450007, China. [2]School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo, Henan 454010, China.

Acknowledgements
Dedicated to Professor Hari M Srivastava.
The authors thank the editor and the referees for their valuable suggestions to improve the quality of this paper. The present investigation was supported, in part, by the Natural Science Foundation of Henan Province of China under Grant 112300410022.

Received: 14 December 2012 **Accepted:** 7 February 2013 **Published:** 22 February 2013

References

1. Feller, W: An Introduction to Probability Theory and Its Applications. Wiley, New York (1966)
2. Andrews, GE, Askey, R, Roy, R: Special Functions. Cambridge University Press, Cambridge (1999)
3. Webster, R: Convexity. Oxford University Press, Oxford (1994)
4. Mitrinović, DS: Analytic Inequalities. Springer, Berlin (1970)
5. Piros, M: A generalization of the Wallis' formula. Miskolc Math. Notes **4**, 151-155 (2003)
6. Abramowitz, M, Stegun, IA: Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables. Dover, New York (1966)
7. Cao, J, Niu, D-W, Qi, F: A Wallis type inequality and a double inequality for probability integral. Aust. J. Math. Anal. Appl. **5**, 3 (2007)
8. Chen, C-P, Qi, F: Best upper and lower bounds in Wallis' inequality. J. Indones. Math. Soc. **11**, 137-141 (2005)
9. Chen, C-P, Qi, F: Completely monotonic function associated with the gamma function and proof of Wallis' inequality. Tamkang J. Math. **36**, 303-307 (2005)
10. Chen, C-P, Qi, F: The best bounds in Wallis' inequality. Proc. Am. Math. Soc. **133**, 397-401 (2005)
11. Qi, F: Bounds for the ratio of two gamma functions. J. Inequal. Appl. **2010**, 493058 (2010)
12. Qi, F, Luo, Q-M: Bounds for the ratio of two gamma functions - from Wendel's and related inequalities to logarithmically completely monotonic functions. Banach J. Math. Anal. **6**, 132-158 (2012)
13. Guo, S: Monotonicity and concavity properties of some functions involving the gamma function with applications. J. Inequal. Pure Appl. Math. **7**, 45 (2006)
14. Guo, S, Qi, F: A logarithmically complete monotonicity property of the gamma function. Int. J. Pure Appl. Math. **43**, 63-68 (2008)
15. Guo, S, Qi, F, Srivastava, HM: Supplements to a class of logarithmically completely monotonic functions associated with the gamma function. Appl. Math. Comput. **197**, 768-774 (2008)
16. Guo, S, Srivastava, HM: A class of logarithmically completely monotonic functions. Appl. Math. Lett. **21**, 1134-1141 (2008)
17. Alzer, H, Batir, N: Monotonicity properties of the gamma function. Appl. Math. Lett. **20**, 778-781 (2007)
18. Batir, N: On some properties of the gamma function. Expo. Math. **26**, 187-196 (2008)
19. Chen, C-P, Qi, F: Logarithmically completely monotonic functions relating to the gamma function. J. Math. Anal. Appl. **321**, 405-411 (2006)
20. Grinshpan, AZ, Ismail, ME-H: Completely monotonic functions involving the gamma and q-gamma functions. Proc. Am. Math. Soc. **134**, 1153-1160 (2006)
21. Guo, B-N, Qi, F: A class of completely monotonic functions involving divided differences of the psi and tri-gamma functions and some applications. J. Korean Math. Soc. **48**, 655-667 (2011)
22. Guo, S: Some properties of completely monotonic sequences and related interpolation. Appl. Math. Comput. **219**, 4958-4962 (2013)
23. Guo, S: Some classes of completely monotonic functions involving the gamma function. Int. J. Pure Appl. Math. **30**, 561-566 (2006)
24. Guo, S, Qi, F: A class of logarithmically completely monotonic functions associated with the gamma function. J. Comput. Appl. Math. **224**, 127-132 (2009)
25. Guo, S, Qi, F: A class of completely monotonic functions related to the remainder of Binet's formula with applications. Tamsui Oxf. J. Math. Sci. **25**, 9-14 (2009)
26. Guo, S, Qi, F, Srivastava, HM: A class of logarithmically completely monotonic functions related to the gamma function with applications. Integral Transforms Spec. Funct. **23**, 557-566 (2012)
27. Guo, S, Qi, F, Srivastava, HM: Necessary and sufficient conditions for two classes of functions to be logarithmically completely monotonic. Integral Transforms Spec. Funct. **18**, 819-826 (2007)
28. Guo, S, Srivastava, HM: A certain function class related to the class of logarithmically completely monotonic functions. Math. Comput. Model. **49**, 2073-2079 (2009)
29. Qi, F: A class of logarithmically completely monotonic functions and application to the best bounds in the second Gautsch-Kershaw's inequality. J. Comput. Appl. Math. **224**, 538-543 (2009)
30. Qi, F: Three classes of logarithmically completely monotonic functions involving gamma and psi functions. Integral Transforms Spec. Funct. **18**, 503-509 (2007)
31. Qi, F, Cui, R-Q, Chen, C-P, Guo, B-N: Some completely monotonic functions involving polygamma functions and an application. J. Math. Anal. Appl. **310**, 303-308 (2005)
32. Qi, F, Guo, B-N: Necessary and sufficient conditions for functions involving the tri- and tetra-gamma functions to be completely monotonic. Adv. Appl. Math. **44**, 71-83 (2010)
33. Qi, F, Guo, B-N: A logarithmically completely monotonic function involving the gamma function. Taiwan. J. Math. **14**, 1623-1628 (2010)
34. Qi, F, Guo, B-N: Some logarithmically completely monotonic functions related to the gamma function. J. Korean Math. Soc. **47**, 1283-1297 (2010)
35. Qi, F, Guo, B-N: Wendel's and Gautschi's inequalities: refinements, extensions, and a class of logarithmically completely monotonic functions. Appl. Math. Comput. **205**, 281-290 (2008)
36. Qi, F, Guo, B-N: A class of logarithmically completely monotonic functions and the best bounds in the second Kershaw's double inequality. J. Comput. Appl. Math. **212**, 444-456 (2008)
37. Qi, F, Guo, B-N, Chen, C-P: Some completely monotonic functions involving the gamma and polygamma functions. J. Aust. Math. Soc. **80**, 81-88 (2006)

38. Qi, F, Guo, S, Guo, B-N: Complete monotonicity of some functions involving polygamma functions. J. Comput. Appl. Math. **233**, 2149-2160 (2010)
39. Sevli, H, Batir, N: Complete monotonicity results for some functions involving the gamma and polygamma functions. Math. Comput. Model. **53**, 1771-1775 (2011)
40. Shemyakova, E, Khashin, SI, Jeffrey, DJ: A conjecture concerning a completely monotonic function. Comput. Math. Appl. **60**, 1360-1363 (2010)
41. Koumandos, S, Pedersen, HL: Completely monotonic functions of positive order and asymptotic expansions of the logarithm of Barnes double gamma function and Euler's gamma function. J. Math. Anal. Appl. **355**, 33-40 (2009)
42. Srivastava, HM, Choi, J: Zeta and q-Zeta Functions and Associated Series and Integrals. Elsevier, Amsterdam (2012)
43. Srivastava, HM, Guo, S, Qi, F: Some properties of a class of functions related to completely monotonic functions. Comput. Math. Appl. **64**, 1649-1654 (2012)

doi:10.1186/1029-242X-2013-67
Cite this article as: Guo et al.: Some exact constants for the approximation of the quantity in the Wallis' formula. *Journal of Inequalities and Applications* 2013 **2013**:67.