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# Approximation by a kind of complex modified $q$-Durrmeyer type operators in compact disks

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**Abstract**

In this paper, in order to make the convergence faster to a function being approximated, we introduce a kind of complex modified $q$-Durrmeyer type operators which can reproduce constant and linear functions. We study the approximation properties of these operators. We obtain the order of simultaneous approximation and a Voronovskaja-type result with a quantitative estimate for these complex modified $q$-Durrmeyer type operators attached to analytic functions on compact disks. More important, our results show the overconvergence phenomenon for these complex operators.
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## 1 Introduction

Let $q > 0$, for each nonnegative integer $k$, the $q$-integer $[k]_q$ and the $q$-factorial $[k]_q!$ are defined by

$$[k]_q := \begin{cases} (1 - q^k)/(1 - q), & q \neq 1, \\ k, & q = 1 \end{cases}$$

and

$$[k]_q! := \begin{cases} [k]_q[k-1]_q \cdots [1]_q, & k \geq 1, \\ 1, & k = 0, \end{cases}$$

respectively.

Then for $q > 0$ and integers $n, k, n \geq k \geq 0$, we have

$$[k+1]_q = 1 + q[k]_q \quad \text{and} \quad [k]_q + q^k[n-k]_q = [n]_q.$$

For the integers $n, k, n \geq k \geq 0$, the $q$-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Let $q > 0$, $q \neq 1$, we can define the derivative $D_q f$ of functions $f$ in the $q$-calculus by

$$
D_q f(x) = \begin{cases} \frac{f(qx) - f(x)}{(q-1)x}, & x \neq 0, \\ f'(0), & x = 0. \end{cases}
$$

Let $a > 0$, the $q$-Jackson integral in the interval $[0, a]$ is defined as

$$
\int_0^a f(t)\, d_q t = (1-q)a \sum_{j=0}^{\infty} f(aq^j)q^j, \quad 0 < q < 1.
$$

The $q$-analogue of the Beta function is defined as

$$
B_q(m, n) = \int_0^1 t^{m-1}(1 - qt)_q^{n-1}\, d_q t, \quad m, n > 0,
$$

where

$$
(a - b)_q^n = \prod_{j=0}^{n-1}(a - q^j b).
$$

Also, it is known that

$$
B_q(m, n) = \frac{[m-1]_q![n-1]_q!}{[m+n-1]_q!}.
$$

All of the previous concepts can be found in [1, 2].

In 1997 Philips [3] firstly introduced and studied $q$ analogue of Bernstein polynomials. After this, the applications of $q$-calculus in the approximation theory became one of the main areas of research; many authors studied new classes of $q$-generalized operators (for instance, see [4–11]). Very recently Gupta and Wang [12] introduced and studied the following $q$-Durrmeyer operators for $0 < q < 1$:

$$
T_{n,q}(f; x) = [n+1]_q \sum_{k=1}^{n} q^{1-k} p_{n,k}(q; x) \int_0^1 p_{n,k-1}(q; qt) f(t)\, d_q t + f(0) p_{n,0}(q; x), \tag{1.1}
$$

where $x \in [0, 1]$, $n = 1, 2, \ldots$, $0 < q < 1$ and

$$
p_{n,k}(q; x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} \left(1 - q^s x\right) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1 - x)_q^{n-k}.
$$

Agarwal and Gupta [13] have extended the operators which were given by (1.1) to a complex space and have studied the approximation properties of these complex operators. They have obtained the order of approximation and a Voronovskaja-type result with a quantitative estimate for these complex operators attached to analytic functions on compact disks.

The moments of the operators $T_{n,q}(f; x)$ were obtained as follows (see [12]):

**Remark 1** Denote $e_k(x) = x^k$, $k = 0, 1, 2$. For $0 < q < 1$, $x \in [0, 1]$, $n \in \mathbb{N}$, we have

$$T_{n,q}(e_0; x) = 1, \qquad T_{n,q}(e_1; x) = \frac{[n]_q x}{[n+2]_q},$$

$$T_{n,q}(e_2; x) = x \frac{(1+q)[n]_q}{[n+3]_q[n+2]_q} + x^2 \frac{q[n]_q([n]_q - 1)}{[n+3]_q[n+2]_q}.$$

It can be observed from the above remark that the operators $T_{n,q}(f; x)$ reproduce only a constant function. To make the convergence faster, we modify these operators so that they reproduce constant as well as linear functions. For this reason, we change the scale of reference by replacing the term $x$ by $\frac{[n+2]_q x}{[n]_q}$, in the definition of $T_{n,q}(f; x)$ given by (1.1). Using the restriction $x \in [0, \frac{1}{[3]_q}]$, we have the following positive linear operators:

$$R_{n,q}(f; x) = [n+1]_q \sum_{k=1}^{n} q^{1-k} t_{n,k}(q; x) \int_0^1 p_{n,k-1}(q; qt) f(t) \, d_q t + f(0) t_{n,0}(q; x), \qquad (1.2)$$

where $x \in [0, \frac{1}{[3]_q}]$, $n \in \mathbb{N}$, $0 < q < 1$, the term $p_{n,k}(q; x)$ is given in (1.1) and

$$t_{n,k}(q; x) := \begin{bmatrix} n \\ k \end{bmatrix}_q \left( \frac{[n+2]_q}{[n]_q} x \right)^k \left( 1 - \frac{[n+2]_q}{[n]_q} x \right)^{n-k}_q.$$

By simple computation, we get the moments of the operators $R_{n,q}(f; x)$.

**Remark 2** Denoting $e_k(x) = x^k$, $k = 0, 1, 2$, for $0 < q < 1$, $x \in [0, \frac{1}{[3]_q}]$, $n \in \mathbb{N}$, we have

$$R_{n,q}(e_0; x) = 1, \qquad R_{n,q}(e_1; x) = x, \qquad R_{n,q}(e_2; x) = x \frac{1+q}{[n+3]_q} + x^2 \frac{q^2[n-1]_q[n+2]_q}{[n+3]_q[n]_q}.$$

The aim of the present article is to obtain approximation results for the complex extension of the $q$-Bernstein-Durrmeyer type modified operator (1.2) defined by

$$M_{n,q}(f; z) = [n+1]_q \sum_{k=1}^{n} q^{1-k} t_{n,k}(q; z) \int_0^1 p_{n,k-1}(q; qt) f(t) \, d_q t + f(0) t_{n,0}(q; z), \qquad (1.3)$$

where $z \in \mathbb{C}$, $n = 1, 2, \ldots$, $0 < q < 1$ and $t_{n,k}(q; z) := \begin{bmatrix} n \\ k \end{bmatrix}_q \left( \frac{[n+2]_q}{[n]_q} z \right)^k \left( 1 - \frac{[n+2]_q}{[n]_q} z \right)^{n-k}_q$, $p_{n,k}(q; z) := \begin{bmatrix} n \\ k \end{bmatrix}_q z^k (1-z)^{n-k}_q$.

## 2 Auxiliary results

In the sequel, we shall need the following auxiliary results.

**Lemma 1** *Let $0 < q < 1$, $m \in \mathbb{N}$. We have $M_{n,q}(t^m; z)$ is a polynomial of degree $\leq \min(m, n)$ and*

$$M_{n,q}(t^m; z) = \frac{[n+1]_q!}{[n+m+1]_q!} \sum_{s=1}^{m} c_s(m) [n]_q^s B_{n,q}(t^s; z),$$

*where $c_s(m) \geq 0$ are constants depending on $m$ and $q$ and*

$$B_{n,q}(f;z) = \sum_{k=0}^{n} t_{n,k}(q;z) f\big([k]_q/[n]_q\big).$$

*Proof* By the definition of $q$-Beta function, we have

$$M_{n,q}\big(t^m;z\big) = [n+1]_q \sum_{k=1}^{n} q^{1-k} t_{n,k}(q;z) \int_0^1 p_{n,k-1}(q;qt) t^m \, d_q t$$

$$= \frac{[n+1]_q!}{[n+m+1]_q!} \sum_{k=1}^{n} t_{n,k}(q;z) \frac{[k+m-1]_q!}{[k-1]_q!}.$$

Considering the definition of the $B_{n,q}(f;z)$, for any $m \in \mathbb{N}$, applying the principle of mathematical induction, we immediately obtain the desired conclusion. □

**Lemma 2** *Let $0 < q < 1$. For all $m, n \in \mathbb{N}$, we can get the inequality*

$$\frac{[n+1]_q!}{[n+m+1]_q!} \sum_{s=1}^{m} c_s(m)[n]_q^s \leq 1.$$

*Proof* By Lemma 1, we have

$$M_{n,q}\left(t^m; \frac{[n]_q}{[n+2]_q}\right) = \frac{[n+1]_q!}{[n+m+1]_q!} \sum_{s=1}^{m} c_s(m)[n]_q^s B_{n,q}\left(t^s; \frac{[n]_q}{[n+2]_q}\right)$$

$$= \frac{[n+1]_q!}{[n+m+1]_q!} \sum_{s=1}^{m} c_s(m)[n]_q^s.$$

On the other hand, we have $t_{n,k}(q; \frac{[n]_q}{[n+2]_q}) = 0$, $k = 0, 1, 2, \ldots, n-1$, also $t_{n,n}(q;z) = (\frac{[n+2]_q}{[n]_q} z)^n$ and $t_{n,n}(q; \frac{[n]_q}{[n+2]_q}) = 1$. So, by formula (1.3) and using the above values, we have

$$M_{n,q}\left(t^m; \frac{[n]_q}{[n+2]_q}\right) = [n+1]_q t_{n,n}\left(q; \frac{[n]_q}{[n+2]_q}\right) q^{1-n} \int_0^1 p_{n,n-1}(q;qt) t^m \, d_q t$$

$$= \frac{[n]_q[n+1]_q}{[n+m]_q[n+m+1]_q} \leq 1,$$

which implies that we get the desired conclusion. □

**Corollary 1** *Denote $e_m(t) = t^m$, let $r \geq 1$ and $0 < q < 1$. Then for all $m \in \mathbb{N} \cup \{0\}$ and $|z| \leq r$, we have $|M_{n,q}(e_m;z)| \leq ([3]_q r)^m$.*

**Lemma 3** *Let $0 < q < 1$, $e_m(t) = t^m$, $m \in \mathbb{N} \cup \{0\}$ and $z \in \mathbb{C}$, we have*

$$M_{n,q}(e_{m+1};z) = \frac{q^m z([n]_q - [n+2]_q z)}{[n]_q[m+n+2]_q} D_q M_{n,q}(e_m;z)$$

$$+ \frac{[m]_q + q^m[n+2]_q z}{[m+n+2]_q} M_{n,q}(e_m;z). \tag{2.1}$$

*Proof* By Lemma 1, we have $M_{n,q}(e_0; z) = 1$ and $M_{n,q}(e_1; z) = z$, therefore, this result is established for $m = 0$. Now, let $m \in \mathbb{N}$, in view of $D_q(f(x)g(x)) = g(x)D_qf(x) + f(qx)D_qg(x)$ and $D_q(a + bx)^n_q = [n]_q b(a + bqx)^{n-1}_q$, by simple calculation, we obtain

$$z\left(1 - \frac{[n+2]_q}{[n]_q}z\right)D_q\big(t_{n,k}(q;z)\big) = \big([k]_q - [n+2]_q z\big)t_{n,k}(q;z),$$

$$z(1 - z)D_q\big(p_{n,k}(q;z)\big) = \big([k]_q - [n]_q z\big)p_{n,k}(q;z).$$

For $u = u(t) = \alpha t$ ($\alpha$ is a constant), since $D_q f(u(t)) = D_q f(u) \cdot D_q u(t)$, therefore, we have

$$t(1 - qt)D_q\big(p_{n,k}(q;qt)\big) = p_{n,k}(q;qt)\big([k]_q - [n]_q qt\big),$$

$$t(1 - qt)D_q\big(p_{n,k-1}(q;qt)\big) = p_{n,k-1}(q;qt)\big([k-1]_q - [n]_q qt\big).$$

It follows that

$$z\left(1 - \frac{[n+2]_q}{[n]_q}z\right)D_q\big(M_{n,q}(e_m; z)\big)$$

$$= [n+1]_q \sum_{k=1}^{n} q^{1-k}\big([k]_q - [n+2]_q z\big)t_{n,k}(q;z)\int_0^1 p_{n,k-1}(q;qt)t^m \, d_q t$$

$$= [n+1]_q \sum_{k=1}^{n} q^{1-k} t_{n,k}(q;z)\int_0^1 \big(1 + q[k-1]_q - [n]_q q^2 t + [n]_q q^2 t\big)p_{n,k-1}(q;qt)t^m \, d_q t$$

$$\quad - z[n+2]_q[n+1]_q \sum_{k=1}^{n} q^{1-k} t_{n,k}(q;z)\int_0^1 p_{n,k-1}(q;qt)t^m \, d_q t$$

$$= q[n+1]_q \sum_{k=1}^{n} q^{1-k} t_{n,k}(q;z)\int_0^1 D_q\big(p_{n,k-1}(q;qt)\big)t(1 - qt)t^m \, d_q t$$

$$\quad + M_{n,q}(e_m; z) + q^2[n]_q M_{n,q}(e_{m+1}; z) - z[n+2]_q M_{n,q}(e_m; z).$$

Letting $\delta(t) = \frac{t}{q}(1 - t)(\frac{t}{q})^m$, using $q$-integrate by parts, we have

$$\int_0^1 D_q\big(f(t)\big)\delta(qt)\, d_q t = \delta(t)f(t)\big|_0^1 - \int_0^1 f(t)D_q\delta(t)\, d_q t.$$

So, the $q$-integral in the above formula becomes

$$\int_0^1 D_q\big(p_{n,k-1}(q;qt)\big)t(1 - qt)t^m \, d_q t$$

$$= \delta(t)p_{n,k-1}(q;qt)\big|_0^1 - \int_0^1 p_{n,k-1}(q;qt)D_q\delta(t)\, d_q t$$

$$= -q^{-m-1}\int_0^1 p_{n,k-1}(q;qt)D_q\big(t^{m+1} - t^{m+2}\big)\, d_q t$$

$$= -q^{-m-1}[m+1]_q \int_0^1 p_{n,k-1}(q;qt)t^m \, d_q t + q^{-m-1}[m+2]_q \int_0^1 p_{n,k-1}(q;qt)t^{m+1} \, d_q t.$$

Thus, we obtain

$$
\begin{aligned}
z\left(1 - \frac{[n+2]_q}{[n]_q} z\right) D_q\big(M_{n,q}(e_m; z)\big) = {}&-q^{-m}[m+1]_q M_{n,q}(e_m; z) \\
&+ q^{-m}[m+2]_q M_{n,q}(e_{m+1}; z) + M_{n,q}(e_m; z) \\
&+ q^2[n]_q M_{n,q}(e_{m+1}; z) - z[n+2]_q M_{n,q}(e_m; z).
\end{aligned}
$$

In view of $[m+2]_q + q^{m+2}[n]_q = [m+n+2]_q$ and $[m+1]_q = [m]_q + q^m$, by simple calculation, we can get the recurrence in the statement. □

**Lemma 4** *Denote* $S_{n,m}(q; z) = M_{n,q}(e_m; z) - z^m$. *Let* $0 < q < 1$, $e_m(t) = t^m$, *for all* $m \in \mathbb{N} \cup \{0\}$ *and* $z \in \mathbb{C}$, *we have*

$$
\begin{aligned}
S_{n,m}(q; z) = {}&\frac{q^{m-1} z([n]_q - [n+2]_q z)}{[n]_q [m+n+1]_q} D_q M_{n,q}(e_{m-1}; z) \\
&+ \frac{[m-1]_q + z q^{m-1}[n+2]_q}{[m+n+1]_q} S_{n,m-1}(q; z) + \frac{[m-1]_q (1-z) z^{m-1}}{[m+n+1]_q}.
\end{aligned}
\tag{2.2}
$$

*Proof* Using formula (2.1), by simple calculation, we can easily get the recurrence (2.2), the proof is omitted here. □

**Lemma 5** *If* $P_m(z)$ *is a polynomial of degree* $m$, *for all* $|z| \le r$, *we have*

$$
\big|D_q P_m(z)\big| \le \big\|P_m'\big\|_r \le \frac{m}{r} \|P_m\|_r,
\tag{2.3}
$$

*where* $\|P_m\|_r = \max\{|P_m(z)|; |z| \le r\}$.

*Proof* The proof is easy by using the Bernstein inequality and the complex mean value theorem, the proof is omitted here. □

Let $e_m(t) = t^m$, $m \in \mathbb{N}$. By Lemma 1, for all $|z| \le r$, we have

$$
\big|D_q M_{n,q}(e_{m-1}; z)\big| \le \frac{m-1}{r} \big\|M_{n,q}(e_{m-1}; \cdot)\big\|_r.
$$

## 3 Main results

The first main result is expressed by the following upper estimates.

**Theorem 1** *Let* $0 < q < 1$, $R > 3$, $D_R = \{z \in \mathbb{C} : |z| < R\}$. *Suppose that* $f : D_R \to \mathbb{C}$ *is analytic in* $D_R$, *i.e.*, $f(z) = \sum_{m=0}^{\infty} c_m z^m$ *for all* $z \in D_R$. *Take* $1 \le r \le \frac{R}{3}$.
  (i) *For all* $|z| \le r$ *and* $n \in \mathbb{N}$, *we have*

$$
\big|M_{n,q}(f; z) - f(z)\big| \le \frac{K_r(f)}{|n|_q},
$$

*where* $K_r(f) = (1+r) \sum_{m=2}^{\infty} |c_m| m(m-1)([3]_q r)^{m-1} < \infty$.

(ii) (*Simultaneous approximation*) *If* $1 \leq r < r_1 < \frac{R}{3}$ *are arbitrary fixed, then for all* $|z| \leq r$ *and* $n, p \in \mathbb{N}$, *we have*

$$\left| M_{n,q}^{(p)}(f; z) - f^{(p)}(z) \right| \leq \frac{K_{r_1}(f) p! r_1}{[n]_q (r_1 - r)^{p+1}},$$

*where* $K_{r_1}(f)$ *is defined as in* (i) *above.*

*Proof* Taking $e_m(z) = z^m$, by the hypothesis that $f(z)$ is analytic in $D_R$, i.e., $f(z) = \sum_{m=0}^{\infty} c_m z^m$ for all $z \in D_R$, it is easy for us to obtain $M_{n,q}(f; z) = \sum_{m=0}^{\infty} c_m M_{n,q}(e_m; z)$, therefore, we get

$$\left| M_{n,q}(f; z) - f(z) \right| \leq \sum_{m=0}^{\infty} |c_m| \cdot \left| M_{n,q}(e_m; z) - e_m(z) \right|$$

$$= \sum_{m=2}^{\infty} |c_m| \cdot \left| M_{n,q}(e_m; z) - e_m(z) \right|,$$

as $M_{n,q}(e_0; z) = 1$, $M_{n,q}(e_1; z) = z$.

(i) By Lemma 4, Lemma 5 and Corollary 1, for all $m \in \mathbb{N}$, we get

$$\left| M_{n,q}(e_m; z) - e_m(z) \right| = \left| S_{n,m}(q; z) \right| \leq \frac{r(1+r)}{[n]_q} \cdot \frac{m-1}{r} \left\| M_{n,q}(e_{m-1}; \cdot) \right\|_r$$

$$+ r \left| S_{n,m-1}(q; z) \right| + \frac{m-1}{[n]_q}(1+r) r^{m-1}$$

$$\leq \frac{2(m-1)}{[n]_q}(1+r)\left([3]_q r\right)^{m-1} + r \left| S_{n,m-1}(q; z) \right|.$$

By writing the last inequality, for $m = 2, 3, \dots$, we easily obtain

$$\left| M_{n,q}(e_m; z) - e_m(z) \right| \leq \frac{1+r}{[n]_q} m(m-1)\left([3]_q r\right)^{m-1}.$$

In conclusion, it follows that

$$\left| M_{n,q}(f; z) - f(z) \right| \leq \frac{1+r}{[n]_q} \sum_{m=2}^{\infty} |c_m| m(m-1)\left([3]_q r\right)^{m-1}.$$

By the hypothesis on $f$, we have $f^{(2)}(z) = \sum_{m=2}^{\infty} c_m m(m-1) z^{m-2}$, and the series is absolutely convergent in $|z| \leq [3]_q r$, so we get $\sum_{m=2}^{\infty} |c_m| m(m-1)([3]_q r)^{m-2} < \infty$, that is $K_r(f) = (1+r) \sum_{m=2}^{\infty} |c_m| m(m-1)([3]_q r)^{m-1} < \infty$.

(ii) Denoting by $\Gamma$ the circle of radius $r_1 > r$ and center 0, since for any $|z| \leq r$ and $v \in \Gamma$ we have $|v - z| \geq r_1 - r$, by the Cauchy's formulas it follows that for all $|z| \leq r$ and $n, p \in \mathbb{N}$, we have

$$\left| M_{n,q}^{(p)}(f; z) - f^p(z) \right| = \frac{p!}{2\pi} \left| \int_{\Gamma} \frac{M_{n,q}(f; v) - f(v)}{(v - z)^{p+1}} \, dv \right|$$

$$\leq \frac{K_{r_1}(f)}{[n]_q} \frac{p!}{2\pi} \frac{2\pi r_1}{(r_1 - r)^{p+1}} = \frac{K_{r_1}(f)}{[n]_q} \cdot \frac{p! r_1}{(r_1 - r)^{p+1}},$$

which proves the theorem. □

**Remark 3** Let $0 < q < 1$ be fixed. Since we have $\frac{1}{[n]_q} \to 1 - q$ as $n \to \infty$, by passing to limit with $n \to \infty$ in the estimates in Theorem 1, we do not obtain the convergence of $M_{n,q}^{(p)}(f;z)$ to $f^{(p)}(z)$, $p = 0, 1, \ldots$. But this situation can be improved by choosing $0 < q_n < 1$ with $q_n \to 1$ as $n \to \infty$. Indeed, since in this case $\frac{1}{[n]_{q_n}} \to 0$ as $n \to \infty$ (see Videnskii [14], formula (2.7)), from Theorem 1 we get that $M_{n,q}^{(p)}(f;z) \to f^{(p)}(z)$, for $p = 0, 1, \ldots$, uniformly for $|z| \le r$, for any $1 \le r < r_1 < \frac{R}{3}$.

The following Voronovskaja-type result with a quantitative estimate holds.

**Theorem 2** *Let $0 < q < 1$, $R > 3$ and suppose that $f : D_R \to \mathbb{C}$ is analytic in $D_R = \{z \in \mathbb{C} : |z| < R\}$, i.e., $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in D_R$. For any fixed $r \in [1, \frac{R}{3}]$ and for all $n \in \mathbb{N}$, $|z| \le r$, we have*

$$\left| M_{n,q}(f;z) - f(z) - \frac{z(1-z)f''(z)}{[n]_q} \right| \le \frac{M_r(f)}{[n]_q^2},$$

*where $M_r(f) = \sum_{k=2}^{\infty} |c_k|(k-1)F_{k,r}([3]_q r)^k < \infty$ and $F_{k,r} = (k-1)(k-2)(2k-3) + 6k(k-1)^2 + 4(k-1)k^2 + 4(k-2)(k-1)^2(1+r)$.*

*Proof* Denoting $e_k(z) = z^k$, $k = 0, 1, 2, \ldots$, by the hypothesis that $f(z)$ is analytic in $D_R$, *i.e.*, $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in D_R$, we can write $M_{n,q}(f;z) = \sum_{k=0}^{\infty} c_k M_{n,q}(e_k;z)$, thus, for all $z \in D_R$, $n \in \mathbb{N}$, we have

$$\left| M_{n,q}(f;z) - f(z) - \frac{z(1-z)f''(z)}{[n]_q} \right|$$

$$\le \sum_{k=2}^{\infty} |c_k| \left| M_{n,q}(e_k;z) - e_k(z) - \frac{k(k-1)(1-z)z^{k-1}}{[n]_q} \right|.$$

Denoting

$$E_{k,n}(q;z) = M_{n,q}(e_k;z) - e_k(z) - \frac{k(k-1)(1-z)z^{k-1}}{[n]_q},$$

it is obvious that $E_{k,n}(q;z)$ is a polynomial of degree less than or equal to $k$. By simple computation and the use of Lemma 3, for all $k \ge 2$, we can get

$$E_{k,n}(q;z) = \frac{q^{k-1}z([n]_q - [n+2]_q z)}{[n]_q[n+k+1]_q} D_q E_{k-1,n}(q;z)$$

$$+ \frac{q^{k-1}[n+2]_q z + [k-1]_q}{[n+k+1]_q} E_{k-1,n}(q;z) + G_{k,n}(q;z),$$

where

$$G_{k,n}(q;z) = \frac{z^{k-2}}{[n]_q^2[n+k+1]_q} \left\{ z^2 \left[ -q^{k-1}[k-1]_q[n]_q[n+2]_q \right. \right.$$

$$+ q^{k-1}(k-1)(k-2)[n+2]_q$$

$$+ q^k(k-1)(k-2)[k-2]_q[n+2]_q + q^{k-1}[n+2]_q[n]_q^2$$

$$- q^{k-1}(k-1)(k-2)[n]_q[n+2]_q - [n+k+1]_q[n]_q^2$$

$$+ k(k-1)[n+k+1]_q[n]_q]$$

$$+ z\big[q^{k-1}[k-1]_q[n]_q^2 - q^{k-1}(k-1)(k-2)[n]_q - q^k(k-1)(k-2)[k-2]_q[n]_q$$

$$- q^{k-1}(k-1)(k-2)[k-2]_q[n+2]_q$$

$$+ [k-1]_q[n]_q^2 - (k-1)(k-2)[k-1]_q[n]_q$$

$$+ q^{k-1}(k-1)(k-2)[n]_q[n+2]_q - k(k-1)[n+k+1]_q[n]_q\big]$$

$$+ \big[q^{k-1}(k-1)(k-2)[k-2]_q[n]_q + (k-1)(k-2)[k-1]_q[n]_q\big]\big\}$$

$$:= \frac{z^{k-2}}{[n]_q^2[n+k+1]_q}\big(z^2 A_{k,n}(q) + z B_{k,n}(q) + C_{k,n}(q)\big).$$

For all $k \geq 2$, we easily obtain $|C_{k,n}(q)| \leq [n]_q(k-1)(k-2)(2k-3)$, it follows that

$$\left|\frac{z^{k-2} C_{k,n}(q)}{[n]_q^2[n+k+1]_q}\right| \leq \frac{(k-1)(k-2)(2k-3)r^k}{[n]_q^2}.$$

In view of $[n+k+1]_q = [k-1]_q + q^{k-1}[n]_q + q^{n+k-1} + q^{n+k}$ and $[n+2]_q = [n]_q + q^n + q^{n+1}$, for all $k \geq 2$, we can get

$$B_{k,n}(q) = [n]_q^2\big[q^{k-1}\big([k-1]_q - (k-1)\big) + \big([k-1]_q - q^{k-1}(k-1)\big)\big]$$

$$+ [n]_q\big[-q^{k-1}(k-1)(k-2)\big(1 + q[k-2]_q + [k-2]_q\big) - 2(k-1)^2[k-1]_q$$

$$- 2q^{n+k-1}(1+q)(k-1)\big] - q^{n+k-1}(1+q)(k-1)(k-2)[k-2]_q.$$

Also, according to $[k-1]_q - (k-1) = (q-1)\sum_{j=0}^{k-2}[j]_q$ and $[k-1]_q - q^{k-1}(k-1) = (1-q)\sum_{j=1}^{k-1}[j]_q q^{k-1-j}$, we have

$$\left|[n]_q^2\big[q^{k-1}\big([k-1]_q - (k-1)\big) + \big([k-1]_q - q^{k-1}(k-1)\big)\big]\right|$$

$$= \left|[n]_q q^{k-1}\big(q^n - 1\big)\sum_{j=0}^{k-2}[j]_q + [n]_q\big(1 - q^n\big)\sum_{j=1}^{k-1}[j]_q q^{k-1-j}\right|$$

$$\leq [n]_q\big[(k-1)[k-2]_q + (k-1)[k-1]_q\big].$$

Thus, through simple calculation, we can get

$$\left|\frac{z^{k-1} B_{k,n}(q)}{[n]_q^2[n+k+1]_q}\right| \leq \frac{6k(k-1)^2 r^k}{[n]_q^2}.$$

Now, we estimate $A_{k,n}(q)$. Similar to the calculation of $B_{k,n}(q)$, for all $k \geq 2$, we have

$$A_{k,n}(q) = -[n]_q\Bigg[q^{k-1}\big(q^n - 1\big)\sum_{j=0}^{k-2}[j]_q + \big(1 - q^n\big)\sum_{j=1}^{k-1}[j]_q q^{k-1-j}\Bigg] + [n]_q\big[k(k-1)[k-1]_q$$

$$+ q^{k-1}(k-1)(k-2)\big(1 + q[k-2]_q\big) - q^{n+k-1}(1+q)\big([k-1]_q - 2k + 2\big)\big]$$

$$+ q^{n+k-1}(1+q)(k-1)(k-2)\big(1 + q[k-2]_q\big).$$

By simple calculation, it follows that

$$\left| \frac{z^k A_{k,n}(q)}{[n]_q^2 [n+k+1]_q} \right| \le \frac{4(k-1)k^2 r^k}{[n]_q^2}.$$

Thus, for all $k \ge 2$, $n \in \mathbb{N}$ and $|z| \le r$, we can obtain

$$\left| G_{k,n}(q;z) \right| \le \frac{r^k}{[n]_q^2}\left[ (k-1)(k-2)(2k-3) + 6k(k-1)^2 + 4(k-1)k^2 \right] := \frac{r^k}{[n]_q^2} D_k,$$

where $D_k = (k-1)(k-2)(2k-3) + 6k(k-1)^2 + 4(k-1)k^2$.

For all $k \ge 2$, $n \in \mathbb{N}$ and $|z| \le r$, $1 \le r$, it follows

$$\left| E_{k,n}(q;z) \right| \le \frac{r(1+r)}{[n]_q}\left| D_q E_{k-1,n}(q;z) \right|$$

$$+ \frac{q^{k-1}[n+2]_q r + [k-1]_q}{[n+k+1]_q}\left| E_{k-1,n}(q;z) \right| + \left| G_{k,n}(q;z) \right|.$$

Since $q^{k-1}[n+2]_q r + [k-1]_q \le [n+k+1]_q r$, it follows

$$\left| E_{k,n}(q;z) \right| \le \frac{r(1+r)}{[n]_q}\left| D_q E_{k-1,n}(q;z) \right| + r\left| E_{k-1,n}(q;z) \right| + \left| G_{k,n}(q;z) \right|.$$

Using the estimate in the proof of Theorem 1(i), we get

$$\left| M_{n,q}(e_k;z) - e_k(z) \right| \le \frac{1+r}{[n]_q} k(k-1)\left( [3]_q r \right)^{k-1}$$

for all $k, n \in \mathbb{N}$, $|z| \le r$, $1 \le r$.

Denote $\|f\|_r = \max\{|f(z)|; |z| \le r\}$, by Lemma 5, we have

$$\left| D_q E_{k-1,n}(q;z) \right|$$

$$\le \frac{k-1}{r}\|E_{k-1,n}\|_r$$

$$\le \frac{k-1}{r}\left[ \left\| M_{n,q}(e_{k-1};\cdot) - e_{k-1} \right\|_r + \left\| \frac{(k-1)(k-2)(1-e_1)e_{k-2}}{[n]_q} \right\|_r \right]$$

$$\le \frac{k-1}{r}\left[ \frac{(k-1)(k-2)(1+r)([3]_q r)^{k-2}}{[n]_q} + \frac{(k-1)(k-2)(1+r)([3]_q r)^{k-2}}{[n]_q} \right]$$

$$\le \frac{4(k-2)(k-1)^2([3]_q r)^{k-1}}{[n]_q}.$$

It follows

$$\left| E_{k,n}(q;z) \right| \le \frac{4(k-2)(k-1)^2(1+r)([3]_q r)^k}{[n]_q^2} + r\left| E_{k-1,n}(q;z) \right| + \frac{([3]_q r)^k}{[n]_q^2} D_k$$

$$:= \frac{([3]_q r)^k}{[n]_q^2} F_{k,r} + r\left| E_{k-1,n}(q;z) \right|,$$

where $F_{k,r}$ is a polynomial of degree 3 in $k$ defined as $F_{k,r} = D_k + 4(k-2)(k-1)^2(1+r)$, $D_k$ is expressed in the above.

Since $E_{0,n}(q;z) = E_{1,n}(q;z) = 0$ for any $z \in \mathbb{C}$, therefore, by writing the last inequality for $k = 2, 3, \dots$, we easily, step by step, obtain the following:

$$\left|E_{k,n}(q;z)\right| \leq \frac{([3]_q r)^k}{[n]_q^2} \sum_{j=2}^{k} F_{j,r} \leq \frac{(k-1)F_{k,r}([3]_q r)^k}{[n]_q^2}.$$

As a conclusion, we have

$$\left|M_{n,q}(f;z) - f(z) - \frac{z(1-z)f''(z)}{[n]_q}\right| \leq \sum_{k=2}^{\infty} |c_k| \left|E_{k,n}(q;z)\right|$$

$$\leq \frac{1}{[n]_q^2} \sum_{k=2}^{\infty} |c_k|(k-1)F_{k,r}([3]_q r)^k.$$

As $f^{(4)}(z) = \sum_{k=4}^{\infty} c_k k(k-1)(k-2)(k-3)z^{k-4}$ and the series is absolutely convergent in $|z| \leq [3]_q r$, it easily follows that $\sum_{k=4}^{\infty} |c_k| k(k-1)(k-2)(k-3)([3]_q r)^{k-4} < \infty$, which implies that $\sum_{k=2}^{\infty} |c_k|(k-1)F_{k,r}([3]_q r)^k < \infty$. This completes the proof of the theorem. □

In the following theorem, we will obtain the exact order in approximation.

**Theorem 3** *Let $0 < q_n < 1$ satisfy $\lim_{n\to\infty} q_n = 1$, $R > 3$, $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$. Suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$. If $f$ is not a polynomial of degree $\leq 1$, then for any $r \in [1, \frac{R}{3})$, we have*

$$\left\|M_{n,q_n}(f;\cdot) - f\right\|_r \geq \frac{C_r(f)}{[n]_{q_n}}, \quad n \in \mathbb{N},$$

*where $\|f\|_r = \max\{|f(z)|; |z| \leq r\}$ and the constant $C_r(f) > 0$ depends on $f$, $r$ and on the sequence $\{q_n\}_{n\in\mathbb{N}}$ but is independent of $n$.*

*Proof* Denote $e_1(z) = z$ and

$$H_{n,q_n}(f;z) = M_{n,q_n}(f;z) - f(z) - \frac{z(1-z)f''(z)}{[n]_{q_n}}.$$

For all $z \in \mathbb{D}_R$ and $n \in \mathbb{N}$, we have

$$M_{n,q_n}(f;z) - f(z) = \frac{1}{[n]_{q_n}} \left\{ z(1-z)f''(z) + \frac{1}{[n]_{q_n}} \left[[n]_{q_n}^2 H_{n,q_n}(f;z)\right] \right\}.$$

Using the property $\|F + G\|_r \geq \left| \|F\|_r - \|G\|_r \right| \geq \|F\|_r - \|G\|_r$, it follows

$$\left\|M_{n,q_n}(f;\cdot) - f\right\|_r \geq \frac{1}{[n]_{q_n}} \left\{ \left\|e_1(1-e_1)f''\right\|_r - \frac{1}{[n]_{q_n}} \left[[n]_{q_n}^2 \left\|H_{n,q_n}(f;\cdot)\right\|_r\right] \right\}.$$

Considering the hypothesis that $f$ is not a polynomial of degree $\leq 1$ in $\mathbb{D}_R$, we get

$$\left\|e_1(1-e_1)f''\right\|_r > 0.$$

Indeed, supposing the contrary, it follows that $z(1 - z)f''(z) = 0$ for all $z \in \overline{D_r}$, that is $f''(z) = 0$ for all $z \in \overline{D_r}$. Thus, $f$ is a polynomial of degree $\leq 1$, a contradiction to the hypothesis.

By Theorem 2, we get $[n]_{q_n}^2 \|H_{n,q_n}(f; \cdot)\|_r \leq M_r(f)$. Taking into account $\frac{1}{[n]_{q_n}} \to 0$ as $n \to \infty$, therefore, there exists an index $n_0$ depending only on $f$, $r$ and on sequence $\{q_n\}_{n \in \mathbb{N}}$ such that for all $n \geq n_0$, we have

$$\left\| e_1(1 - e_1)f'' \right\|_r - \frac{1}{[n]_{q_n}} \left[ [n]_{q_n}^2 \left\| H_{n,q_n}(f; \cdot) \right\|_r \right] \geq \frac{1}{2} \left\| e_1(1 - e_1)f'' \right\|_r,$$

which implies

$$\left\| M_{n,q_n}(f; \cdot) - f \right\|_r \geq \frac{1}{2[n]_{q_n}} \left\| e_1(1 - e_1)f'' \right\|_r, \quad \forall n \geq n_0.$$

On the other hand, for $n \in \{1, 2, \ldots, n_0 - 1\}$, we have

$$\left\| M_{n,q_n}(f; \cdot) - f \right\|_r \geq \frac{W_{r,n}(f)}{[n]_{q_n}},$$

where $W_{r,n}(f) = [n]_{q_n} \cdot \|M_{n,q_n}(f; \cdot) - f\|_r > 0$.

As a conclusion, we have

$$\left\| M_{n,q_n}(f; \cdot) - f \right\|_r \geq \frac{C_r(f)}{[n]_{q_n}}, \quad \forall n \in \mathbb{N},$$

where $C_r(f) = \min\{W_{r,1}(f), W_{r,2}(f), \ldots, W_{r,n_0-1}(f), \frac{1}{2}\|e_1(1 - e_1)f''\|_r\}$, this completes the proof. $\square$

Combining Theorem 3 with Theorem 1, we get the following result.

**Corollary 2** *Let $0 < q_n < 1$ satisfy $\lim_{n \to \infty} q_n = 1$, $R > 3$, $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$. Suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$. If $f$ is not a polynomial of degree $1$, then for any $r \in [1, \frac{R}{3})$, we have*

$$\left\| M_{n,q_n}(f; \cdot) - f \right\|_r \asymp \frac{1}{[n]_{q_n}}, \quad n \in \mathbb{N},$$

*where $\|f\|_r = \max\{|f(z)|; |z| \leq r\}$ and the constants in the equivalence depend on $f$, $r$ and on the sequence $\{q_n\}_{n \in \mathbb{N}}$ but are independent of $n$.*

Considering the derivatives of complex modified $q$-Durrmeyer type operators, we can prove the following result.

**Theorem 4** *Let $0 < q_n < 1$ satisfy $\lim_{n \to \infty} q_n = 1$, $R > 3$, $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$. Suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$. Also, let $1 \leq r < r_1 < \frac{R}{3}$ and $p \in \mathbb{N}$ be fixed. If $f$ is not a polynomial of degree $\leq \max(1, p - 1)$, then we have*

$$\left\| M_{n,q_n}^{(p)}(f; \cdot) - f^{(p)} \right\|_r \asymp \frac{1}{[n]_{q_n}}, \quad n \in \mathbb{N},$$

where $\|f\|_r = \max\{|f(z)|; |z| \leq r\}$ *and the constants in the equivalence depend on $f$, $r$, $r_1$, $p$* *and on the sequence $\{q_n\}_{n \in \mathbb{N}}$ but are independent of $n$.*

*Proof* Taking into account the upper estimate in Theorem 1, it remains to prove the lower estimate only.

Denoting by $\Gamma$ the circle of radius $r_1 > r$ and center 0, by the Cauchy's formula, it follows that for all $|z| \leq r$ and $n \in \mathbb{N}$, we have

$$M_{n,q_n}^{(p)}(f;z) - f^{(p)}(z) = \frac{p!}{2\pi i} \int_\Gamma \frac{M_{n,q_n}(f;v) - f(v)}{(v-z)^{p+1}}\, dv.$$

Keeping the notation there for $H_{n,q_n}(f;z)$, for all $n \in \mathbb{N}$, we have

$$M_{n,q_n}(f;z) - f(z) = \frac{1}{[n]_{q_n}}\left\{ z(1-z)f''(z) + \frac{1}{[n]_{q_n}}\left[ [n]_{q_n}^2 H_{n,q_n}(f;z) \right] \right\}.$$

By using Cauchy's formula, for all $v \in \Gamma$, we get

$$M_{n,q_n}^{(p)}(f;z) - f^{(p)}(z) = \frac{1}{[n]_{q_n}}\left\{ \left[ z(1-z)f''(z) \right]^p + \frac{1}{[n]_{q_n}}\frac{p!}{2\pi i}\int_\Gamma \frac{[n]_{q_n}^2 H_{n,q_n}(f;v)}{(v-z)^{p+1}}\, dv \right\}.$$

Passing now to $\|\cdot\|_r$ and denoting $e_1(z) = z$, we get the following:

$$\left\| M_{n,q_n}^{(p)}(f;\cdot) - f^{(p)} \right\|_r \geq \frac{1}{[n]_{q_n}}\Bigg[ \left\| \left[ e_1(1-e_1)f'' \right]^{(p)} \right\|_r$$

$$- \frac{1}{[n]_{q_n}}\left\| \frac{p!}{2\pi i}\int_\Gamma \frac{[n]_{q_n}^2 H_{n,q_n}(f;v)}{(v-\cdot)^{p+1}}\, dv \right\|_r \Bigg].$$

Since for any $|z| \leq r$ and $v \in \Gamma$ we have $|v - z| \geq r_1 - r$, so, by using Theorem 2, we get

$$\left\| \frac{p!}{2\pi i}\int_\Gamma \frac{[n]_{q_n}^2 H_{n,q_n}(f;v)}{(v-\cdot)^{p+1}}\, dv \right\|_r \leq \frac{p!}{2\pi}\frac{2\pi r_1 [n]_{q_n}^2 \|H_{n,q_n}(f;\cdot)\|_{r_1}}{(r_1 - r)^{p+1}} \leq \frac{M_{r_1}(f)p!r_1}{(r_1 - r)^{p+1}}.$$

By the hypothesis on $f$, we have

$$\left\| \left[ e_1(1-e_1)f'' \right]^{(p)} \right\|_r > 0.$$

Indeed, supposing the contrary, it follows that $\|[e_1(1-e_1)f'']^{(p)}\|_r = 0$, that is, $z(1-z)f''(z)$ is a polynomial of degree $\leq p - 1$. Let $p = 1$ and $p = 2$, then the analyticity of $f$ obviously implies that $f$ is a polynomial of degree $\leq 1 = \max(1, p-1)$, a contradiction.

Now let $p \geq 3$, then the analyticity of $f$ obviously implies that $f$ is a polynomial of degree $\leq p - 1 = \max(1, p-1)$, a contradiction to the hypothesis.

In conclusion, $\|[e_1(1-e_1)f'']^{(p)}\|_r > 0$, and in continuation reasoning exactly as in the proof of Theorem 3, we can get the desired conclusion. □

**Remark 4** If we use King's approach to consider a King-type modification of the complex extension of the operators which was given by (1.1), we will obtain better approximation (*cf.* [15–17]).

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