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Some results on asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense and equilibrium problems

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Abstract
In this paper, we investigate a common fixed point problem of a finite family of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense and an equilibrium problem. Strong convergence theorems of common solutions are established in a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property.
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1 Introduction-preliminaries

Let E be a real Banach space. Recall that E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U_E$. It is said to be uniformly smooth if the above limit is attained uniformly for $x, y \in U_E$.

Recall that E has Kadec-Klee property if for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. For more details of the Kadec-Klee property, the readers can refer to [1] and the references therein. It is well known that if E is a uniformly convex Banach space, then E enjoys the Kadec-Klee property.

Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that if E is uniformly smooth if and only if E^* is uniformly convex.

Next, we assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x,y \in E.$$

Observe that, in a Hilbert space H, the equality is reduced to $\phi(x,y) = \|x-y\|^2$, $x,y \in H$. As we all know if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [2] recently introduced a generalized projection operator Π_C in a Banach space E which is an analog of the metric projection P_C in Hilbert spaces. Recall that the generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x},x) = \min_{y\in C} \phi(y,x).$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J; see, for example, [1, 2]. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

$$\bigl(\|x\| - \|y\|\bigr)^2 \le \phi(x,y) \le \bigl(\|y\| + \|x\|\bigr)^2, \quad \forall x,y \in E, \tag{1.1}$$

and

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x-z, Jz - Jy\rangle, \quad \forall x,y,z \in E. \tag{1.2}$$

Remark 1.1 If E is a reflexive, strictly convex, and smooth Banach space, then $\phi(x,y) = 0$ if and only if $x = y$; for more details, see [1, 2] and the references therein.

Let C be a nonempty subset of E and let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the fixed point set of T. T is said to be asymptotically regular on C if for any bounded subset K of C,

$$\limsup_{n\to\infty} \bigl\{ \bigl\| T^{n+1}x - T^n x \bigr\| : x \in K \bigr\} = 0.$$

T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$. In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

Recall that a point p in C is said to be an asymptotic fixed point of T [3] iff C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$.

A mapping T is said to be relatively nonexpansive iff

$$\widetilde{F}(T) = F(T) \neq \emptyset, \qquad \phi(p, Tx) \le \phi(p,x), \quad \forall x \in C, \forall p \in F(T).$$

A mapping T is said to be relatively asymptotically nonexpansive iff

$$\widetilde{F}(T) = F(T) \neq \emptyset, \qquad \phi(p, T^n x) \leq (1 + \mu_n)\phi(p, x), \quad \forall x \in C, \forall p \in F(T), \forall n \geq 1,$$

where $\{\mu_n\} \subset [0, \infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$.

Remark 1.2 The class of relatively asymptotically nonexpansive mappings were first considered in [4]; see also, [5] and the references therein.

Recall that a mapping T is said to be quasi-ϕ-nonexpansive iff

$$F(T) \neq \emptyset, \qquad \phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \forall p \in F(T).$$

Recall that a mapping T is said to be asymptotically quasi-ϕ-nonexpansive iff there exists a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$F(T) \neq \emptyset, \qquad \phi(p, T^n x) \leq (1 + \mu_n)\phi(p, x), \quad \forall x \in C, \forall p \in F(T), \forall n \geq 1.$$

Remark 1.3 The class of quasi-ϕ-nonexpansive mappings was considered in [6]. The class of asymptotically quasi-ϕ-nonexpansive mappings which was investigated in [7] and [8] includes the class of quasi-ϕ-nonexpansive mappings as a special case.

Remark 1.4 The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are more general than the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings. Quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive do not require the restriction $F(T) = \widetilde{F}(T)$.

Remark 1.5 The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are generalizations of the class of quasi-nonexpansive mappings and the class of asymptotically quasi-nonexpansive mappings in Banach spaces.

Recall that T is said to be asymptotically quasi-ϕ-nonexpansive in the intermediate sense iff $F(T) \neq \emptyset$ and the following inequality holds:

$$\limsup_{n \to \infty} \sup_{p \in F(T), x \in C} \bigl(\phi(p, T^n x) - \phi(p, x)\bigr) \leq 0. \tag{1.3}$$

Putting

$$\xi_n = \max\Bigl\{0, \sup_{p \in F(T), x \in C} \bigl(\phi(p, T^n x) - \phi(p, x)\bigr)\Bigr\},$$

it follows that $\xi_n \to 0$ as $n \to \infty$. Then (1.3) is reduced to the following:

$$\phi(p, T^n x) \leq \phi(p, x) + \xi_n, \quad \forall p \in F(T), \forall x \in C. \tag{1.4}$$

Remark 1.6 The class of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense was first considered by Qin and Wang [9].

Remark 1.7 The class of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense is a generalization of the class of asymptotically quasi-nonexpansive mappings in the intermediate sense, which was considered by Kirk [10], in the framework of Banach spaces.

Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem. Find $p \in C$ such that

$$f(p,y) \geq 0, \quad \forall y \in C. \tag{1.5}$$

We use $EP(f)$ to denote the solution set of the equilibrium problem (1.5). That is,

$$EP(f) = \{p \in C : f(p,y) \geq 0, \forall y \in C\}.$$

We remark here that the equilibrium problem was first introduced by Fan [11]. Given a mapping $Q : C \to E^*$, let

$$f(x,y) = \langle Qx, y - x\rangle, \quad \forall x, y \in C.$$

Then $p \in EP(f)$ if and only if p is a solution of the following variational inequality. Find p such that

$$\langle Qp, y - p\rangle \geq 0, \quad \forall y \in C. \tag{1.6}$$

To study the equilibrium problems (1.5), we may assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F\big(tz + (1-t)x, y\big) \leq F(x,y);$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and weakly lower semi-continuous.

Numerous problems in physics, optimization, and economics reduce to find a solution of (1.5). Recently, many authors have investigated common solutions of fixed point and equilibrium problems in Banach spaces; see, for example, [12–33] and the references therein.

In this paper, we consider a projection algorithm for treating the equilibrium problem and fixed point problems of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense.

In order to prove our main results, we need the following lemmas.

Lemma 1.8 [2] *Let E be a reflexive, strictly convex and smooth Banach space. Let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 1.9 [2] *Let C be a nonempty closed convex subset of a smooth Banach space E and let $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 1.10 *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4). *Let $r > 0$ and $x \in E$. Then*

(a) [34] *There exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.$$

(b) [6, 24] *Define a mapping $T_r : E \to C$ by*

$$S_r x = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle, \forall y \in C \right\}.$$

Then the following conclusions hold:

(1) *S_r is single-valued*;

(2) *S_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle S_r x - S_r y, JS_r x - JS_r y \rangle \leq \langle S_r x - S_r y, Jx - Jy \rangle$$

(3) *$F(S_r) = EP(f)$*;

(4) *S_r is quasi-ϕ-nonexpansive*;

(5) *$\phi(q, S_r x) + \phi(S_r x, x) \leq \phi(q, x)$, $\forall q \in F(S_r)$*;

(6) *$EP(f)$ is closed and convex.*

Lemma 1.11 [35] *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|)$$

for all $x, y \in B_r = \{x \in E : \|x\| \leq r\}$ and $t \in [0, 1]$.

2 Main results

Theorem 2.1 *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property and let C be a nonempty closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and let N be some positive integer. Let $T_i : C \to C$ an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense for every $1 \leq i \leq N$. Assume that T_i is closed asymptotically regular on C and $\bigcap_{i=1}^N F(T_i) \cap$*

$EF(f)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_{n,0} J x_n + \sum_{i=1}^N \alpha_{n,i} J T_i^n x_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $\xi_n = \max\{0, \sup_{p \in F(T_i), x \in C}(\phi(p, T_i^n x) - \phi(p, x))\}$, $\{\alpha_{n,i}\}$ is a real number sequence in $(0,1)$ for every $1 \leq i \leq N$, $\{r_n\}$ is a real number sequence in $[k, \infty)$, where k is some positive real number. Assume that $\sum_{i=0}^N \alpha_{n,i} = 1$ and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ for every $1 \leq i \leq N$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{\bigcap_{i=1}^N F(T_i) \cap EF(f)} x_1$, where $\Pi_{\bigcap_{i=1}^N F(T_i) \cap EF(f)}$ is the generalized projection from E onto $\bigcap_{i=1}^N F(T_i) \cap EF(f)$.

Proof First, we show that $\bigcap_{i=1}^N F(T_i) \cap EF(f)$ is closed and convex. From [9], we find that $\bigcap_{i=1}^N F(T_i)$ is closed and convex, which combines with Lemma 1.10 shows that $\bigcap_{i=1}^N F(T_i) \cap EF(f)$ is closed and convex. Next, we show that C_n is closed and convex. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_h is closed and convex for some positive integer h. For $z \in C_h$, we see that $\phi(z, u_h) \leq \phi(z, x_h) + \xi_h$ is equivalent to

$$2\langle z, J x_h - J u_h \rangle \leq \|x_k\|^2 - \|u_k\|^2 + \xi_h.$$

It is to see that C_{h+1} is closed and convex. This proves that C_n is closed and convex. This in turn shows that $\Pi_{C_{n+1}} x_1$ is well defined. Putting $u_n = S_{r_n} y_n$, we from Lemma 1.10 see that S_{r_n} is quasi-ϕ-nonexpansive. Now, we are in a position to prove that $\bigcap_{i=1}^N F(T_i) \cap EF(f) \subset C_n$. Indeed, $\bigcap_{i=1}^N F(T_i) \cap EF(f) \subset C_1 = C$ is obvious. Assume that $\bigcap_{i=1}^N F(T_i) \cap EF(f) \subset C_h$ for some positive integer h. Then, for $\forall w \in \bigcap_{i=1}^N F(T_i) \cap EF(f) \subset C_h$, we have

$$\begin{aligned} \phi(w, u_h) &= \phi(w, S_{r_h} y_h) \\ &\leq \phi(w, y_h) \\ &= \phi\left(w, J^{-1}\left(\alpha_{h,0} J x_h + \sum_{i=1}^N \alpha_{h,i} J T_i^h x_h\right)\right) \\ &= \|w\|^2 - 2\left\langle w, \alpha_{h,0} J x_h + \sum_{i=1}^N \alpha_{h,i} J T_i^h x_h \right\rangle + \left\|\alpha_{h,0} J x_h + \sum_{i=1}^N \alpha_{h,i} J T_i^h x_h\right\|^2 \\ &\leq \|w\|^2 - 2\alpha_{h,0}\langle w, J x_h \rangle - 2\sum_{i=1}^N \alpha_{h,i}\langle w, J T_i^h x_h \rangle + \alpha_{h,0}\|x_h\|^2 + \sum_{i=1}^N \alpha_{h,i}\left\|T_i^h x_h\right\|^2 \\ &= \alpha_{h,0}\phi(w, x_h) + \sum_{i=1}^N \alpha_{h,i}\phi\left(w, T_i^h x_h\right) \\ &\leq \alpha_{h,0}\phi(w, x_h) + \sum_{i=1}^N \alpha_{h,i}\phi(w, x_h) + \sum_{i=1}^N \alpha_{h,i}\xi_h \end{aligned}$$

$$= \phi(w, x_h) + \sum_{i=1}^{N} \alpha_{h,i}\xi_h$$

$$\leq \phi(w, x_h) + \sum_{i=1}^{N} \xi_h, \tag{2.1}$$

which shows that $w \in C_{h+1}$. This implies that $\bigcap_{i=1}^{N} F(T_i) \cap EF(f) \subset C_n$.

Next, we prove that the sequence $\{x_n\}$ is bounded. Notice that $x_n = \Pi_{C_n} x_1$. We find from Lemma 1.9 that $\langle x_n - z, Jx_1 - Jx_n\rangle \geq 0$, for any $z \in C_n$. Since $\bigcap_{i=1}^{N} F(T_i) \cap EF(f) \subset C_n$, we find that

$$\langle x_n - w, Jx_1 - Jx_n\rangle \geq 0, \quad \forall w \in \bigcap_{i=1}^{N} F(T_i) \cap EF(f). \tag{2.2}$$

It follows from Lemma 1.8 that

$$\begin{aligned} \phi(x_n, x_1) &\leq \phi(\Pi_{\bigcap_{i=1}^{N} F(T_i)\cap EF(f)} x_1, x_1) - \phi(\Pi_{\bigcap_{i=1}^{N} F(T_i)\cap EF(f)} x_1, x_n) \\ &\leq \phi(\Pi_{\bigcap_{i=1}^{N} F(T_i)\cap EF(f)} x_1, x_1). \end{aligned}$$

This implies that the sequence $\{\phi(x_n, x_1)\}$ is bounded. It follows from (1.1) that the sequence $\{x_n\}$ is also bounded. Since the space is reflexive, we may assume, without loss of generality, that $x_n \rightharpoonup \bar{x}$. Next, we prove that $\bar{x} \in \bigcap_{i=1}^{N} F(T_i) \cap EF(f)$. Since C_n is closed and convex, we find that $\bar{x} \in C_n$. This implies from $x_n = \Pi_{C_n} x_1$ that $\phi(x_n, x_1) \leq \phi(\bar{x}, x_1)$. On the other hand, we see from the weakly lower semicontinuity of $\|\cdot\|$ that

$$\begin{aligned} \phi(\bar{x}, x_1) &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jx_1\rangle + \|x_1\|^2 \\ &\leq \liminf_{n\to\infty} \bigl(\|x_n\|^2 - 2\langle x_n, Jx_1\rangle + \|x_1\|^2\bigr) \\ &= \liminf_{n\to\infty} \phi(x_n, x_1) \\ &\leq \limsup_{n\to\infty} \phi(x_n, x_1) \\ &\leq \phi(\bar{x}, x_1), \end{aligned}$$

which implies that $\lim_{n\to\infty} \phi(x_n, x_1) = \phi(\bar{x}, x_1)$. Hence, we have $\lim_{n\to\infty} \|x_n\| = \|\bar{x}\|$. In view of the Kadec-Klee property of E, we find that $x_n \to \bar{x}$ as $n \to \infty$. Since $x_n = \Pi_{C_n} x_1$, and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we find that $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$. This shows that $\{\phi(x_n, x_1)\}$ is nondecreasing. We find from its boundedness that $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. It follows that

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_1) \\ &\leq \phi(x_{n+1}, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\ &= \phi(x_{n+1}, x_1) - \phi(x_n, x_1). \end{aligned}$$

This implies that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{2.3}$$

In the light of $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1}$, we find that

$$\phi(x_{n+1}, u_n) \le \phi(x_{n+1}, x_n) + \xi_n.$$

It follows from (2.3) that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \tag{2.4}$$

In view of (1.1), we see that $\lim_{n\to\infty}(\|x_{n+1}\| - \|u_n\|) = 0$. This implies that $\lim_{n\to\infty} \|u_n\| = \|\bar{x}\|$. That is,

$$\lim_{n\to\infty} \|Ju_n\| = \lim_{n\to\infty} \|u_n\| = \|J\bar{x}\|. \tag{2.5}$$

This implies that $\{Ju_n\}$ is bounded. Note that both E and E^* are reflexive. We may assume, without loss of generality, that $Ju_n \rightharpoonup u^* \in E^*$. In view of the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists an element $u \in E$ such that $Ju = u^*$. It follows that

$$\begin{aligned}\phi(x_{n+1}, u_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|u_n\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|Ju_n\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on both sides of the equality aboven yields

$$\begin{aligned}0 &\ge \|\bar{x}\|^2 - 2\langle \bar{x}, u^*\rangle + \|u^*\|^2 \\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Ju\rangle + \|Ju\|^2 \\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Ju\rangle + \|u\|^2 \\ &= \phi(\bar{x}, u).\end{aligned}$$

That is, $\bar{x} = u$, which in turn implies that $u^* = J\bar{x}$. It follows that $Ju_n \rightharpoonup J\bar{x} \in E^*$. Since E^* enjoys the Kadec-Klee property, we obtain from (2.5) that $\lim_{n\to\infty} Ju_n = J\bar{x}$. Since $J^{-1} : E^* \to E$ is demicontinuous and E enjoys the Kadec-Klee property, we obtain $u_n \to \bar{x}$, as $n \to \infty$. Note that

$$\|x_n - u_n\| \le \|x_n - \bar{x}\| + \|\bar{x} - u_n\|.$$

It follows that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{2.6}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{2.7}$$

On the other hand, we have

$$\begin{aligned}\phi(w, x_n) - \phi(w, u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle w, Jx_n - Ju_n\rangle \\ &\le \|x_n - u_n\|\big(\|x_n\| + \|u_n\|\big) + 2\|w\|\|Jx_n - Ju_n\|.\end{aligned}$$

We, therefore, find that

$$\lim_{n\to\infty}\big(\phi(w,x_n)-\phi(w,u_n)\big)=0. \tag{2.8}$$

Since E is uniformly smooth, we know that E^* is uniformly convex. In view of Lemma 1.11, we find that

$$\begin{aligned}
\phi(w,u_n) &= \phi(w,S_{r_n}y_n)\\
&\le \phi(w,y_n)\\
&= \phi\left(w,J^{-1}\left(\alpha_{n,0}Jx_n+\sum_{i=1}^{N}\alpha_{n,i}JT_i^nx_n\right)\right)\\
&= \|w\|^2-2\left\langle w,\alpha_{n,0}Jx_n+\sum_{i=1}^{N}\alpha_{n,i}JT_i^nx_n\right\rangle+\left\|\alpha_{n,0}Jx_n+\sum_{i=1}^{N}\alpha_{n,i}JT_i^nx_n\right\|^2\\
&\le \|w\|^2-2\alpha_{n,0}\langle w,Jx_n\rangle-2\sum_{i=1}^{N}\alpha_{n,i}\langle w,JT_i^nx_n\rangle+\alpha_{n,0}\|x_n\|^2+\sum_{i=1}^{N}\alpha_{n,i}\|T_i^nx_n\|^2\\
&\quad -\alpha_{n,0}\alpha_{n,1}g\big(\|Jx_n-JT_1^nx_n\|\big)\\
&= \alpha_{n,0}\phi(w,x_n)+\sum_{i=1}^{N}\alpha_{n,i}\phi\big(w,T_i^nx_n\big)-\alpha_{n,0}\alpha_{n,1}g\big(\|Jx_n-JT_1^nx_n\|\big)\\
&\le \alpha_{n,0}\phi(w,x_n)+\sum_{i=1}^{N}\alpha_{n,i}\phi(w,x_n)+\sum_{i=1}^{N}\alpha_{n,i}\xi_h\\
&\quad -\alpha_{n,0}\alpha_{n,1}g\big(\|Jx_n-JT_1^nx_n\|\big)\\
&= \phi(w,x_n)+\sum_{i=1}^{N}\alpha_{n,i}\xi_n-\alpha_{n,0}\alpha_{n,1}g\big(\|Jx_n-JT_1^nx_n\|\big)\\
&\le \phi(w,x_n)+\xi_n-\alpha_{n,0}\alpha_{n,1}g\big(\|Jx_n-JT_1^nx_n\|\big).
\end{aligned}$$

It follows that

$$\alpha_{n,0}\alpha_{n,1}g\big(\|Jx_n-JT_1^nx_n\|\big)\le\phi(w,x_n)-\phi(w,u_n)+\xi_n.$$

In view of the restriction on the sequences, we find from (2.8) that $\lim_{n\to\infty}g(\|Jx_n-JT_1^nx_n\|)=0$. It follows that

$$\lim_{n\to\infty}\|Jx_n-JT_1^nx_n\|=0.$$

In the same way, we obtain

$$\lim_{n\to\infty}\|Jx_n-JT_i^nx_n\|=0,\quad \forall 1\le i\le N.$$

Notice that $\|JT_i^nx_n-J\bar{x}\|\le\|JT_i^nx_n-Jx_n\|+\|Jx_n-J\bar{x}\|$. It follows that

$$\lim_{n\to\infty}\|JT_i^nx_n-J\bar{x}\|=0. \tag{2.9}$$

The demicontinuity of $J^{-1}: E^* \to E$ implies that $T_i^n x_n \rightharpoonup \bar{x}$. Note that

$$\big| \| T_i^n x_n \| - \|\bar{x}\| \big| = \big| \| J T_i^n x_n \| - \|J\bar{x}\| \big| \le \| J T_i^n x_n - J\bar{x} \|.$$

This implies from (2.9) that $\lim_{n\to\infty} \|T_i^n x_n\| = \|\bar{x}\|$. Since E has the Kadec-Klee property, we obtain $\lim_{n\to\infty} \|T_i^n x_n - \bar{x}\| = 0$. On the other hand, we have

$$\| T_i^{n+1} x_n - \bar{x} \| \le \| T_i^{n+1} x_n - T_i^n x_n \| + \| T_i^n x_n - \bar{x} \|.$$

It follows from the uniformly asymptotic regularity of T_i that

$$\lim_{n\to\infty} \| T_i^{n+1} x_n - \bar{x} \| = 0.$$

That is, $T_i T_i^n x_n \to \bar{x}$. From the closedness of T_i, we find $\bar{x} = T_i \bar{x}$ for every $1 \le i \le N$. This proves $\bar{x} \in \bigcap_{i=1}^N F(T_i)$.

Next, we show that $\bar{x} \in EF(f)$. In view of Lemma 1.8, we find that

$$\begin{aligned}
\phi(u_n, y_n) &\le \phi(w, y_n) - \phi(w, u_n) \\
&\le \phi(w, x_n) + \mu_n - \phi(w, u_n).
\end{aligned}$$

It follows from (2.8) that $\lim_{n\to\infty} \phi(u_n, y_n) = 0$. This implies that $\lim_{n\to\infty}(\|u_n\| - \|y_n\|) = 0$. It follows from (2.6) that

$$\lim_{n\to\infty} \|y_n\| = \|\bar{x}\|.$$

It follows that

$$\lim_{n\to\infty} \|Jy_n\| = \lim_{n\to\infty} \|y_n\| = \|\bar{x}\| = \|J\bar{x}\|.$$

This shows that $\{Jy_n\}$ is bounded. Since E^* is reflexive, we may assume that $Jy_n \rightharpoonup v^* \in E^*$. In view of $J(E) = E^*$, we see that there exists $v \in E$ such that $Jv = v^*$. It follows that

$$\begin{aligned}
\phi(u_n, y_n) &= \|u_n\|^2 - 2\langle u_n, Jy_n\rangle + \|y_n\|^2 \\
&= \|u_n\|^2 - 2\langle u_n, Jy_n\rangle + \|Jy_n\|^2.
\end{aligned}$$

Taking $\liminf_{n\to\infty}$ the both sides of equality above yields that

$$\begin{aligned}
0 &\ge \|\bar{x}\|^2 - 2\langle \bar{x}, v^*\rangle + \|v^*\|^2 \\
&= \|\bar{x}\|^2 - 2\langle \bar{x}, Jv\rangle + \|Jv\|^2 \\
&= \|\bar{x}\|^2 - 2\langle \bar{x}, Jv\rangle + \|v\|^2 \\
&= \phi(\bar{x}, v).
\end{aligned}$$

That is, $\bar{x} = v$, which in turn implies that $v^* = J\bar{x}$. It follows that $Jy_n \rightharpoonup J\bar{x} \in E^*$. Since E^* enjoys the Kadec-Klee property, we obtain $Jy_n - J\bar{x} \to 0$ as $n \to \infty$. Note that $J^{-1}: E^* \to E$

is demicontinuous. It follows that $y_n \rightharpoonup \bar{x}$. Since E enjoys the Kadec-Klee property, we obtain $y_n \to \bar{x}$ as $n \to \infty$. Note that

$$\|u_n - y_n\| \le \|u_n - \bar{x}\| + \|\bar{x} - y_n\|.$$

This implies that $\lim_{n\to\infty} \|u_n - y_n\| = 0$. Since J is uniformly norm-to-norm continuous on any bounded sets, we have $\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0$. From the assumption $r_n \ge k$, we see that

$$\lim_{n\to\infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0. \tag{2.10}$$

Since $u_n = S_{r_n} y_n$, we find that

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall y \in C.$$

It follows from (A2) that

$$\|y - u_n\| \frac{\|Ju_n - Jy_n\|}{r_n} \ge \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge f(y, u_n), \quad \forall y \in C.$$

In view of (A4), we find from (2.10) that

$$f(y, \bar{x}) \le 0, \quad \forall y \in C.$$

For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)\bar{x}$. It follows that $y_t \in C$, which yields $f(y_t, \bar{x}) \le 0$. It follows from (A1) and (A4) that

$$0 = f(y_t, y_t) \le tf(y_t, y) + (1-t)f(y_t, \bar{x}) \le tf(y_t, y).$$

That is,

$$f(y_t, y) \ge 0.$$

Letting $t \downarrow 0$, we obtain from (A3) that $f(\bar{x}, y) \ge 0$, $\forall y \in C$. This implies that $\bar{x} \in EP(f)$.

Finally, we turn our attention to proving that $\bar{x} = \Pi_{\bigcap_{i=1}^N F(T_i) \cap EF(f)} x_1$.

Letting $n \to \infty$ in (2.2), we obtain

$$\langle \bar{x} - w, Jx_1 - J\bar{x}\rangle \ge 0, \quad \forall w \in \bigcap_{i=1}^N F(T_i) \cap EF(f).$$

In view of Lemma 1.9, we find that $\bar{x} = \Pi_{\bigcap_{i=1}^N F(T_i) \cap EF(f)} x_1$. This completes the proof. □

From the definition of quasi-ϕ-nonexpansive mappings, we see that every quasi-ϕ-nonexpansive mapping is asymptotically quasi-ϕ-nonexpansive in the intermediate sense. We also know that every uniformly smooth and uniformly convex space is a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property (note that every uniformly convex Banach space enjoys the Kadec-Klee property).

Remark 2.2 Theorem 2.1 can be viewed an extension of the corresponding results in Qin *et al.* [6], Kim [12], Qin *et al.* [22], Takahashi and Zembayashi [24], respectively. The space L^p, where $p > 1$, satisfies the restriction in Theorem 2.1.

3 Applications

Theorem 3.1 *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property and let C be a nonempty closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4). *Let $T : C \to C$ an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense. Assume that T is closed asymptotically regular on C and $F(T) \cap EF(f)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_{n,0} J x_n + \alpha_{n,1} J T^n x_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $\xi_n = \max\{0, \sup_{p \in F(T), x \in C}(\phi(p, T^n x) - \phi(p, x))\}$, $\{r_n\}$ is a real number sequence in $[k, \infty)$, where k is some positive real number, $\{\alpha_{n,0}\}$ and $\{\alpha_n n, 1\}$ are two real number sequence in $(0, 1)$. Assume that $\liminf_{n \to \infty} \alpha_{n,0} \alpha_{n,1} > 0$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T) \cap EF(f)} x_1$, where $\Pi_{F(T) \cap EF(f)}$ is the generalized projection from E onto $F(T) \cap EF(f)$.

Proof Putting $N = 1$, we draw from Theorem 2.1 the desired conclusion immediately. □

Remark 3.2 If the mapping T in Theorem 3.1 is quasi-ϕ-nonexpansive, then the restrictions that T is closed asymptotically regular on C and $F(T) \cap EF(f)$ is bounded will not be required anymore.

If $T_i = I$, where I is the identity for every $1 \leq i \leq N$, then we find from Theorem 2.1 the following.

Theorem 3.3 *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property and let C be a nonempty closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4). *Assume that $EF(f)$ is nonempty. Let*

$\{x_n\}$ *be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $\{r_n\}$ *is a real number sequence in* $[k, \infty)$, *where k is some positive real number. Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{EF(f)} x_1$, *where* $\Pi_{EF(f)}$ *is the generalized projection from E onto* $EF(f)$.

Competing interests
The authors declare that they have no competing interests.

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