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Fourier spectral method for the modified Swift-Hohenberg equation

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Abstract

In this paper, we consider the Fourier spectral method for numerically solving the modified Swift-Hohenberg equation. The semi-discrete and fully discrete schemes are established. Moreover, the existence, uniqueness and the optimal error bound are also considered.

1 Introduction

In [1], Doelman *et al.* studied the modified Swift-Hohenberg equation

$$u_t = -k(1+\Delta)^2 u + \mu u - b|\nabla u|^2 - u^3. \tag{1}$$

Setting $a = k - \mu$, considering (1) in 1D case, we find that

$$u_t + ku_{xxxx} + 2ku_{xx} + au + b|u_x|^2 + u^3 = 0, \quad (x,t) \in (0,1) \times (0,T). \tag{2}$$

On the basis of physical considerations, as usual, Eq. (2) is supplemented with the following boundary value conditions:

$$u(x,t) = u_{xx}(x,t) = 0, \quad x = 0,1 \tag{3}$$

and the initial condition

$$u(x,0) = u_0(x), \quad x \in (0,1), \tag{4}$$

where $k > 0$ and a, b are constants. $u_0(x)$ is a given function from a suitable phase space.

The Swift-Hohenberg equation is one of the universal equations used in the description of pattern formation in spatially extended dissipative systems (see [2]), which arise in the study of convective hydrodynamics [3], plasma confinement in toroidal devices [4], viscous film flow and bifurcating solutions of the Navier-Stokes [5]. Note that the usual Swift-Hohenberg equation [3] is recovered for $b = 0$. The additional term $b|u_x|^2$, reminiscent of the Kuramoto-Sivashinsky equation, which arises in the study of various pattern formation phenomena involving some kind of phase turbulence or phase transition (see [6–8]), breaks the symmetry $u \to -u$.

During the past years, many authors have paid much attention to the Swift-Hohenberg equation (see, *e.g.*, [3, 9, 10]). However, only a few people have been devoted to the modified Swift-Hohenberg equation. It was Doelman *et al.* [1] who first studied the modified Swift-Hohenberg equation for a pattern formation system with two unbounded spatial directions that are near the onset to instability. Polat [7] also considered the modified Swift-Hohenberg equation. In his paper, the existence of a global attractor is proved for the modified Swift-Hohenberg equation as (2)-(4). Recently, Song *et al.* [2] studied the long time behavior for the modified Swift-Hohenberg equation in an H^k ($k \geq 0$) space. By using an iteration procedure, regularity estimates for the linear semigroups and a classical existence theorem of a global attractor, they proved that problem (2)-(4) possesses a global attractor in the Sobolev space H^k for all $k \geq 0$, which attracts any bounded subset of $H^k(\Omega)$ in the H^k-norm.

The spectral methods are essentially discretization methods for the approximate solution of partial differential equations. They have the natural advantage in keeping the physical properties of primitive problems. During the past years, many papers have already been published to study the spectral methods, for example, [11–14]. However, for the other boundary condition, can we also use the Fourier spectral method? The answer is 'Yes'. Choose a good finite dimensional subspace S_N (here, we set $S_N = \operatorname{span}\{\sin k\pi x; k = 0, 1, \ldots, N\}$), we can also have the basic results for the orthogonal projecting operator P_N.

In this paper, we consider the Fourier spectral method for the modified Swift-Hohenberg equation. The existence of a solution locally in time is proved by the standard Picard iteration, global existence results are obtained by proving *a priori* estimate for the appropriate norms of $u(x,t)$. Adjusted to our needs, the results are given in the following form.

Theorem 1.1 *Assume that* $u_0 \in H_E^2(0,1) = \{v; v \in H^2(0,1), v(0,t) = v(1,t) = 0\}$ *and* $b^2 \leq 8k$, *then there exists a unique global solution* $u(x,t)$ *of the problem* (2)-(4) *for all* $T \geq 0$ *such that*

$$u(x,t) \in L^\infty\big(0,T; H_E^2(0,1)\big) \cap L^2\big(0,T; H^4(0,1)\big).$$

Furthermore, it satisfies

$$\begin{aligned} &\left(\frac{\partial u}{\partial t}, v\right) + k(u_{xx}, v_{xx}) - 2k(u_x, v_x) + \gamma(u,v) \\ &\quad + b\big(|u_x|^2, v\big) + \big(u^3, v\big) = 0, \quad \forall v \in H_E^2(0,1), \\ &\big(u(\cdot,0), v\big) = (u_0, v), \quad \forall v \in H_E^2(0,1). \end{aligned} \tag{5}$$

This paper is organized as follows. In the next section, we consider a semi-discrete Fourier spectral approximation, prove its existence and uniqueness of the numerical solution and derive the error bound. In Section 3, we consider the full-discrete approximation for problem (2)-(4). Furthermore, we prove convergence to the solution of the associated continuous problem. In the last section, some numerical experiments which confirm our results are performed.

Throughout this paper, we denote L^2, L^p, L^∞, H^k norm in Ω simply by $\|\cdot\|$, $\|\cdot\|_p$, $\|\cdot\|_\infty$ and $\|\cdot\|_{H^k}$.

2 Semi-discrete approximation

In this section, we consider the semi-discrete approximation for problem (2)-(4). First of all, we recall some basic results on the Fourier spectral method which will be used throughout this paper. For any integer $N > 0$, introduce the finite dimensional subspace of $H_E^2(0,1)$

$$S_N = \operatorname{span}\{\sin k\pi x; k = 1, \dots, N\}.$$

Let $P_N : L^2(0,1) \to S_N$ be an orthogonal projecting operator which satisfies

$$(u - P_N u, v) = 0, \quad \forall v \in S_N. \tag{6}$$

For the operator P_N, we have the following result (see [13, 15]):

(B1) P_N commutes with derivation on $H_E^2(0,1)$, *i.e.*,

$$P_N u_{xx} = (P_N u)_{xx}, \quad \forall u \in H_E^2(0,1).$$

Using the same method as [15, 16], we can obtain the following result (B2) for problem (2)-(4):

(B2) For any real $0 \le \mu \le 2$, there is a constant c such that

$$\|u - P_N u\|_\mu \le cN^{\mu-2}\|u_{xx}\|, \quad \forall u \in H_E^2(0,1).$$

Define the Fourier spectral approximation: Find $u_N(t) = \sum_{j=1}^N a_j(t)\cos j\pi x \in S_N$ such that

$$\begin{aligned}&\left(\frac{\partial u_N}{\partial t}, v_N\right) + k(u_{Nxx}, v_{Nxx}) + 2k(u_N, v_{Nxx}) + a(u_N, v_N)\\ &\quad + b\big(|u_{Nx}|^2, v_N\big) + \big(u_N^3, v_N\big) = 0, \quad \forall v_N \in S_N\end{aligned} \tag{7}$$

for all $T \ge 0$ with $u_N(0) = P_N u_0$.

Now, we are going to establish the existence, uniqueness *etc.* of the Fourier spectral approximation solution $u_N(t)$ for all $T \ge 0$.

Lemma 2.1 *Let $u_0 \in L^2(0,1)$ and $b^2 \le 8k$, then problem* (7) *has a unique solution $u_N(t)$ satisfying the following inequalities*:

$$\|u_N(t)\|^2 \le c_1\|u_0\|^2, \qquad \int_0^T \|u_{Nxx}(\tau)\|^2\,d\tau \le c_1'\|u_0\|^2, \tag{8}$$

where $c_1 = e^{(|a|+2)T}$ and $c_1' = \frac{8(|a|+2)Tc_1+8}{8k-b^2}$ for all $T \ge 0$.

Proof Set $v_N = \cos j\pi x$ in (7) for each j $(1 \le j \le N)$ to obtain

$$\frac{d}{dt}a_j(t) = f_j\big(a_1(t), a_2(t), \dots, a_N(t)\big), \quad j = 1, 2, \dots, N, \tag{9}$$

where all $f_j : \mathbb{R}^N \to \mathbb{R}$ $(1 \le j \le N)$ are smooth and locally Lipschitz continuous. Noticing that $u_N(0) = P_N u_0$, then

$$a_j(0) = (u_0, \cos j\pi x), \quad j = 1, 2, \ldots, N. \tag{10}$$

Using the theory of initial-value problems of the ordinary differential equations, there is a time $T_N > 0$ such that the initial-value problem (9)-(10) has a unique smooth solution $(a_1(t), \ldots, a_N(t))$ for $t \in [0, T_N]$.

Setting $v_N = u_N$ in (7), we have

$$\frac{1}{2}\frac{d}{dt}\|u_N\|^2 + k\|u_{Nxx}\|^2 + \|u_N\|_4^4 \le |a|\|u_N\|^2 + 2k\|u_{Nx}\|^2 + |b|\big(|u_{Nx}|^2, u_N\big). \tag{11}$$

Noticing that

$$|b|\big(|u_{Nx}|^2, u_N\big) = -\frac{|b|}{2}\big(u_N^2, u_{Nxx}\big) \le \|u_N\|_4^4 + \frac{b^2}{16}\|u_{Nxx}\|^2$$

and

$$2k\|u_{Nx}\|^2 = -2k(u_N, u_{Nxx}) \le \frac{k}{2}\|u_{Nxx}\|^2 + 2k\|u_N\|^2.$$

Summing up, we get

$$\frac{d}{dt}\|u_N\|^2 + \left(k - \frac{b^2}{8}\right)\|u_{Nxx}\|^2 \le 2\big(|a| + 2k\big)\|u_N\|^2. \tag{12}$$

Using Gronwall's inequality, we deduce that

$$\|u_N\|^2 \le e^{2(|a|+2k)t}\big\|u_N(0)\big\|^2 \le e^{2(|a|+2k)T}\|u_0\|^2 = c_1\|u_0\|^2.$$

Integrating (12) from 0 to t, we derive that

$$\|u_N\|^2 - \big\|u_N(0)\big\|^2 + \left(k - \frac{b^2}{8}\right)\int_0^T \|u_{Nxx}\|^2\,dt \le \big(2|a| + 4k\big)\int_0^T \|u_N\|^2\,dt.$$

Hence

$$\left(k - \frac{b^2}{8}\right)\int_0^T \|u_{Nxx}\|^2\,dt \le 2\big(|a| + 2k\big)c_1 T\|u_0\|^2 + \|u_0\|^2.$$

From the above inequality, we obtain the second inequality of (8) immediately. Therefore, Lemma 2.1 is proved. □

Lemma 2.2 *Let $u_0 \in H_0^1(0,1)$ and $b^2 \le 8k$, then the solution $u_N(t)$ of problem* (7) *satisfying*

$$\big\|u_{Nx}(t)\big\|^2 \le c_2, \qquad \int_0^T \big\|u_{nxxx}(\tau)\big\|^2\,d\tau \le c_2' \tag{13}$$

for all $T \ge 0$, where c_2 and c_2' are positive constants depending only on k, a, b, T and $\|u_0\|_{H^1}$.

Proof Setting $v_N = u_{Nxx}$ in (7), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_{Nx}\|^2 + k\|u_{Nxxx}\|^2 + a\|u_{Nx}\|^2 = 2k\|u_{Nxx}\|^2 + b\big(|u_{Nx}|^2, u_{Nxx}\big) + \big(u_N^3, u_{Nxx}\big). \tag{14}$$

Notice that

$$\big(u_N^3, u_{Nxx}\big) = -\big(3u_N^2 u_{Nx}, u_{Nx}\big) = -3\|u_N u_{Nx}\|^2 \le 0$$

and

$$2k\|u_{Nxx}\|^2 = -2k(u_{Nx}, u_{Nxxx}) \le \frac{k}{4}\|u_{Nxxx}\|^2 + 4\|u_{Nx}\|^2.$$

On the other hand, by Nirenberg's inequality, we have

$$\|u_{Nx}\|_4 \le C\|u_{Nxxx}\|^{\frac{5}{12}}\|u_N\|^{\frac{7}{12}},$$

where C is a positive constant independent of N. Hence

$$\begin{aligned}
|b|\big(|u_{Nx}|^2, u_{Nxx}\big) &\le \frac{|b|}{2}\|u_{Nxx}\|^2 + \frac{|b|}{2}\|u_{Nx}\|_4^4 \\
&\le \frac{k}{8}\|u_{Nxxx}\|^2 + \frac{|b|^2}{2k}\|u_{Nx}\|^2 + C^4\|u_{Nxxx}\|^{\frac{5}{3}}\|u_N\|^{\frac{7}{3}} \\
&\le \frac{k}{4}\|u_{Nxxx}\|^2 + \frac{|b|^2}{2k}\|u_{Nx}\|^2 + c\big(\|u_N\|\big) \\
&\le \frac{k}{4}\|u_{Nxxx}\|^2 + \frac{|b|^2}{2k}\|u_{Nx}\|^2 + c\big(a, k, T, \|u_0\|\big).
\end{aligned} \tag{15}$$

Summing up, we get

$$\frac{d}{dt}\|u_{Nx}\|^2 + k\|u_{Nxxx}\|^2 \le \left(2|a| + 8 + \frac{b^2}{k}\right)\|u_{Nx}\|^2 + 2c\big(a, k, T, \|u_0\|\big). \tag{16}$$

Using Gronwall's inequality, we immediately obtain

$$\begin{aligned}
\|u_{Nx}\|^2 &\le e^{(2|a|+8+\frac{b^2}{k})t}\big\|u_{Nx}(0)\big\|^2 + \frac{2kc(a, k, T, \|u_0\|)}{2|a|k + b^2 + 8k} \\
&\le e^{(2|a|+8+\frac{b^2}{k})T}\|u_{x0}\|^2 + \frac{2kc(a, k, T, \|u_0\|)}{2|a|k + b^2 + 8k} \le c_2\big(k, a, b, T, \|u_0\|_{H^1}\big).
\end{aligned}$$

Integrating (16) from 0 to t, we obtain

$$\begin{aligned}
\int_0^T \big\|u_{Nxxx}(\tau)\big\|^2\,d\tau &\le \frac{1}{k}\left(\left(2|a| + 8 + \frac{b^2}{k}\right)c_2 T + 2cT + \|u_{x0}\|^2\right) \\
&= c_2'\big(k, a, b, T, \|u_0\|_{H^1}\big).
\end{aligned}$$

Then Lemma 2.2 is proved. □

Lemma 2.3 *Let $u_0 \in H_E^2(0,1)$ and $b^2 \le 8k$, then the solution $u_N(t)$ of problem* (7), *satisfying*

$$\|u_{Nxx}(t)\|^2 \le c_3, \qquad \int_0^T \|u_{Nxxxx}(\tau)\|^2\,d\tau \le c_3' \tag{17}$$

for all $T \ge 0$, where c_3 and c_3' are positive constants, depending only on k, a, b, T and $\|u_0\|_{H^2}$.

Proof Setting $v_N = u_{Nxxxx}$ in (7), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_{Nxx}\|^2 + k\|u_{Nxxxx}\|^2\\ &\quad = 2\|u_{Nxxx}\|^2 - a\|u_{Nxx}\|^2 - b\bigl(|u_{Nx}|^2, u_{Nxxxx}\bigr) - \bigl(u_N^3, u_{Nxxxx}\bigr).\end{aligned} \tag{18}$$

Using Nirenberg's inequality, we obtain

$$\|u_N\|_6 \le C\|u_{Nxxxx}\|^{\frac{1}{12}}\|u_N\|^{\frac{11}{12}}, \qquad \|u_{Nx}\|_4 \le C\|u_{Nxxxx}\|^{\frac{1}{12}}\|u_{Nx}\|^{\frac{11}{12}},$$

where $C > 0$ is a constant depending only on the domain. Therefore

$$\begin{aligned}|b|\bigl(|u_{Nx}|^2, u_{Nxxxx}\bigr) &\le \frac{k}{10}\|u_{Nxxxx}\|^2 + \frac{5b^2}{2k}\|u_{Nx}\|_4^4\\ &\le \frac{k}{10}\|u_{Nxxxx}\|^2 + \frac{k}{10}\|u_{Nxxxx}\|^2 + c\bigl(a,k,b,T,\|u_0\|_{H^1}\bigr)\end{aligned}$$

and

$$\begin{aligned}\bigl(u_N^3, u_{Nxxxx}\bigr) &\le \frac{k}{10}\|u_{Nxxxx}\|^2 + \frac{5}{2k}\|u_N\|_6^6\\ &\le \frac{k}{10}\|u_{Nxxxx}\|^2 + \frac{k}{10}\|u_{Nxxxx}\|^2 + c\bigl(a,k,T,\|u_0\|\bigr).\end{aligned}$$

On the other hand, we have

$$2k\|u_{Nxxx}\|^2 = -2k(u_{Nxx}, u_{Nxxxx}) \le \frac{k}{10}\|u_{Nxxxx}\|^2 + 10\|u_{Nxx}\|^2.$$

Summing up, we get

$$\frac{d}{dt}\|u_{Nxx}\|^2 + k\|u_{Nxxxx}\|^2 \le \bigl(20 + 2|a|\bigr)\|u_{Nxx}\|^2 + 4c\bigl(a,k,b,T,\|u_0\|_{H^1}\bigr). \tag{19}$$

Using Gronwall's inequality, we have

$$\begin{aligned}\|u_{Nxx}\|^2 &\le e^{(20+2|a|)t}\|u_{Nxx}(0)\|^2 + \frac{2c(a,k,b,T,\|u_0\|_{H^1})}{10+|a|}\\ &\le e^{(20+2|a|)T}\|u_{xx0}\|^2 + \frac{2c(a,k,b,T,\|u_0\|_{H^1})}{10+|a|} = c_3\bigl(a,k,b,T,\|u_{xx0}\|\bigr).\end{aligned} \tag{20}$$

Integrating (19) from 0 to t, we obtain

$$\int_0^T \|u_{Nxxxx}(\tau)\|^2\,d\tau \le \frac{1}{k}\big((20+2|a|)c_3T + 4Tc(a,k,b,T,\|u_0\|_{H^1}) + \|u_{Nxx}(0)\|^2\big)$$
$$= c_3'(k,a,b,T,\|u_0\|_{H^2}).$$

Therefore, Lemma 2.3 is proved. □

Remark 2.1 Basing on the above Lemmas 2.1-2.3, we can get the H^2-norm estimate for problem (7). Then, by Sobolev's embedding theorem, we immediately conclude that

$$\sup_{x\in[0,1]}|u_N(x,t)| = \|u_N(x,t)\|_\infty \le c_4(k,a,b,T,\|u_0\|_{H^1}), \tag{21}$$

$$\sup_{x\in[0,1]}|u_{Nx}(x,t)| = \|u_{Nx}(x,t)\|_\infty \le c_5(k,a,b,T,\|u_0\|_{H^2}). \tag{22}$$

Now, we give the following theorem.

Theorem 2.1 *Suppose that* $u_0 \in H_E^2(0,1)$ *and* $b^2 \le 8k$. *Suppose further that* $u(x,t)$ *is the solution of problem* (2)-(4) *and* $u_N(x,t)$ *is the solution of semi-discrete approximation* (7). *Then there exist a constant* c *depending on* k, a, b, T *and* $\|u_0\|_{H^2}$ *such that*

$$\|u(x,t) - u_N(x,t)\| \le c\big(N^{-2} + \|u_0 - u_N(0)\|\big).$$

Proof Denote $\eta_N = u(t) - P_Nu(t)$ and $e_N = P_Nu(t) - u_N(t)$. From (2) and (7), we get

$$(e_{Nt},v_N) + k(e_{Nxx},v_{Nxx}) - 2k(e_{Nx},v_{Nx}) + a(e_N,v_N)$$
$$+ b\big(|u_x|^2 - |u_{Nx}|^2, v_N\big) + \big(u^3 - u_N^3, v_N\big) = 0, \quad \forall v_N \in S_N. \tag{23}$$

Set $v_N = e_N$ in (23), we derive that

$$\frac{1}{2}\frac{d}{dt}\|e_N\|^2 + k\|e_{Nxx}\|^2 = 2k\|e_{Nx}\|^2 - a\|e_N\|^2 - b\big(|u_x|^2 - |u_{Nx}|^2, e_N\big) - \big(u^3 - u_N^3, e_N\big).$$

By Theorem 1.1, we have

$$\sup_{x\in[0,1]}|u(x,t)| \le c_6(k,a,b,\|u_0\|_{H^1}).$$

Then

$$\begin{aligned}
&-\big(u^3 - u_N^3, e_N\big)\\
&\quad= -\big((e_N + \eta_N)\big(u^2 + uu_N + u_N^2\big), e_N\big)\\
&\quad\le \sup_{x\in[0,1]}\big(|u|^2 + |uu_N| + |u_N|^2\big)\cdot\big(\|e_N\|^2 + \|\eta_N\|\|e_N\|\big)\\
&\quad\le \big(c_4^2 + c_4c_6 + c_6^2\big)\bigg(\|e_N\|^2 + \frac{1}{2}\|e_N\|^2 + \frac{1}{2}\|\eta_N\|^2\bigg).
\end{aligned}$$

By Theorem 1.1, we have

$$\sup_{x\in[0,1]}\bigl|u_x(x,t)\bigr| \le c_7\bigl(k,a,b,\|u_0\|_{H^2}\bigr), \qquad \bigl\|u_{xx}(x,t)\bigr\|^2 \le c_8\bigl(k,a,b,\|u_0\|_{H^2}\bigr). \tag{24}$$

Using Sobolev's embedding theorem, we have

$$\sup_{x\in[0,1]}|e_{Nx}| \le C'\|e_N\|_{H^2} \le C\|e_{Nxx}\|, \tag{25}$$

where C' and C are positive constants depending only on the domain. Then, using the method of integration by parts, we have

$$\begin{aligned} -b\bigl(|u_x|^2-|u_{Nx}|^2,e_N\bigr) &= -b\bigl((e_{Nx}+\eta_{Nx})(u_x+u_{Nx}),e_N\bigr) \\ &= b\bigl((e_N+\eta_N)(u_x+u_{Nx}),e_{Nx}\bigr) \\ &\quad + b\bigl((e_N+\eta_N)(u_{xx}+u_{Nxx}),e_N\bigr). \end{aligned} \tag{26}$$

Hence, by (24)-(26) and Hölder's inequality, we get

$$\begin{aligned} &-b\bigl(|u_x|^2-|u_{Nx}|^2,e_N\bigr) \\ &\quad\le |b|\sup_{x\in[0,1]}|u_x+u_{Nx}|\cdot\bigl(\|e_N\|\|e_{Nx}\|+\|\eta_N\|\|e_{Nx}\|\bigr) \\ &\qquad + |b|\sup_{x\in[0,1]}|e_N|\cdot\|e_N+\eta_N\|\|u_{xx}+u_{Nxx}\| \\ &\quad\le |b|\sup_{x\in[0,1]}|u_x+u_{Nx}|\cdot\bigl(\|e_N\|\|e_{Nx}\|+\|\eta_N\|\|e_{Nx}\|\bigr) \\ &\qquad + C|b|\|e_{Nx}\|\cdot\|e_N+\eta_N\|\|u_{xx}+u_{Nxx}\| \\ &\quad\le (c_7+c_5)|b|\left(\|e_{Nx}\|^2+\frac{1}{2}\|e_N\|^2+\frac{1}{2}\|\eta_N\|^2\right) \\ &\qquad + 2C(c_3+c_8)|b|\left(\varepsilon\|e_{Nx}\|^2+\frac{1}{2\varepsilon}\|e_N\|^2+\frac{1}{2\varepsilon}\|\eta_N\|^2\right) \\ &\quad\le (c_7+c_5)|b|\left(\varepsilon\|e_{Nxx}\|^2+\left(\frac{1}{2}+\frac{1}{4\varepsilon}\right)\|e_N\|^2+\frac{1}{2}\|\eta_N\|^2\right) \\ &\qquad + 2C(c_3+c_8)|b|\left(\varepsilon\|e_{Nxx}\|^2+\left(\frac{1}{2\varepsilon}+\frac{\varepsilon}{4}\right)\|e_N\|^2+\frac{1}{2\varepsilon}\|\eta_N\|^2\right), \end{aligned}$$

where $\varepsilon\in\mathbb{R}^+$ is a constant. Summing up, we get

$$\begin{aligned} &\frac{d}{dt}\|e_N\|^2+2\bigl(k-\bigl[(c_5+c_7)|b|+2C(c_3+c_8)|b|\bigr]\varepsilon\bigr)\|e_{Nxx}\|^2 \\ &\quad\le 4k\|e_{Nx}\|^2+2c_9\|e_N\|^2+2c_{10}\|\eta_N\|^2 \\ &\quad= -4k(e_N,e_{Nxx})+2c_9\|e_N\|^2+2c_{10}\|\eta_N\|^2 \\ &\quad\le \varepsilon\|e_{Nxx}\|^2+\left(\frac{4k^2}{\varepsilon}+2c_9\right)\|e_N\|^2+2c_{10}\|\eta_N\|^2, \end{aligned} \tag{27}$$

where

$$c_9 = |\gamma| + \frac{3}{2}\left(c_4^2 + c_4c_6 + c_6^2\right) + \left(\frac{1}{2} + \frac{1}{4\varepsilon}\right)(c_5 + c_7)|b| + \left(\frac{1}{\varepsilon} + \frac{\varepsilon}{2}\right)C(c_3 + c_8)|b|,$$
$$c_{10} = \frac{1}{2}\left(c_4^2 + c_4c_6 + c_6^2\right) + \frac{1}{2}(c_7 + c_5)|b| + \frac{1}{\varepsilon}C(c_3 + c_8)|b|.$$

From Theorem 1.1 and (B2), we have

$$\|\eta_N\| \le cN^{-2}\|u_{xx}\| \le c_{11}\left(k, a, b, T, \|u_0\|_{H^2}\right)N^{-2}. \tag{28}$$

Then a simple calculation shows that

$$\begin{aligned}&\frac{d}{dt}\|e_N\|^2 + \left[2\left(k - \left[(c_5 + c_7)|b| + 2C(c_3 + c_8)|b|\right]\varepsilon\right) - \varepsilon\right]\|e_{Nxx}\|^2 \\ &\quad\le \left(\frac{4k^2}{\varepsilon} + 2c_9\right)\|e_N\|^2 + 2c_{10}\|\eta_N\|^2 \le \left(\frac{4k^2}{\varepsilon} + 2c_9\right)\|e_N\|^2 + 2c_{10}c_{11}^2N^{-4},\end{aligned} \tag{29}$$

where ε is small enough, it satisfies $2(k - [(c_5 + c_7)|b| + 2C(c_3 + c_8)|b|]\varepsilon) - \varepsilon > 0$. Therefore, by Gronwall's inequality, we deduce that

$$\|e_N\|^2 \le e^{(\frac{2}{\varepsilon}+2c_9)T}\left\|e_N(0)\right\|^2 + \frac{c_{10}c_{11}^2}{(1 + c_9)\varepsilon}N^{-4}. \tag{30}$$

Hence, the proof is completed. □

3 Fully discrete scheme

In this section, we set up a full-discretization scheme for problem (2)-(4) and consider the fully discrete scheme which implies the pointwise boundedness of the solution.

Let Δt be the time-step. The full-discretization spectral method for problem (2)-(4) is read as: find $u_N^j \in S_N$ ($j = 0, 1, 2, \ldots, N$) such that for any $v_N \in S_N$, the following holds:

$$\begin{aligned}&\left(\frac{u_N^{j+1} - u_N^j}{\Delta t}, v_N\right) + k\left(\bar{u}_{Nxx}^{j+\frac{1}{2}}, v_{Nxx}\right) - 2k\left(\bar{u}_{Nx}^{j+\frac{1}{2}}, v_{Nx}\right) \\ &\quad + a\left(\bar{u}_N^{j+\frac{1}{2}}, v_N\right) + b\left(\left(\bar{u}_{Nx}^{j+\frac{1}{2}}\right)^2, v_N\right) + \left(\left(\bar{u}_N^{j+\frac{1}{2}}\right)^3, v_N\right) = 0,\end{aligned} \tag{31}$$

with $u_N(0) = P_Nu_0$, where $\bar{u}_N^{j+\frac{1}{2}} = \frac{1}{2}(u_N^j + u_N^{j+1})$.

The solution u_N^j has the following property.

Lemma 3.1 *Assume that $u_0 \in H_E^2(0,1)$ and $b^2 \le 8k$. Suppose that u_N^j is a solution of problem* (31), *then there exist positive constants $c_{12}, c_{13}, c_{14}, c_{15}, c_{16}$ depending only on k, a, b, T and $\|u_0\|_{H^2}$ such that*

$$\left\|u_N^j\right\| \le c_{12}, \qquad \left\|u_{Nx}^j\right\| \le c_{13}, \qquad \left\|u_{Nxx}^j\right\| \le c_{14}.$$

Furthermore, we have

$$\sup_{x\in[0,1]}\left|u_N^j\right| \le c_{15}, \qquad \sup_{x\in[0,1]}\left|u_{Nx}^j\right| \le c_{16}.$$

Proof It can be proved the same as Lemmas 2.1-2.3. Since the proof is so easy, we omit it. □

In the following, we analyze the error estimates between numerical solution u_N^j and exact solution $u(t_j)$. According to the properties of the projection operator P_N, we only need to analyze the error between $P_N u(t_j)$ and u_N^j. Denoted by $u^j = u(t_j)$, $e^j = P_N u^j - u_N^j$ and $\eta^j = u^j - P_N u^j$. Therefore

$$u^j - u_N^j = \eta^j + e^j.$$

If no confusion occurs, we denote the average of the two instant errors e^n and e^{n+1} by $\bar{e}^{n+\frac{1}{2}}$, where $\bar{e}^{n+\frac{1}{2}} = \frac{e^n+e^{n+1}}{2}$. On the other hand, we let $\bar{\eta}^{j+\frac{1}{2}} = \frac{\eta^j+\eta^{j+1}}{2}$.

Firstly, we give the following error estimates for the full discretization scheme.

Lemma 3.2 *For the instant errors e^{j+1} and e^j, we have*

$$\begin{aligned}\|e^{j+1}\|^2 \leq \|e^j\|^2 &+ 2\Delta t\left(u_t(t_{j+\frac{1}{2}}) - \frac{u_N^{j+1} - u_N^j}{\Delta t}, \bar{e}^{j+\frac{1}{2}}\right) \\ &+ \frac{1}{320}(\Delta t)^4 \int_{t_j}^{t_{j+1}} \|u_{ttt}\|^2\,dt + \Delta t\|\bar{e}^{j+\frac{1}{2}}\|^2. \end{aligned} \tag{32}$$

Proof Applying Taylor's expansion about $t_{j+\frac{1}{2}}$, using Hölder's inequality, we can prove the lemma immediately. Since the proof is the same as [11], we omit it. □

Taking the inner product of (2) with $\bar{e}^{j+\frac{1}{2}}$, and letting $t = t_{j+\frac{1}{2}}$, we obtain

$$\begin{aligned}&\big(u_t^{j+\frac{1}{2}}, \bar{e}^{j+\frac{1}{2}}\big) + \big(ku_{xx}^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}\big) - 2k\big(u_x^{j+\frac{1}{2}}, \bar{e}_x^{j+\frac{1}{2}}\big) + a\big(u^{j+\frac{1}{2}}, \bar{e}^{j+\frac{1}{2}}\big) \\ &\quad + b\big(\big(u_x^{j+\frac{1}{2}}\big)^2, \bar{e}^{j+\frac{1}{2}}\big) + \big(\big(u^{j+\frac{1}{2}}\big)^3, \bar{e}^{j+\frac{1}{2}}\big) = 0.\end{aligned}$$

Taking $v_N = \bar{e}^{n+\frac{1}{2}}$ in (31), we obtain

$$\begin{aligned}&\left(\frac{u_N^{j+1} - u_N^j}{\Delta t}, \bar{e}^{j+\frac{1}{2}}\right) + k\big(\bar{u}_{Nxx}^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}\big) - 2k\big(\bar{u}_{Nx}^{j+\frac{1}{2}}, \bar{e}_x^{j+\frac{1}{2}}\big) + a\big(\bar{u}_N^{j+\frac{1}{2}}, \bar{e}^{j+\frac{1}{2}}\big) \\ &\quad + b\big(\big(\bar{u}_{Nx}^{j+\frac{1}{2}}\big)^2, \bar{e}^{j+\frac{1}{2}}\big) + \big(\big(\bar{u}_N^{j+\frac{1}{2}}\big)^3, \bar{e}^{j+\frac{1}{2}}\big) = 0.\end{aligned}$$

Comparing the above two equations, we get

$$\begin{aligned}&\left(u_t^{j+\frac{1}{2}} - \frac{u_N^{j+1} - u_N^j}{\Delta t}, \bar{e}^{j+\frac{1}{2}}\right) \\ &\quad = -k\big(u_{xx}^{j+\frac{1}{2}} - \bar{u}_{Nxx}^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}\big) + 2k\big(u_x^{j+\frac{1}{2}} - \bar{u}_{Nx}^{j+\frac{1}{2}}, \bar{e}_x^{j+\frac{1}{2}}\big) - a\big(u^{j+\frac{1}{2}} - \bar{u}_N^{j+\frac{1}{2}}, \bar{e}^{j+\frac{1}{2}}\big) \\ &\qquad - b\big(\big(u_x^{j+\frac{1}{2}}\big)^2 - \big(\bar{u}_{Nx}^{j+\frac{1}{2}}\big)^2, \bar{e}^{j+\frac{1}{2}}\big) - \big(\big(u^{j+\frac{1}{2}}\big)^3 - \big(\bar{u}_N^{j+\frac{1}{2}}\big)^3, \bar{e}^{j+\frac{1}{2}}\big).\end{aligned}$$

So, we investigate the error estimates of the five items on the right-hand side of the previous equation.

Lemma 3.3 *Suppose that* $u_0 \in H_E^2(0,1)$ *and* $b^2 \leq 8k$, *u is the solution for problem* (2)-(4) *and* u_N^j *is the solution for problem* (31), *then*

$$-k\big(u_{xx}^{j+\frac{1}{2}} - \bar{u}_{Nxx}^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}\big) \leq -\frac{k}{2}\big\|\bar{e}_{xx}^{j+\frac{1}{2}}\big\|^2 + \frac{k(\Delta t)^3}{192}\int_{t_j}^{t_{j+1}} \|u_{xxtt}\|^2\,dt.$$

Proof Using Taylor's expansion, we obtain

$$u^j = u^{j+\frac{1}{2}} - \frac{\Delta t}{2}u_t^{j+\frac{1}{2}} + \int_{t_j}^{t_{j+\frac{1}{2}}} (t-t_j)u_{tt}\,dt,$$

$$u^{j+1} = u^{j+\frac{1}{2}} + \frac{\Delta t}{2}u_t^{j+\frac{1}{2}} + \int_{t_{j+\frac{1}{2}}}^{t_{j+1}} (t_j-t)u_{tt}\,dt.$$

Hence

$$\frac{1}{2}\big(u^j + u^{j+1}\big) - u^{j+\frac{1}{2}} = \frac{1}{2}\bigg(\int_{t_j}^{t_{j+\frac{1}{2}}} (t-t_j)u_{tt}\,dt + \int_{t_{j+\frac{1}{2}}}^{t_{j+1}} (t_j-t)u_{tt}\,dt\bigg).$$

By Hölder's inequality, we have

$$\begin{aligned}\bigg\|u_{xx}^{j+\frac{1}{2}} - \frac{1}{2}\big(u_{xx}^j + u_{xx}^{j+1}\big)\bigg\|^2 &= \frac{1}{4}\bigg\|\bigg(\int_{t_j}^{t_{j+\frac{1}{2}}} (t-t_j)u_{tt}\,dt + \int_{t_{j+\frac{1}{2}}}^{t_{j+1}} (t_j-t)u_{tt}\,dt\bigg)_{xx}\bigg\|^2 \\ &\leq \frac{(\Delta t)^3}{96}\int_{t_j}^{t_{j+1}} \|u_{xxtt}\|^2\,dt.\end{aligned}$$

Noticing that $(\bar{\eta}_{xx}^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}) = 0$. Therefore

$$\begin{aligned}-\big(u_{xx}^{j+\frac{1}{2}} - \bar{u}_{Nxx}^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}\big) &= -\bigg(u_{xx}^{j+\frac{1}{2}} - \frac{u_{xx}^j + u_{xx}^{j+1}}{2}, \bar{e}_{xx}^{j+\frac{1}{2}}\bigg) - \bigg(\frac{u_{xx}^{j+1} + u_{xx}^j}{2} - \frac{u_{Nxx}^{j+1} + u_{Nxx}^j}{2}, \bar{e}_{xx}^{j+\frac{1}{2}}\bigg) \\ &\leq \bigg\|u_{xx}^{j+\frac{1}{2}} - \frac{u_{xx}^j + u_{xx}^{j+1}}{2}\bigg\|\big\|\bar{e}_{xx}^{j+\frac{1}{2}}\big\| - \big(\bar{\eta}_{xx}^{j+\frac{1}{2}} + \bar{e}_{xx}^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}\big) \\ &\leq \bigg(\frac{(\Delta t)^3}{96}\int_{t_j}^{t_{j+1}} \|u_{xxtt}\|^2\,dt\bigg)^{\frac{1}{2}}\big\|\bar{e}_{xx}^{j+\frac{1}{2}}\big\| - \big(\bar{\eta}_{xx}^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}\big) - \big\|\bar{e}_{xx}^{j+\frac{1}{2}}\big\|^2 \\ &\leq \frac{(\Delta t)^3}{192}\int_{t_j}^{t_{j+1}} \|u_{xxtt}\|^2\,dt - \frac{1}{2}\big\|\bar{e}_{xx}^{j+\frac{1}{2}}\big\|^2.\end{aligned}$$

Then Lemma 3.3 is proved. □

Lemma 3.4 *Suppose that* $u_0 \in H_E^2(0,1)$ *and* $b^2 \leq 8k$, *u is the solution for problem* (2)-(4) *and* u_N^j *is the solution for problem* (31), *then*

$$\begin{aligned}&2k\big(u_x^{j+\frac{1}{2}} - \bar{u}_{Nx}^{j+\frac{1}{2}}, \bar{e}_x^{j+\frac{1}{2}}\big) \\ &\quad\leq \frac{k}{16}\big\|\bar{e}_{xx}^{j+\frac{1}{2}}\big\|^2 + 192k\big\|\bar{e}^{j+\frac{1}{2}}\big\|^2 + 192kc_{11}N^{-4} + \frac{k(\Delta t)^3}{2}\int_{t_j}^{t_{j+1}} \|u_{tt}\|^2\,dt,\end{aligned}$$

where c_{11} *is the same constant as* (28).

Proof Noticing that $\|\bar{\eta}^{j+\frac{1}{2}}\| \le cN^{-2}$. Hence

$$\begin{aligned}
&2\big(u_x^{j+\frac{1}{2}} - \bar{u}_{Nx}^{j+\frac{1}{2}}, \bar{e}_x^{j+\frac{1}{2}}\big) \\
&\quad = -2\big(u^{j+\frac{1}{2}} - \bar{u}_N^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}\big) \\
&\quad = -2\big(u^{j+\frac{1}{2}} - \bar{u}^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}\big) - 2\left(\frac{u^{j+1}+u^j}{2} - \frac{u_N^{j+1}+u_N^j}{2}, \bar{e}_{xx}^{j+\frac{1}{2}}\right) \\
&\quad \le 2\left\| u^{j+\frac{1}{2}} - \frac{u^{j+1}+u^j}{2}\right\| \big\|\bar{e}_{xx}^{j+\frac{1}{2}}\big\| + 2\left\|\frac{u^{j+1}+u^j}{2} - \frac{u_N^{j+1}+u_N^j}{2}\right\| \big\|\bar{e}_{xx}^{j+\frac{1}{2}}\big\| \\
&\quad \le 2\left\|\int_{t_j}^{t_{j+\frac{1}{2}}}(t-t_j)u_{tt}\,dt + \int_{t_{j+\frac{1}{2}}}^{t_{j+1}}(t_{j+1}-t)u_{tt}\,dt\right\| \big\|\bar{e}_{xx}^{j+\frac{1}{2}}\big\| + 2\big\|\bar{e}^{j+\frac{1}{2}} + \bar{\eta}^{j+\frac{1}{2}}\big\| \big\|\bar{e}_{xx}^{j+\frac{1}{2}}\big\| \\
&\quad \le 2\left(\frac{(\Delta t)^3}{96}\int_{t_j}^{t_{j+1}}\|u_{tt}\|^2\,dt\right)^{\frac{1}{2}} \big\|\bar{e}_{xx}^{j+\frac{1}{2}}\big\| + 4\big\|\bar{e}^{j+\frac{1}{2}}\big\| \big\|\bar{e}_{xx}^{j+\frac{1}{2}}\big\| + 4\big\|\bar{\eta}^{j+\frac{1}{2}}\big\| \big\|\bar{e}_{xx}^{j+\frac{1}{2}}\big\| \\
&\quad \le 3\varepsilon\big\|\bar{e}_{xx}^{j+\frac{1}{2}}\big\|^2 + \frac{4}{\varepsilon}\big(\big\|\bar{e}^{j+\frac{1}{2}}\big\|^2 + c_{11}N^{-4}\big) + \frac{(\Delta t)^3}{96\varepsilon}\int_{t_j}^{t_{j+1}}\|u_{tt}\|^2\,dt.
\end{aligned}$$

In the above inequality, setting $\varepsilon = \frac{1}{48}$, we get the conclusion. □

Lemma 3.5 *Suppose that $u_0 \in H_E^2(0,1)$ and $b^2 \le 8k$, u is the solution for problem* (2)-(4) *and u_N^j is the solution for problem* (31), *then*

$$\begin{aligned}
&-a\big(u^{j+\frac{1}{2}} - \bar{u}_N^{j+\frac{1}{2}}, \bar{e}^{j+\frac{1}{2}}\big) \\
&\quad \le 4|a|\big\|\bar{e}^{j+\frac{1}{2}}\big\|^2 + |a|c_{11}N^{-4} + \frac{|a|(\Delta t)^3}{384}\int_{t_j}^{t_{j+1}}\|u_{tt}\|^2\,dt,
\end{aligned}$$

where c_{11} is the same constant as (28).

Proof We have

$$\begin{aligned}
&-a\big(u^{j+\frac{1}{2}} - \bar{u}_N^{j+\frac{1}{2}}, \bar{e}^{j+\frac{1}{2}}\big) \\
&\quad = -a\left(u^{j+\frac{1}{2}} - \frac{u^{j+1}+u^j}{2}, \bar{e}^{j+\frac{1}{2}}\right) + a\left(\frac{u^{j+1}-u^j}{2} - \bar{u}_N^{j+\frac{1}{2}}, \bar{e}^{j+\frac{1}{2}}\right) \\
&\quad \le |a|\left\|u^{j+\frac{1}{2}} - \frac{u^{j+1}+u^j}{2}\right\| \big\|\bar{e}^{j+\frac{1}{2}}\big\| + |a|\left\|\frac{u^{j+1}-u^j}{2} - \bar{u}_N^{j+\frac{1}{2}}\right\| \big\|\bar{e}^{j+\frac{1}{2}}\big\| \\
&\quad \le |a|\left\|\int_{t_j}^{t_{j+\frac{1}{2}}}(t-t_j)u_{tt}\,dt + \int_{t_{j+\frac{1}{2}}}^{t_{j+1}}(t_{j+1}-t)u_{tt}\,dt\right\| \big\|\bar{e}^{j+\frac{1}{2}}\big\| + |a|\big\|\bar{e}^{j+\frac{1}{2}} + \bar{\eta}^{j+\frac{1}{2}}\big\| \big\|\bar{e}^{j+\frac{1}{2}}\big\| \\
&\quad \le |a|\left(\frac{(\Delta t)^3}{96}\int_{t_j}^{t_{j+1}}\|u_{tt}\|^2\,dt\right)^{\frac{1}{2}} \big\|\bar{e}^{j+\frac{1}{2}}\big\| + 2|a|\big(\big\|\bar{e}^{j+\frac{1}{2}}\big\|^2 + \big\|\bar{\eta}^{j+\frac{1}{2}}\big\| \big\|\bar{e}^{j+\frac{1}{2}}\big\|\big) \\
&\quad \le 4|a|\big\|\bar{e}^{j+\frac{1}{2}}\big\|^2 + |a|c_{11}N^{-4} + \frac{|a|(\Delta t)^3}{384}\int_{t_j}^{t_{j+1}}\|u_{tt}\|^2\,dt.
\end{aligned}$$

Then Lemma 3.5 is proved. □

Lemma 3.6 *Suppose that $u_0 \in H_E^2(0,1)$ and $b^2 \leq 8k$, u is the solution for problem* (2)-(4) *and u_N^j is the solution for problem* (31), *then*

$$\begin{aligned}
&-b\big(\big(u_x^{j+\frac{1}{2}}\big)^2 - \big(\bar{u}_{Nx}^{j+\frac{1}{2}}\big)^2, \bar{e}^{j+\frac{1}{2}}\big) \\
&\quad \leq \frac{k}{16}\big\|\bar{e}_{xx}^{j+\frac{1}{2}}\big\|^2 + \frac{c_7|b|(\Delta t)^3}{96}\int_{t_j}^{t_{j+1}} \|u_{xtt}\|^2\,dt + c_{17}c_{11}N^{-4} + c_{18}\big\|\bar{e}^{j+\frac{1}{2}}\big\|^2,
\end{aligned}$$

where $C \in \mathbb{R}^+$, $c_{17} = c_7|b| + c_{16}|b| + C|b|(c_8 + c_{14})$ and $c_{18} = c_7|b| + c_{17} + \frac{8c_{17}}{k}$.

Proof Notice that

$$\sup_{x\in[0,1]}\big|u_x(x,t)\big| \leq c_7, \qquad \big\|u_{xx}(x,t)\big\|^2 \leq c_8, \qquad \big\|u_{Nxx}^j\big\| \leq c_{14}, \qquad \sup_{x\in[0,1]}\big|u_{Nx}^j\big| \leq c_{16}.$$

Hence

$$\begin{aligned}
&-b\big(\big(u_x^{j+\frac{1}{2}}\big)^2 - \big(\bar{u}_{Nx}^{j+\frac{1}{2}}\big)^2, \bar{e}^{j+\frac{1}{2}}\big) \\
&\quad = -b\left(\left(u_x^{j+\frac{1}{2}} + \frac{u_x^{j+1} + u_x^j}{2}\right)\left(u_x^{j+\frac{1}{2}} - \frac{u_x^{j+1} + u_x^j}{2}\right), \bar{e}^{j+\frac{1}{2}}\right) \\
&\qquad - b\left(\left(\frac{u_x^{j+1} + u_x^j}{2} + \bar{u}_{Nx}^{j+\frac{1}{2}}\right)\left(\frac{u_x^{j+1} + u_x^j}{2} - \bar{u}_{Nx}^{j+\frac{1}{2}}\right), \bar{e}^{j+\frac{1}{2}}\right) \\
&\quad = -b\left(\left(u_x^{j+\frac{1}{2}} + \frac{u_x^{j+1} + u_x^j}{2}\right)\left(u_x^{j+\frac{1}{2}} - \frac{u_x^{j+1} + u_x^j}{2}\right), \bar{e}^{j+\frac{1}{2}}\right) \\
&\qquad + b\left(\left(\frac{u_x^{j+1} + u_x^j}{2} + \bar{u}_{Nx}^{j+\frac{1}{2}}\right)\left(\frac{u^{j+1} + u^j}{2} - \frac{u_N^{j+1} + u_N^j}{2}\right), \bar{e}_x^{j+\frac{1}{2}}\right) \\
&\qquad + b\left(\left(\frac{u_{xx}^{j+1} + u_{xx}^j}{2} + \frac{u_{Nxx}^{j+1} + u_{Nxx}^j}{2}\right)\left(\frac{u^{j+1} + u^j}{2} - \frac{u_N^{j+1} + u_N^j}{2}\right), \bar{e}^{j+\frac{1}{2}}\right).
\end{aligned} \tag{33}$$

We have used the method of integration by parts in (33). Then

$$\begin{aligned}
&-b\big(\big(u_x^{j+\frac{1}{2}}\big)^2 - \big(\bar{u}_{Nx}^{j+\frac{1}{2}}\big)^2, \bar{e}^{j+\frac{1}{2}}\big) \\
&\quad \leq |b|\sup_{x\in[0,1]}\left|u_x^{j+\frac{1}{2}} + \frac{u_x^{j+1} + u_x^j}{2}\right| \cdot \left\|u_x^{j+\frac{1}{2}} - \frac{u_x^{j+1} + u_x^j}{2}\right\| \big\|\bar{e}^{j+\frac{1}{2}}\big\| \\
&\qquad + |b|\sup_{x\in[0,1]}\left|\frac{u_x^{j+1} + u_x^j}{2} + \bar{u}_{Nx}^{j+\frac{1}{2}}\right| \cdot \left\|\frac{u^{j+1} + u^j}{2} - \frac{u_N^{j+1} + u_N^j}{2}\right\| \big\|\bar{e}_x^{j+\frac{1}{2}}\big\| \\
&\qquad + |b|\sup_{x\in[0,1]}\big|\bar{e}^{j+\frac{1}{2}}\big| \cdot \left\|\frac{u_{xx}^{j+1} + u_{xx}^j}{2} + \frac{u_{Nxx}^{j+1} + u_{Nxx}^j}{2}\right\| \left\|\frac{u^{j+1} + u^j}{2} - \frac{u_N^{j+1} + u_N^j}{2}\right\| \\
&\quad \leq 2c_7|b|\left\|\int_{t_j}^{t_{j+\frac{1}{2}}}(t - t_j)u_{xtt}\,dt + \int_{t_{j+\frac{1}{2}}}^{t_{j+1}}(t_{j+1} - t)u_{xtt}\,dt\right\| \big\|\bar{e}^{j+\frac{1}{2}}\big\| \\
&\qquad + \big(c_7|b| + c_{16}|b| + C|b|(c_8 + c_{14})\big)\big\|\bar{e}^{j+\frac{1}{2}} + \bar{\eta}^{j+\frac{1}{2}}\big\| \big\|\bar{e}_x^{j+\frac{1}{2}}\big\| \\
&\quad \leq c_7|b|\frac{(\Delta t)^3}{96}\int_{t_j}^{t_{j+1}}\|u_{xtt}\|^2\,dt + c_7|b|\big\|\bar{e}^{j+\frac{1}{2}}\big\|^2 + \varepsilon\big\|\bar{e}_{xx}^{j+\frac{1}{2}}\big\|^2
\end{aligned}$$

$$+ c_{17}\left(\left\|\bar{e}^{j+\frac{1}{2}}\right\|^2 + \left\|\bar{\eta}^{j+\frac{1}{2}}\right\|^2\right) + \frac{c_{17}}{2\varepsilon}\left\|\bar{e}^{j+\frac{1}{2}}\right\|^2$$
$$\le \varepsilon\left\|\bar{e}_{xx}^{j+\frac{1}{2}}\right\|^2 + \frac{c_7|b|(\Delta t)^3}{96}\int_{t_j}^{t_{j+1}}\|u_{xtt}\|^2\,dt + \left(c_7|b| + c_{17} + \frac{c_{17}}{2\varepsilon}\right)\left\|\bar{e}^{j+\frac{1}{2}}\right\|^2 + c_{17}c_{11}N^{-4}.$$

Setting $\varepsilon = \frac{k}{16}$ in the above inequality, we obtain the conclusion. □

Lemma 3.7 *Suppose that* $u_0 \in H_E^2(0,1)$ *and* $b^2 \le 8k$, u *is the solution for problem* (2)-(4) *and* u_N^j *is the solution for problem* (31), *then*

$$-\left(\left(u^{j+\frac{1}{2}}\right)^3 - \left(\bar{u}_N^{j+\frac{1}{2}}\right)^3, \bar{e}^{j+\frac{1}{2}}\right) \le \frac{(\Delta t)^3}{96}\int_{t_j}^{t_{j+1}}\|u_{tt}\|^2\,dt + c_{19}\left\|\bar{e}^{j+\frac{1}{2}}\right\|^2 + c_{20}c_{11}N^{-4},$$

where $c_{19} = \frac{9c_6^2}{4} + 3(c_6^2 + c_6c_{12} + c_{12}^2)$, $c_{20} = c_6^2 + c_6c_{12} + c_{12}^2$.

Proof Notice that

$$\sup_{x\in[0,1]}\left|u^j\right| \le c_6, \qquad \sup_{x\in[0,1]}\left|u_N^j\right| \le c_{12}.$$

Hence

$$-\left(\left(u^{j+\frac{1}{2}}\right)^3 - \left(\bar{u}_N^{j+\frac{1}{2}}\right)^3, \bar{e}^{j+\frac{1}{2}}\right)$$
$$= -\left(\left(u^{j+\frac{1}{2}}\right)^3 - \left(\frac{u^{j+1}+u^j}{2}\right)^3, \bar{e}^{j+\frac{1}{2}}\right) - \left(\left(\frac{u^{j+1}+u^j}{2}\right)^3 - \left(\bar{u}_N^{j+\frac{1}{2}}\right)^3, \bar{e}^{j+\frac{1}{2}}\right)$$
$$\le \sup_{x\in[0,1]}\left|\left(u^{j+\frac{1}{2}}\right)^2 + u^{j+\frac{1}{2}}\frac{u^{j+1}+u^j}{2} + \left(\frac{u^{j+1}+u^j}{2}\right)^2\right|\left\|u^{j+\frac{1}{2}} - \frac{u^{j+1}+u^j}{2}\right\|\left\|\bar{e}^{j+\frac{1}{2}}\right\|$$
$$+ \sup_{x\in[0,1]}\left|\left(\frac{u^{j+1}+u^j}{2}\right)^2 + \bar{u}_N^{j+\frac{1}{2}}\frac{u^{j+1}+u^j}{2} + \left(\bar{u}_N^{j+\frac{1}{2}}\right)^2\right|\left\|\frac{u^{j+1}+u^j}{2} - \bar{u}_N^{j+\frac{1}{2}}\right\|\left\|\bar{e}^{j+\frac{1}{2}}\right\|$$
$$\le 3c_6^2\left\|\int_{t_j}^{t_{j+\frac{1}{2}}}(t-t_j)u_{tt}\,dt + \int_{t_{j+\frac{1}{2}}}^{t_{j+1}}(t_{j+1}-t)u_{tt}\,dt\right\|\left\|\bar{e}^{j+\frac{1}{2}}\right\|$$
$$+ \left(c_6^2 + c_6c_{12} + c_{12}^2\right)\left\|\bar{e}^{j+\frac{1}{2}} + \bar{\eta}^{j+\frac{1}{2}}\right\|\left\|\bar{e}^{j+\frac{1}{2}}\right\|$$
$$\le \frac{(\Delta t)^3}{96}\int_{t_j}^{t_{j+1}}\|u_{tt}\|^2\,dt + \left(\frac{9c_6^2}{4} + 3\left(c_6^2 + c_6c_{12} + c_{12}^2\right)\right)\left\|\bar{e}^{j+\frac{1}{2}}\right\|^2$$
$$+ \left(c_6^2 + c_6c_{12} + c_{12}^2\right)\left\|\bar{\eta}^{j+\frac{1}{2}}\right\|^2$$
$$\le \frac{(\Delta t)^3}{96}\int_{t_j}^{t_{j+1}}\|u_{tt}\|^2\,dt + c_{19}\left\|\bar{e}^{j+\frac{1}{2}}\right\|^2 + c_{20}c_{11}N^{-4}.$$

Then Lemma 3.7 is proved. □

Now, we obtain the following theorem.

Theorem 3.1 *Suppose that* $u_0 \in H_E^2(0,1)$ *and* $b^2 \le 8k$, $u(x,t)$ *is the solution for problem* (2)-(4) *satisfying*

$$u \in L^\infty\left(0,T;H^2(0,1)\right), \qquad u_{tt} \in L^2\left(0,T;H^2(0,1)\right), \qquad u_{ttt} \in L^2\left(0,T;L^2(0,1)\right).$$

Suppose further that u_N^j *is the solution for problem* (31). *Then if* Δt *is sufficiently small, there exist positive constants* c_{21} *depending on* k, a, b, T, $\|u_0\|_{H^2}$ *and* c_{22} *depending on* k, a, b, T, $\|u_0\|_{H^2}$, $\int_0^T \|u_{tt}\|_{H^2}^2\,dt$ *and* $\int_0^T \|u_{ttt}\|^2\,dt$ *such that, for* $j = 0, 1, 2, \ldots, N$,

$$\|e^{j+1}\| \le c_{21}\big(N^{-2} + \|e^0\|\big) + c_{22}(\Delta t)^2.$$

Proof By Lemmas 3.2-3.7, we obtain

$$\begin{aligned}\|e^{j+1}\|^2 &\le \|e^j\|^2 + \Delta t c_{23}\big(\|e^{j+1}\|^2 + \|e^j\|^2 + N^{-4}\big)\\ &\quad + (\Delta t)^4 c_{24}\int_{t_j}^{t_{j+1}}\big(\|u_{tt}\|^2 + \|u_{xtt}\|^2 + \|u_{xxtt}\|^2 + \|u_{ttt}\|^2\big)\,dt,\end{aligned}$$

where c_{23} and c_{24} are positive constants depending only on k, a, b, T and $\|u_0\|_{H^2}$. For Δt being sufficiently small such that $c_{24}\Delta t \le \frac{1}{2}$, setting $c_{25} = 2(c_{23} + c_{24})$, we get

$$\|e^{j+1}\|^2 \le (1 + c_{25}\Delta t)\|e^j\|^2 + c_{25}\big(\Delta t N^{-4} + (\Delta t)^4 B^j\big),$$

where

$$B^j = \int_{t_j}^{t_{j+1}}\big(\|u_{tt}\|^2 + \|u_{xtt}\|^2 + \|u_{xxtt}\|^2 + \|u_{ttt}\|^2\big)\,dt.$$

Using Gronwall's inequality for the discrete form, we have

$$\|e^{j+1}\|^2 \le e^{c_{25}(j+1)\Delta t}\left(\|e^0\|^2 + c_{25}\left(j\Delta t N^{-4} + (\Delta t)^4\sum_{i=0}^{j} B^i\right)\right).$$

Direct computation shows that

$$\sum_{i=0}^{j} B^i \le \int_0^{t_{j+1}}\big(\|u_{tt}\|^2 + \|u_{xtt}\|^2 + \|u_{xxtt}\|^2 + \|u_{ttt}\|^2\big)\,dt.$$

Thus, Theorem 3.1 is proved. □

Furthermore, we have the following theorem.

Theorem 3.2 *Suppose that* $u_0 \in H_E^2(0,1)$ *and* $b^2 \le 8k$, $u(x,t)$ *is the solution for problem* (2)-(4) *satisfying*

$$u \in L^\infty\big(0,T;H^2(0,1)\big), \qquad u_{tt} \in L^2\big(0,T;H^2(0,1)\big), \qquad u_{ttt} \in L^2\big(0,T;L^2(0,1)\big).$$

Suppose further that $u_N^j \in S_N$ $(j = 0, 1, 2, \ldots)$ *is the solution for problem* (31) *and the initial value* u_N^0 *satisfies* $\|e^0\| = \|P_N u_0 - u_N^0\| \le cN^{-2}\|u_{xx}\|$. *Then there exist positive constants* c' *depending on* k, a, b, T, $\|u_0\|_{H^2}$ *and* c'' *depending on* k, a, b, T, $\|u_0\|_{H^2}$, $\int_0^T \|u_{tt}\|_{H^2}^2\,dt$, $\int_0^T \|u_{ttt}\|^2\,dt$ *such that*

$$\|u(x,t_j) - u_N^j\| \le c'N^{-2} + c''(\Delta t)^2, \quad j = 0, 1, 2, \ldots, N.$$

4 Numerical results

In this section, using the spectral method described in (31), we carry out some numerical computations to illustrate out results in previous section. The full-discretization spectral method is read as: For $v_l = \sin l\pi x$, $l = 1, \dots, N$, find

$$u_N^n = \sum_{i=1}^{N} \alpha_i^n \sin i\pi x, \quad i = 1, \dots, N,$$

such that (31) holds.

Noticing that under the inner product $(\cdot,\cdot)$, $\{\sin i\pi x, i = 1, 2, \dots\}$ is the system of orthogonal functions, then

$$\int_0^1 \sin i_1\pi x \cdot \sin i_2\pi x\, dx = \begin{cases} 0, & i_1 \neq i_2, \\ \frac{1}{2}, & i_1 = i_2. \end{cases} \tag{34}$$

Therefore, the terms of (31) are

$$\begin{aligned}
&\left(\frac{u_N^{j+1} - u_N^j}{\Delta t}, v_l\right) = \frac{\alpha_l^{j+1} - \alpha_l^j}{2\Delta t}, \\
&k\left(\bar{u}_{Nxx}^{j+\frac{1}{2}}, v_{lxx}\right) = \frac{k}{2}\left(u_{Nxx}^{j+1} + u_{Nxx}^j, v_{lxx}\right) = \frac{k}{4}(l\pi)^4\left(\alpha_l^{j+1} + \alpha_l^j\right), \\
&-2k\left(\bar{u}_{Nx}^{j+\frac{1}{2}}, v_{lx}\right) = k\left(u_N^{j+1} + u_N^j, v_{lxx}\right) = -\frac{k}{2}(l\pi)^2\left(\alpha_l^{j+1} + \alpha_l^j\right), \\
&a\left(\bar{u}_N^{j+\frac{1}{2}}, v_l\right) = \frac{a}{2}\left(u_N^{j+1} + u_N^j, v_l\right) = \frac{a}{4}\left(\alpha_l^{j+1} + \alpha_l^j\right), \\
&b\left(\left(\bar{u}_{Nx}^{j+\frac{1}{2}}\right)^2, v_l\right) = \frac{b}{4}\left(\left(u_{Nx}^{j+1} + u_{Nx}^j\right)^2, v_l\right) \\
&\qquad = \frac{b}{4}\sum_{p_1,p_2=1}^{N} p_1 p_2 \pi \left(\alpha_{p_1}^{j+1}\alpha_{p_2}^{j+1} + \alpha_{p_1}^j\alpha_{p_2}^j + 2\alpha_{p_1}^{j+1}\alpha_{p_2}^j\right)\xi_{p_1p_2l}
\end{aligned}$$

and

$$\begin{aligned}
&\left(\left(\bar{u}_N^{j+\frac{1}{2}}\right)^3, v_l\right) \\
&\quad = \frac{1}{8}\sum_{p_1,p_2,p_3=1}^{N} \left(\alpha_{p_1}^{j+1}\alpha_{p_2}^{j+1}\alpha_{p_3}^{j+1} + 3\alpha_{p_1}^{j+1}\alpha_{p_2}^{j+1}\alpha_{p_3}^j + 3\alpha_{p_1}^{j+1}\alpha_{p_2}^j\alpha_{p_3}^j + \alpha_{p_1}^j\alpha_{p_2}^j\alpha_{p_3}^j\right)\eta_{p_1p_2p_3l},
\end{aligned}$$

where

$$\begin{aligned}
\xi_{p_1p_2l} &= \int_0^1 \cos p_1\pi x \cdot \cos p_2\pi x \cdot \sin l\pi x\, dx, \\
\eta_{p_1p_2p_3l} &= \int_0^1 \sin p_1\pi x \cdot \sin p_2\pi x \cdot \sin p_3\pi x \cdot \sin l\pi x\, dx.
\end{aligned}$$

Thus, (31) can be transformed as

$$\frac{\alpha_l^{j+1} - \alpha_l^j}{2\Delta t} + \left(\frac{k}{4}(l\pi)^4 - \frac{k}{2}(l\pi)^2 + \frac{a}{4}\right)\left(\alpha_l^{j+1} + \alpha_l^j\right) + \frac{b}{4}\rho_l + \frac{1}{8}\sigma_l = 0, \tag{35}$$

where $l = 1, \dots, N$, and

$$\rho_l = \sum_{p_1,p_2=1}^{N} p_1 p_2 \pi^2 \left(\alpha_{p_1}^{j+1}\alpha_{p_2}^{j+1} + \alpha_{p_1}^{j}\alpha_{p_2}^{j} + 2\alpha_{p_1}^{j+1}\alpha_{p_2}^{j}\right)\xi_{p_1 p_2 l},$$

$$\sigma_l = \sum_{p_1,p_2,p_3=1}^{N} \left(\alpha_{p_1}^{j+1}\alpha_{p_2}^{j+1}\alpha_{p_3}^{j+1} + 3\alpha_{p_1}^{j+1}\alpha_{p_2}^{j+1}\alpha_{p_3}^{j} + 3\alpha_{p_1}^{j+1}\alpha_{p_2}^{j}\alpha_{p_3}^{j} + \alpha_{p_1}^{j}\alpha_{p_2}^{j}\alpha_{p_3}^{j}\right)\eta_{p_1 p_2 p_3 l}.$$

If α_k^n $(k = 1, 2, \dots, N)$ is known, there exists an N variable nonlinear system of equations for α_l^{j+1} $(l = 1, 2, \dots, N)$ which can be seen as

$$\mathbf{F}\left(\alpha^{j+1}\right) = \begin{pmatrix} f_1(\alpha^{j+1}) \\ f_2(\alpha^{j+1}) \\ \vdots \\ f_N(\alpha^{j+1}) \end{pmatrix} = \mathbf{0}.$$

We use the simple Newton method to seek the solutions. Initialization yields

$$\alpha_{(0)}^{j+1} = \left(\alpha_1^j, \alpha_2^j, \dots, \alpha_N^j\right)^T. \tag{36}$$

The iterative formulation is as follows:

$$\left.\begin{array}{l} \alpha_{(k+1)}^{j+1} = \alpha_{(k)}^{j+1} + \Delta\alpha_{(k)}^{j+1}, \\ \mathbf{F}'(\alpha_{(0)}^{j+1}) \cdot \Delta\alpha_{(k)}^{j+1} + \mathbf{F}(\alpha_{(k)}^{j+1}) = 0, \quad k = 0, 1, 2, \dots, \end{array}\right\} \tag{37}$$

where $\mathbf{F}'(\alpha_{(0)}^{j+1})$ is the $N \times N$ order Jacobi matrix for $\mathbf{F}(\alpha^{j+1})$ when $\alpha^{j+1} = \alpha_{(0)}^{j+1}$,

$$\mathbf{F}'\left(\alpha_{(0)}^{j+1}\right) = \begin{pmatrix} \partial_1 f_1(\alpha_{(0)}^{j+1}), & \dots, & \partial_N f_1(\alpha_{(0)}^{j+1}) \\ \vdots & \ddots & \vdots \\ \partial_1 f_N(\alpha_{(0)}^{j+1}), & \dots, & \partial_N f_N(\alpha_{(0)}^{j+1}) \end{pmatrix}. \tag{38}$$

Give accuracy $\epsilon > 0$, when $\|\alpha_{(k+1)}^{j+1} - \alpha_{(k)}^{j+1}\| < \epsilon$, stop the iteration, $\alpha^{j+1} \approx \alpha_{(k+1)}^{j+1}$.

As an example, we choose $k = 2$, $a = 1$, $b = 1$, $u_0 = (1-x)^5 x^5$, $\Delta t = 0.0005$, $N = 32$, and get the solution which evolves from $t = 0$ to $t = 0.025$ (*cf.* Figure 1).

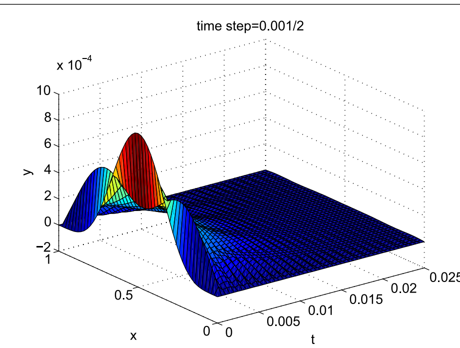

Figure 1 Expanded property of solution when $N = 32$, $\Delta t = 0.0005$.

Table 1 Errors of different time steps at $t = 0.1$

Δt	$\mathrm{err}(0.1, \Delta t)$	$\frac{\mathrm{err}(0.1,\Delta t)}{(\Delta t)^2}$
0.001	1.6808×10^{-6}	1.6808
$0.001 \times \frac{1}{2}$	2.2439×10^{-7}	0.8976
$0.001 \times \frac{1}{4}$	2.6068×10^{-8}	0.4171
$0.001 \times \frac{1}{8}$	2.8494×10^{-9}	0.1824
$0.001 \times \frac{1}{16}$	2.8754×10^{-10}	0.0736

Table 2 Errors of different basic function numbers at $t = 0.1$

N	$\mathrm{err}(0.1, \Delta t_0)$	$\frac{\mathrm{err}(0.1,\Delta t_0)}{N^{-2}}$
24	2.28708×10^{-10}	1.32×10^{-7}
28	1.3588×10^{-10}	1.07×10^{-7}
32	7.32182×10^{-11}	7.50×10^{-8}
36	3.87087×10^{-11}	5.02×10^{-8}
40	2.03256×10^{-11}	3.25×10^{-8}

Now, we consider the variation of error. Since there is no exact solution for (2)-(4) known to us, we make a comparison between the solution of (31) on coarse meshes and a fine mesh.

Choose $\Delta t = 0.001, 0.001 \times \frac{1}{2}, 0.001 \times \frac{1}{4}, 0.001 \times \frac{1}{8}, 0.001 \times \frac{1}{10}, 0.001 \times \frac{1}{16}$, respectively, to solve (31). Set $u_N^{\min}(x, 0.1)$ as the solution for $\Delta t_{\min} = 0.001 \times \frac{1}{32}$. Denote

$$\mathrm{err}(0.1, \Delta t) = \left(\int_0^1 \left(u_N^k(x, 0.1) - u_N^{\min}(x, 0.1) \right)^2 dx \right)^{\frac{1}{2}}, \quad k = 1, 2, \ldots, 6. \tag{39}$$

Then the error is showed in Table 1 at $t = 0.1$.

On the other hand, choose $N = 24, 28, 32, 36, 40$, $\Delta t_0 = 0.001 \times \frac{1}{16}$, respectively, to solve (31). Then the error is showed in Table 2 at $t = 0.1$.

It is easy to see that the third column $\frac{\mathrm{err}(0.1,\Delta t)}{(\Delta t)^2}$ of Table 1 is monotone decreasing along with the time step's waning, the third column $\frac{\mathrm{err}(0.1,\Delta t_0)}{N^{-2}}$ of Table 2 is monotone decreasing along with N's magnifying. Hence, we can find positive constants $C_1 = 1.6808$, $C_2 = 1.32 \times 10^{-7}$ such that

$$\frac{\mathrm{err}(0.1, \Delta t)}{(\Delta t)^2} \le C, \quad k = 1, 2, \ldots, 6$$

and

$$\frac{\mathrm{err}(0.1, \Delta t_0)}{N^{-2}} \le C, \quad N = 24, 28, 32, 36, 40.$$

Thus, the order of error estimates is $O((\Delta t)^2 + N^{-2})$ proved in Theorem 3.2.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

XZ wrote the first draft, PZ made the figure of numerical solution and results on errors of different time steps, WZ made the results on errors of different basic function numbers, BL and FL corrected and improved the final version. All authors read and approved the final draft.

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