Frasin *Journal of Inequalities and Applications* 2012, **2012**:24
http://www.journalofinequalitiesandapplications.com/content/2012/1/24

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

New subclasses of analytic functions

Basem Aref Frasin

Correspondence: bafrasin@yahoo.com
Faculty of Science, Department of Mathematics, Al al-Bayt University, P.O. Box 130095, Mafraq, Jordan

Abstract

For analytic functions $f(z)$ in the open unit disk $\mathcal{U}$, subclasses $\mathcal{T}(\beta_1, \beta_2, \beta_3 : \lambda), \mathcal{P}(\theta, \alpha)$, and $\mathcal{K}(\theta, \alpha)$ are introduced. The object of the present article is to discuss some interesting properties of functions $f(z)$ associated with classes $\mathcal{T}(\beta_1, \beta_2, \beta_3 : \lambda), \mathcal{P}(\theta, \alpha)$, and $\mathcal{K}(\theta, \alpha)$.

Mathematics Subject Classification (2010): 30C45.

Keywords: analytic, univalent functions, Cauchy-Schwarz inequality

1. Introduction and Definitions

Let $\mathcal{A}$ denotes the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also, a function $f(z)$ belonging to $\mathcal{A}$ is said to be convex of order α if it satisfies

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in \mathcal{U}) \tag{1.2}$$

for some $\alpha (0 \le \alpha < 1)$. We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are convex of order α in $\mathcal{U}$ (see, [1,2]). Further, a function $f(z)$ belonging to $\mathcal{A}$ is said to be in the class $\mathcal{P}(\alpha)$ iff

$$\text{Re}\,(zf''(z) + f'(z)) > \alpha, \quad (z \in \mathcal{U}). \tag{1.3}$$

for some $\alpha (0 \le \alpha < 1)$.

For analytic functions $f(z)$, Uyanik and Owa [3], obtained some interesting properties for analytic functions in the subclass $\mathcal{A}(\beta_1, \beta_2, \beta_3; \lambda)$ defined by

$$\left|\beta_1 z\left(\frac{f(z)}{z}\right)' + \beta_2 z\left(\frac{f(z)}{z}\right)'' + \beta_3 z\left(\frac{f(z)}{z}\right)'''\right| \le \lambda$$
$$(\beta_1, \beta_2, \beta_3 \in \mathbb{C}; \lambda > 0; z \in \mathcal{U}),$$

associated with close-to-convex functions and starlike functions of order α.

In this article, we define the following subclass of analytic functions.

Definition 1.1. A function $f(z)$ belonging to $\mathcal{A}$ is said to be in the class $\mathcal{T}(\beta_1, \beta_2, \beta_3; \lambda)$, if it satisfies

$$\left|\beta_1 z f''(z) + \beta_2 z^2 f'''(z) + \beta_3 z^3 f^{(4)}(z)\right| \leq \lambda \quad (z \in \mathcal{U}), \tag{1.4}$$

for some complex numbers β_1, β_2, β_3, and for some real $\lambda > 0$.

Example 1.2. *Let us consider the function* $f_\gamma(z)$, $\gamma \in \mathbb{R}$, *given by*

$$f_\gamma(z) = z(1+z)^\gamma.$$

Then, we observe that

$$\begin{aligned}&\left|\beta_1 z f_\gamma''(z) + \beta_2 z^2 f_\gamma'''(z) + \beta_3 z^3 f_\gamma^{(4)}(z)\right| \\ &\quad = \left|\sum_{n=2}^{\infty} n(n-1)\binom{\gamma}{n-1}(\beta_1 + (n-2)\beta_2 + (n-2)(n-3)\beta_3) z^{n-1}\right|,\end{aligned}$$

where

$$\binom{\gamma}{n-1} = \frac{\gamma(\gamma-1)(\gamma-2)\cdots(\gamma-n+2)}{(n-1)!}.$$

Therefore, if $\gamma = 1$, then

$$\left|\beta_1 z f_1''(z) + \beta_2 z^2 f_1'''(z) + \beta_3 z^3 f_1^{(4)}(z)\right| = |2\beta_1 z| \leq 2\,|\beta_1|\,.$$

This implies that $f_1(z) \in \mathcal{T}(\beta_1, \beta_2, \beta_3; \lambda)$ for $\lambda \geq 2\ |\beta_1|$. If $\gamma = 2$, then

$$\left|\beta_1 z f_2''(z) + \beta_2 z^2 f_2'''(z) + \beta_3 z^3 f_2^{(4)}(z)\right| = \left|4\beta_1 z + 6(\beta_1 + \beta_2) z^2\right| \leq 10\,|\beta_1| + 6\,|\beta_2|\,.$$

Therefore, $f_2(z) \in \mathcal{T}(\beta_1, \beta_2, \beta_3; \lambda)$ for $\lambda \geq 10\ |\beta_1| + 6\ |\beta_2|$. Further, if $\gamma = 3$; then we have

$$\begin{aligned}&\left|\beta_1 z f_3''(z) + \beta_2 z^2 f_3'''(z) + \beta_3 z^3 f_3^{(4)}(z)\right| \\ &\quad = \left|6\beta_1 z + 18(\beta_1 + \beta_2) z^2 + 12(\beta_1 + 2\beta_2 + 2\beta_3) z^3\right| \\ &\quad \leq 36\,|\beta_1| + 42\,|\beta_2| + 24\,|\beta_3|\,.\end{aligned}$$

Thus, $f_3(z) \in \mathcal{T}(\beta_1, \beta_2, \beta_3; \lambda)$ for $\lambda \geq 36\ |\beta_1| + 42\ |\beta_2| + 24\ |\beta_3|$.

Now, let $\mathcal{A}_\theta$ denotes the subclass of $\mathcal{A}$ consisting of functions $f(z)$ with

$$a_n = |a_n|\, e^{i((n-1)\theta + \pi)} \quad (n = 2, 3, \ldots).$$

Also, we introduce the subclasses $\mathcal{P}(\theta, \alpha)$ and $\mathcal{K}(\theta, \alpha)$ of $\mathcal{A}_\theta$ as follows:

$$\mathcal{P}(\theta, \alpha) = \mathcal{A}_\theta \cap \mathcal{P}(\alpha) \quad \text{and } \mathcal{K}(\theta, \alpha) = \mathcal{A}_\theta \cap \mathcal{K}(\alpha).$$

2. Properties of the class $\mathcal{T}(\beta_1, \beta_2, \beta_3; \lambda)$

We first prove

Theorem 2.1. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)\,|\beta_2| + (n-2)(n-3)\,|\beta_3|)\,|a_n| \leq \lambda \tag{2.1}$$

for some complex numbers β_1*,* β_2*,* β_3 *and for some real* $\lambda > 0$*, then* $f(z) \in \mathcal{T}(\beta_1, \beta_2, \beta_3; \lambda)$.

Proof. We observe that

$$\begin{aligned}
&\left|\beta_1 z f_3''(z) + \beta_2 z^2 f_3'''(z) + \beta_3 z^3 f_3^{(4)}(z)\right| \\
&\quad = \left|\sum_{n=2}^{\infty} n(n-1)(\beta_1 + (n-2)\beta_2 + (n-2)(n-3)\beta_3) a_n z^{n-1}\right| \\
&\quad \le \sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n||z|^{n-1} \\
&\quad < \sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n|.
\end{aligned}$$

Therefore, if $f(z)$ satisfies the inequality (2.1), then $f(z) \in \mathcal{T}(\beta_1, \beta_2, \beta_3; \lambda)$.

Next, we prove

Theorem 2.2. *if* $f(z) \in \mathcal{T}(\beta_1, \beta_2, \beta_3; \lambda)$ *with* $\arg \beta_1 = \arg \beta_2 = \arg \beta_3 = \varphi$ *and* $a_n = |a_n| e^{i((n-1)\theta-\varphi)}$ $(n = 2, 3, \ldots)$*, then we have*

$$\sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n| \le \lambda.$$

Proof. For $f(z) \in \mathcal{T}(\beta_1, \beta_2, \beta_3; \lambda)$, we see that

$$\begin{aligned}
&\left|\beta_1 z f_3''(z) + \beta_2 z^2 f_3'''(z) + \beta_3 z^3 f_3^{(4)}(z)\right| \\
&\quad = \left|\sum_{n=2}^{\infty} n(n-1)(\beta_1 + (n-2)\beta_2 + (n-2)(n-3)\beta_3) a_n z^{n-1}\right| \\
&\quad = \left|\sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n| e^{i(n-1)\theta} z^{n-1}\right| \\
&\quad \le \lambda.
\end{aligned}$$

for all $z \in \mathcal{U}$. Let us consider a point $z \in \mathcal{U}$ such that $z = |z| e^{-i\theta}$.

Then we have

$$\left|\sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n||z|^{n-1}\right| \le \lambda.$$

Letting $|z| \to 1^-$, we obtain

$$\sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n| \le \lambda.$$

Corollary 2.3. *If* $f(z) \in \mathcal{T}(\beta_1, \beta_2, \beta_3; \lambda)$ *with* $\arg \beta_1 = \arg \beta_2 = \arg \beta_3 = \varphi$ *and* $a_n = |a_n| e^{i((n-1)\theta-\varphi)}$ $(n = 2, 3, \ldots)$*, then we have*

$$|a_n| \le \frac{\lambda}{n(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)} \qquad (n = 2, 3, \ldots).$$

Example 2.4. *Let us consider the function* $f(z) \in \mathcal{T}(\beta_1, \beta_2, \beta_3; \lambda)$ *with* $\arg \beta_1 = \arg \beta_2 = \arg \beta_3 = \varphi$ *and*

$$a_n = \frac{\lambda e^{i((n-1)\theta - \phi)}}{n^2(n-1)^2(|\beta_1| + (n-2)\,|\beta_2| + (n-2)(n-3)\,|\beta_3|)} \qquad (n = 2, 3\ldots).$$

Then, we see that

$$\sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)\,|\beta_2| + (n-2)(n-3)\,|\beta_3|)\,|a_n|$$
$$= \lambda \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \lambda \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right) = \lambda.$$

Corollary 2.5. *If* $f(z) \in \mathcal{T}(\beta_1, \beta_2, \beta_3; \lambda)$ *with* $\arg \beta_1 = \arg \beta_2 = \arg \beta_3 = \varphi$ *and* $a_n = |a_n|\, e^{i((n-1)\theta-\varphi)}$ (n = 2, 3,...), *then we have*

$$|z| - \sum_{n=2}^{j} |a_n||z|^n - A_j|z|^{j+1} \le \left|f(z)\right| \le |z| + \sum_{n=2}^{j} |a_n||z|^n + A_j|z|^{j+1}$$

with

$$A_j = \frac{\left(\lambda - \sum_{n=2}^{j} n(n-1)(|\beta_1| + (n-2)\,|\beta_2| + (n-2)(n-3)\,|\beta_3|)\,|a_n|\right)}{j(j+1)(|\beta_1|) + (j-1)\,|\beta_2| + (j-1)(j-2)\,|\beta_3|)}$$

and

$$1 - \sum_{n=2}^{j} |a_n|\,|z|^{n-1} - B_j|z|^j \le \left|f'(z)\right| \le 1 + \sum_{n=2}^{j} |a_n||z|^{n-1} + B_j|z|^j$$

with

$$B_j = \frac{\left(\lambda - \sum_{n=2}^{j} n(n-1)(|\beta_1| + (n-2)\,|\beta_2| + (n-2)(n-3)\,|\beta_3|)\,|a_n|\right)}{j(|\beta_1| + (j-1)\,|\beta_2| + (j-1)(j-2)\,|\beta_3|)}$$

Proof. In view of Theorem 2.1, we know that

$$\sum_{n=j+1}^{\infty} n(n-1)(|\beta_1| + (n-2)\,|\beta_2| + (n-2)(n-3)\,|\beta_3|)\,|a_n|$$
$$\le \lambda - \sum_{n=2}^{j} n(n-1)(|\beta_1| + (n-2)\,|\beta_2| + (n-2)(n-3)\,|\beta_3|)\,|a_n|.$$

Further, we note that

$$j(j+1)(|\beta_1| + (j-1)\,|\beta_2| + (j-1)(j-2)\,|\beta_3|) \sum_{n=j+1}^{\infty} |a_n|$$
$$\le \sum_{n=j+1}^{\infty} n(n-1)(|\beta_1| + (n-2)\,|\beta_2| + (n-2)(n-3)\,|\beta_3|)\,|a_n|\,,$$

which is equivalent to

$$\begin{aligned}\sum_{n=j+1}^{\infty}|a_n| &\le \frac{\left(\lambda-\sum\limits_{n=2}^{j}n(n-1)(|\beta_1|+(n-2)\,|\beta_2|+(n-2)(n-3)\,|\beta_3|)\,|a_n|\right)}{j(j+1)(|\beta_1|+(j-1)\,|\beta_2|+(j-1)(j-2)\,|\beta_3|)}\\ &= A_j.\end{aligned}$$

Thus, we have

$$\left|f(z)\right| \le |z| + \sum_{n=2}^{j}|a_n||z|^n + \sum_{n=j+1}^{\infty}|a_n|\,|z|^n \le |z| + \sum_{n=2}^{j}|a_n|\,|z|^n + A_j|z|^{j+1}$$

and

$$\left|f(z)\right| \ge |z| - \sum_{n=2}^{j}|a_n|\,|z|^n - \sum_{n=j+1}^{\infty}|a_n|\,|z|^n \ge |z| - \sum_{n=2}^{j}|a_n|\,|z|^n - A_j|z|^{j+1}.$$

Next, we observe that

$$\begin{aligned}&j(|\beta_1|+(j-1)\,|\beta_2|+(j-1)(j-2)\,|\beta_3|)\sum_{n=j+1}^{\infty}n\,|a_n|\\ &\quad\le \sum_{n=j+1}^{\infty}n(n-1)(|\beta_1|+(n-2)\,|\beta_2|+(n-2)(n-3)\,|\beta_3|)\,|a_n|\\ &\quad\le \lambda\sum_{n=2}^{j}n(n-1)(|\beta_1|+(n-1)\,|\beta_2|+(n-2)(n-3)\,|\beta_3|)\,|a_n|,\end{aligned}$$

which implies that

$$\begin{aligned}\sum_{n=j+1}^{\infty}n\,|a_n| &\le \frac{\left(\lambda-\sum\limits_{n=2}^{j}n(n-1)(|\beta_1|+(n-2)\,|\beta_2|+(n-2)(n-3)\,|\beta_3|)\,|a_n|\right)}{j(|\beta_1|+(j-1)\,|\beta_2|+(j-1)(j-2)\,|\beta_3|)}\\ &= B_j.\end{aligned}$$

Therefore, we obtain that

$$\left|f'(z)\right| \le 1 + \sum_{n=2}^{j}n\,|a_n|\,|z|^{n-1} + \sum_{n=j+1}^{\infty}n\,|a_n|\,|z|^{n-1} \le 1 + \sum_{n=2}^{j}|a_n||z|^{n-1} + B_j|z|^j$$

and

$$\left|f'(z)\right| \ge 1 - \sum_{n=2}^{j}n\,|a_n||z|^{n-1} - \sum_{n=j+1}^{\infty}n\,|a_n|\,|z|^{n-1} \ge 1 - \sum_{n=2}^{j}|a_n|\,|z|^{n-1} - B_j|z|^j.$$

3. Radius problem for the class $\mathcal{P}(\theta,\alpha)$

To obtain the radius problem for the class $\mathcal{P}(\theta,\alpha)$, we need the following lemma.

Lemma 3.1. *If* $f(z) \in \mathcal{P}(\theta, \alpha)$, *then*

$$\sum_{n=2}^{\infty} n^2 \left|a_n\right| \leq 1 - \alpha. \tag{3.1}$$

Proof. Let $f(z) \in \mathcal{P}(\theta, \alpha)$. Then, we have

$$\begin{aligned} \operatorname{Re}\left\{(zf''(z) + f'(z))\right\} &= \operatorname{Re}\left\{1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1}\right\} \\ &= \operatorname{Re}\left\{1 + \sum_{n=2}^{\infty} n^2 \left|a_n\right| e^{i((n-1)\theta + \pi)} z^{n-1}\right\} \\ &= \operatorname{Re}\left\{1 - \sum_{n=2}^{\infty} n^2 \left|a_n\right| e^{i((n-1)\theta)} z^{n-1}\right\} > \alpha \end{aligned}$$

for all $z \in \mathcal{U}$. Let us consider a point $z \in \mathcal{U}$ such that $z = |z| \, e^{-i\theta}$.
Then we have

$$1 - \sum_{n=2}^{\infty} n^2 \left|a_n\right| |z|^{n-1} > \alpha$$

Letting $|z| \to 1^-$, we obtain the inequality (3.1).

Corollary 3.2. *If* $f(z) \in \mathcal{P}(\theta, \alpha)$, *then*

$$\left|a_n\right| \leq \frac{1-\alpha}{n^2} \quad (n = 2, 3, \ldots).$$

Remark 3.3. By Lemma 3.1, we observe that if $f(z) \in \mathcal{P}(\theta, \alpha)$, then

$$\sum_{n=2}^{\infty} n(n-1) \left|a_n\right| \leq \sum_{n=2}^{\infty} n^2 \left|a_n\right| \leq 1 - \alpha.$$

Applying Theorem 2.1 and Lemma 3.1, we derive

Theorem 3.4. *Let* $f(z) \in \mathcal{P}(\theta, \alpha)$, *and* $\delta \in \mathbb{C}$ $(0 < |\delta| < 1)$. *Then the function* $\frac{1}{\delta} f(\delta z) \in \mathcal{T}(\beta_1, \beta_2, \beta_3; \lambda)$*for* $(0 < |\delta| \leq |\delta_0(\lambda)|$, *where* $|\delta_0(\lambda)|$ *is the smallest positive root of the equation*

$$\begin{aligned} &|\beta_1| \frac{|\delta| \sqrt{2(1-\alpha)}}{\left(1 - |\delta|^2\right)^{3/2}} + |\beta_2| \frac{|\delta|^2 \sqrt{(6 + 18|\delta|^2)(1 - \alpha - 2|a_2|^2)}}{\left(1 - |\delta|^2\right)^{5/2}} \\ &+ |\beta_3| \frac{4\sqrt{3}|\delta|^3 \sqrt{(1 + 8|\delta|^2 + 6|\delta|^4)(1 - \alpha - 2)|a_2|^2 - 6|a_3|^2)}}{\left(1 - |\delta|^2\right)^{7/2}} = \lambda \end{aligned} \tag{3.2}$$

in $0 < |\delta| < 1$.

Proof. For $f(z) \in \mathcal{P}(\theta, \alpha)$, we see that

$$\frac{1}{\delta} f(\delta z) = z + \sum_{n=2}^{\infty} \delta^{n-1} a_n z^n$$

and

$$\sum_{n=2}^{\infty} n(n-1)|a_n|^2 \leq 1-\alpha.$$

Thus, to show that $\frac{1}{\delta}f(\delta z) \in \mathcal{T}(\beta_1, \beta_2, \beta_3; \lambda)$, from Theorem 2.1, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)\,|\beta_2| + (n-2)(n-3)\,|\beta_3|)|\delta|^{n-1}\,|a_n| \leq \lambda.$$

Applying Cauchy-Schwarz inequality, we note that

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)\,|\beta_2| + (n-2)(n-3)\,|\beta_3|)|\delta|^{n-1}\,|a_n| \\
&\quad \leq \frac{|\beta_1|}{|\delta|}\left(\sum_{n=2}^{\infty} n(n-1)|\delta|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty} n(n-1)|a_n|^2\right)^{\frac{1}{2}} \\
&\qquad + \frac{|\beta_2|}{|\delta|}\left(\sum_{n=3}^{\infty} n(n-1)(n-2)^2|\delta|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=3}^{\infty} n(n-1)|a_n|^2\right)^{\frac{1}{2}} \\
&\qquad + \frac{|\beta_3|}{|\delta|}\left(\sum_{n=4}^{\infty} n(n-1)(n-2)^2(n-3)^2|\delta|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=4}^{\infty} n(n-1)|a_n|^2\right)^{\frac{1}{2}} \qquad (3.3) \\
&\quad \leq \frac{|\beta_1|}{|\delta|}\left(\sum_{n=2}^{\infty} n(n-1)|\delta|^{2n}\right)^{\frac{1}{2}}\sqrt{1-\alpha} \\
&\qquad + \frac{|\beta_2|}{|\delta|}\left(\sum_{n=3}^{\infty} n(n-1)(n-2)^2|\delta|^{2n}\right)^{\frac{1}{2}}\sqrt{1-\alpha-2|a_2|^2} \\
&\qquad + \frac{|\beta_3|}{|\delta|}\left(\sum_{n=4}^{\infty} n(n-1)(n-2)^2(n-3)^2|\delta|^{2n}\right)^{\frac{1}{2}}\sqrt{1-\alpha-2|\alpha_2|^2-6|a_3|^2}.
\end{aligned}$$

We note that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad (|x| < 1),$$

thus, we have

$$\sum_{n=2}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}. \qquad (3.4)$$

Since

$$\sum_{n=3}^{\infty} (n-2)x^{n-1} = x^2\left(\sum_{n=3}^{\infty} (n-2)x^{n-3}\right) = x^2\left(\sum_{n=3}^{\infty} x^{n-2}\right)' = \frac{x^2}{(1-x)^2},$$

we see that

$$\sum_{n=3}^{\infty}(n-1)(n-2)^2x^n = x^3\left(\frac{x^2}{(1-x)^2}\right)'' = \frac{2x^3+4x^4}{(1-x)^4}.$$

and thus, we obtain

$$\sum_{n=3}^{\infty}n(n-1)(n-2)^2x^n = \frac{6x^3+18x^4}{(1-x)^5}. \tag{3.5}$$

Furthermore, we have

$$\begin{aligned}\sum_{n=4}^{\infty}(n-1)(n-2)^2(n-3)^2x^n &= x^4\left(\sum_{n=4}^{\infty}(n-1)(n-2)^2(n-3)^2x^{n-4}\right)\\ &= x^4\left(\sum_{n=4}^{\infty}(n-2)(n-3)x^{n-1}\right)''',\end{aligned}$$

but

$$\sum_{n=4}^{\infty}(n-2)(n-3)x^{n-1} = x^3\left(\sum_{n=4}^{\infty}(n-2)(n-3)x^{n-4}\right) = \frac{2x^3}{(1-x)^3}$$

thus, we have

$$\sum_{n=4}^{\infty}(n-1)(n-2)^2(n-3)^2x^n = \frac{12x^4+72x^5+36x^6}{(1-x)^6},$$

which yields

$$\sum_{n=4}^{\infty}n(n-1)(n-2)^2(n-3)^2x^n = \frac{48x^4(1+8x+6x^2)}{(1-x)^7}. \tag{3.6}$$

Therefore, from (3.3)-(3.6) with $|\delta|^2 = x$, we obtain

$$\begin{aligned}&\sum_{n=2}^{\infty}n(n-1)(|\beta_1|+(n-2)\,|\beta_2|+(n-2)(n-3)\,|\beta_3|)|\delta|^{n-1}\,|a_n|\\ &\quad\le |\beta_1|\,\frac{|\delta|\sqrt{2(1-\alpha)}}{(1-|\delta|^2)^{3/2}}+|\beta_2|\,\frac{|\delta|^2\sqrt{(6+18|\delta|^2)(1-\alpha-2|a_2|^2)}}{(1-|\delta|^2)^{5/2}}\\ &\qquad|\beta_3|\,\frac{4\sqrt{3}|\delta|^3\sqrt{(1+8|\delta|^2+6|\delta|^4)(1-\alpha-2|a_2|^2-6|a_3|^2)}}{(1-|\delta|^2)^{7/2}}\end{aligned}$$

Now, let us consider the complex number δ $(0 < |\delta| < 1)$ such that

$$\begin{aligned}&|\beta_1|\,\frac{|\delta|\sqrt{2(1-\alpha)}}{(1-|\delta|^2)^{3/2}}+|\beta_2|\,\frac{|\delta|^2\sqrt{(6+18|\delta|^2)(1-\alpha-2|a_2|^2)}}{(1-|\delta|^2)^{5/2}}\\ &\quad|\beta_3|\,\frac{4\sqrt{3}|\delta|^3\sqrt{(1+8|\delta|^2+6|\delta|^4)(1-\alpha-2|a_2|^2-6|a_3|^2)}}{(1-|\delta|^2)^{7/2}} = \lambda.\end{aligned}$$

If we define the function $h(|\delta|)$ by

$$\begin{aligned} h(|\delta|) = {} & |\beta_1|\,|\delta|\,(1-|\delta|^2)^2\sqrt{2(1-\alpha)} \\ & + |\beta_2|\,|\delta|^2(1-|\delta|^2)\sqrt{(6+18|\delta|^2)(1-\alpha-2|a_2|^2)} \\ & + 4\sqrt{3}\,|\beta_3|\,|\delta|^3\sqrt{(1+8|\delta|^2+6|\delta|^4)(1-\alpha-2|a_2|^2-6|a_3|^2)} \\ & - \lambda(1-|\delta|^2)^{7/2}, \end{aligned}$$

then we have $h(0) = -\lambda < 0$ and $h(1) = 12\sqrt{5}\,|\beta_3|\,\sqrt{1-\alpha-2|a_2|^2-6|a_3|^2} > 0$. This means that there exists some δ_0 such that $h(|\delta_0|) = 0$ $(0 < |\delta_0| < 1)$. This completes the proof of the theorem.

4. Radius problem for the class $\mathcal{K}(\theta,\alpha)$

For the class $\mathcal{K}(\theta,\alpha)$, we prove the following lemma.

Lemma 4.1. *If* $f(z) \in \mathcal{K}(\theta,\alpha)$, *then*

$$\sum_{n=2}^{\infty} n(n-\alpha)\,|a_n| \leq 1-\alpha. \tag{4.1}$$

Proof. Let $f(z) \in \mathcal{K}(\theta,\alpha)$. Then, we have

$$\begin{aligned} \mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} &= \mathrm{Re}\left\{\frac{1+\sum_{n=2}^{\infty} n^2 a_n z^{n-1}}{1+\sum_{n=2}^{\infty} n a_n z^{n-1}}\right\} \\ &= \mathrm{Re}\left\{\frac{1-\sum_{n=2}^{\infty} n^2\,|a_n|\,e^{i(n-1)\theta} z^{n-1}}{1-\sum_{n=2}^{\infty} n\,|a_n|\,e^{i(n-1)\theta} z^{n-1}}\right\} > \alpha \end{aligned}$$

for all $z \in \mathcal{U}$. Let us consider a point $z \in \mathcal{U}$. such that $z = |z|e^{-i\theta}$.
Then we have

$$\frac{1-\sum_{n=2}^{\infty} n^2\,|a_n|\,|z|^{n-1}}{1-\sum_{n=2}^{\infty} n\,|a_n|\,|z|^{n-1}} > \alpha$$

Letting $|z| \to 1^-$, we obtain the inequality (4.1).

Corollary 4.2. *If* $f(z) \in \mathcal{K}(\theta,\alpha)$, *then*

$$|a_n| \leq \frac{1-\alpha}{n(n-\alpha)} \quad (n = 2, 3, \ldots).$$

Remark 4.3. If $f(z) \in \mathcal{K}(\theta,\alpha)$, then

$$\sum_{n=2}^{\infty} n(n-1)\,|a_n| \leq \sum_{n=2}^{\infty} n(n-\alpha)\,|a_n| \leq 1-\alpha.$$

Applying Theorem 2.1, Lemma 4.1 and using the same technique as in the proof of Theorem 3.4, we derive

Theorem 4.4. *Let* $f(z) \in \mathcal{K}(\theta, \alpha)$, *and* $\delta \in \mathbb{C}$ $(0 < |\delta| < 1)$. *Then the function* $\frac{1}{\delta} f(\delta z) \in \mathcal{T}(\beta_1, \beta_2, \beta_3; \lambda)$ *for* $(0 < |\delta| \leq |\delta_0(\lambda)|$, *where* $|\delta_0(\lambda)|$ *is the smallest positive root of the equation*

$$\begin{aligned} &|\beta_1| \frac{|\delta| \sqrt{2(1-\alpha)}}{(1-|\delta|^2)^{3/2}} + |\beta_2| \frac{|\delta|^2 \sqrt{(6+18|\delta|^2)(1-\alpha-2|a_2|^2)}}{(1-|\delta|^2)^{5/2}} \\ &\quad + |\beta_3| \frac{4\sqrt{3}|\delta|^3 \sqrt{(1+8|\delta|^2+6|\delta|^4)(1-\alpha-2|\alpha_2|^2-6|a_3|^2)}}{(1-|\delta|^2)^{7/2}} = \lambda \end{aligned} \tag{4.2}$$

in $0 < |\delta| < 1$.

Competing interests
The author declares that they have no competing interests.

Received: 13 July 2011 Accepted: 9 February 2012 Published: 9 February 2012

References

1. Duren, PL: Univalent Functions. Springer-Verlag, Berlin (1983)
2. Goodman, AW: Univalent Functions. Mariner, Tampa**1-2** (1983)
3. Uyanik, N, Owa, S: New extensions for classes of analytic functions associated with close-to-convex and starlike of order α. Math Comput Model. **54**, 359–366 (2011). doi:10.1016/j.mcm.2011.02.020

doi:10.1186/1029-242X-2012-24
Cite this article as: Frasin: New subclasses of analytic functions. *Journal of Inequalities and Applications* 2012 2012:24.