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On the solvability conditions of the first boundary value problem for a system of elliptic equations that strongly degenerate at a point

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Abstract

A system of elliptic equations which are irregularly degenerate at an inner point is considered in this article. The equations are weakly coupled by a matrix that has multiple zero eigenvalue and corresponding to it adjoint vectors. Two statements of a well-posed Dirichlet type problem in the class of smooth functions are given and sufficient conditions on the existence and uniqueness of the solutions are obtained.

Keywords: systems of elliptic equations, degenerate elliptic equations, boundary value problems, Dirichlet type problem

1 Introduction and statement of the problems

The first results in the area of boundary value problems for an elliptic equation with degeneracy at an inner point of the considered domain are obtained in [1]. In that study, the Dirichlet problem for a weakly (regularly) degenerating elliptic equation with the main part of Laplace's operator is studied. These results are developed in [2], where the degenerate elliptic operator is generalized and, over and above, the second boundary value problem is investigated. In [3], the existence of a weak solution to the Dirichlet problem for an elliptic equation degenerating at isolated points in the class of Hölder functions is proved. In the case of the strong (irregular) degeneracy, can new effects emerge which influence the well-posedness of the boundary value problems. For instance, in [4], it is shown that in a well-posed Dirichlet type problem the asymptotic of the solution near the degeneracy point is supposed to be known. Many more difficulties come into being in the investigation of the systems of degenerate elliptic equations. Some results for weakly related degenerate elliptic systems are obtained in [5-7]. Particularly, these articles deal with Dirichlet type problems for the elliptic system

$$a(r)\Delta u + \sum_{i=1}^{n} B_i(x)u_{x_i} + C(x)u = 0, \quad x \in D, \tag{1}$$

where $r = |x|$, a is a continuous function such that $a(r) = o(1)$ as $r \to 0$, and $a(r) > 0$ for $r > 0$, $x = 0$ is an inner point of domain D, Δ is Laplace's operator, $B_i(x)$ and $C(x)$ are diagonal and square matrices, consequently, which are smooth enough in $\overline{D}$. In [5,6], the Dirichlet problem in the class of vector functions u bounded in $D_0 = D\backslash\{x = 0\}$ is

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solved under the assumption that elements of the matrices $B_i(x)$ tend to zero, as $x \to 0$, fast enough. In [7], a weighted Dirichlet problem with supplementary weighted condition of the shape

$$\lim_{x \to 0} \left(\Psi(x)u(x) - h\left(\frac{x}{r}\right) \right) = 0 \tag{2}$$

is considered under the condition $a(r) = O(r^{2\alpha})$, $\alpha > 1$, as $r \to 0$. In the same study, $\Psi(x)$ is some matrix entries of which are decreasing as $x \to 0$, and h is a given vector function smooth on the unit sphere. It is noteworthy that the matrix $C(x)$ is assumed to be negatively definite in D, i.e., it does not have any zero eigenvalue. Moreover, $C(0)$ should be a normal matrix for the weighted Dirichlet problem to be well-posed. (If coefficients $B_i(x)$ have the main influence to the asymptotic of the solutions of system (1), then the last requirement is dispensable [8,9]). Therefore, it is important to consider the case where $C(0)$ has multiple zero eigenvalue and corresponding to it adjoint vectors.

Hence, the present article deals with a particular case of system (1) of the shape

$$\Delta u - q(r)\Lambda u = 0 \tag{3}$$

in the ball $\Sigma_R = \{x : |x| < R\} \subset R^3$ with the Dirichlet condition

$$u|_{S_R} = f. \tag{4}$$

In this article, Λ is a real constant non-negative definite $N \times N$ matrix having the eigenvalue $\lambda = 0$, q is scalar continuous function positive for $r \neq 0$ and such that

$$q(r) = O(r^{-2\alpha}), \alpha > 1, \text{as} \quad r \to 0, \tag{5}$$

$S_R = \partial\Sigma_R$, $f = (f_1, f_2, ..., f_N)$ and $u = (u_1, u_2, ..., u_N)$ are the given and unknown vector functions, respectively. (Condition (5) means with respect to system (1) that $a(r)$ vanishes as $r \to 0$ not faster than any power of r.) Hence, the order of system (3) is strongly degenerate at the point $x = 0$ because of $\alpha > 1$.

Let S be a non-degenerate matrix such that

$$S\Lambda S^{-1} = J_\Lambda = \text{diag}\left(L_{m_0}(\lambda_0) L_{m_1}(\lambda_1) \ldots L_{m_p}(\lambda_p) \right)$$

is the canonical Jordan form of Λ with $m_i \times m_i$ lower blocks

$$L_{m_i}(\lambda_i) = \begin{pmatrix} \lambda_i & 0 & \ldots & 0 & 0 \\ 1 & \lambda_i & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_i & 0 \\ 0 & 0 & \ldots & 1 & \lambda_i \end{pmatrix}, \quad i = \overline{0, p}.$$

Multiplying both (3) and (4) from the left by S, we get the system

$$\Delta v - q(r)J_\Lambda v = 0 \tag{6}$$

and the Dirichlet condition

$$v|_{S_R} = g, \tag{7}$$

where $v = Su$, and $g = Sf$. Therefore, system (6) and Dirichlet condition (7) can be split into $p + 1$ separate systems

$$\Delta v^i - q(r)L_{m_i}(\lambda_i)v^i = 0,$$

and Dirichlet conditions

$$v^i|_{S_R} = g^i, i = \overline{0, p},$$

which correspond to the blocks of Jordan matrix J_Λ, where both v^i and g^i are m_i-dimensional vector functions. If Λ is a matrix of simple structure, then all $m_i = 1$, i.e., (6) splits into N separate equations, obviously.

Let $\lambda_0 = 0$ and, for convenience, only one eigen vector corresponds to this eigenvalue of Λ. Then, Re $\lambda_i < 0$ for the rest $i = \overline{1, p}$, since the matrix Λ is non-negatively defined. As mentioned above, the solvability of a Dirichlet type problem under the condition Re $\lambda_i < 0$ is investigated in [6,7].

The main aim of this article is to give a well-posedness of the Dirichlet type problems to the system

$$\Delta v^0 - q(r)L_{m_0}(0)v^0 = 0, \tag{8}$$

which is in accordance with eigenvalue $\lambda_0 = 0$ of Λ. In order to avoid the complicated notations, instead of (8), we consider the system

$$\Delta v - q(r)L_s(0)v = 0, \tag{9}$$

where $v = (v_1, v_2, ..., v_s)$ and $L_s(0)$ is a $s \times s$ lower Jordan block with zero diagonal entries. It is easily seen that

$$\begin{cases} \Delta v_1 = 0, \\ \Delta v_{k+1} = q(r)v_k, \quad k = \overline{1, s-1}, \end{cases}$$

is the outspread form of (9).

Denote $\Sigma_R^0 = \Sigma_R \backslash \{x = 0\}$. Let $||v||$ be the Euclidian norm of a vector v. We propose the two following statements of the Dirichlet type problem to system (9).

Problem D_1. Find a solution $v = (v_1, v_2, \ldots, v_s) \in C^2(\Sigma_R^0) \cup C(\Sigma_R^0 \cup S_R)$ of Equation 9 that satisfies Dirichlet condition

$$v|_{S_R} = g, \tag{10}$$

and relation

$$r \|v(x)\| = o(1), \quad \text{as} \quad x \to 0 \tag{11}$$

Problem D_2. Find a solution $v \in C^2(\Sigma_R^0) \cup C(\Sigma_R^0 \cup S_R)$ of Equation 9, such that it satisfies Dirichlet condition (11) and is bounded in Σ_R^0.

2 The properties of particular solutions of Equation 8

Let $H_n^m(x)$ be mth the harmonic of a homogeneous harmonic polynomial of degree n,[a] i.e., $H_n^m(\lambda x) = \lambda^n H_n^m(x)$ and $\Delta H_n^m(x) = 0$. Then, $r^{-n}H_n^m(x) = H_n^m(\omega)$ (here $\omega = x/r$) is the mth spherical harmonic of order n continuous on the unit sphere S_1. Let c_{nm} be any constant vector, and let $Q_n(r)$ be a matrix solution of ODEs system

$$l_n(w) - q(r)L_s(0)w = 0, \quad 0 < r < R, \tag{12}$$

where

$$l_n = \frac{d^2}{dr^2} + \frac{2(n+1)}{r}\frac{d}{dr}.$$

and w is an unknown s-dimensional vector function. Then, the functions

$$v_{nm}(x) = r^n H_n^m(\omega)Q_n(r)c_{nm}, n = 0, 1, 2, \ldots, \quad m = 0, \pm 1, \ldots, \pm n, \tag{13}$$

represent the particular solutions of system (9).

We seek for a solution $Q_n(r)$ of system (12), which satisfies condition $Q(R) = E$, where E is the unit matrix. To this end, on the set of functions ψ bounded on the interval $(0, R)$, we consider the integral operator

$$K_n(\psi)(r) = \int_0^R K_n(r, t)q(t)\psi(t)dt,$$

$$K_n(r, t) = -\frac{t^{2n+2}}{2n+1} \times \begin{cases} r^{-2n-1} - R^{-2n-1}, 0 < t \le r, \\ t^{-2n-1} - R^{-2n-1}, r \le t \le R, \end{cases}$$

and its integer powers

$$K_n^\sigma(\psi)(r) = K_n\left(K_n^{\sigma-1}(\psi)\right)(r),$$

where by definition $K^0(\psi)(x) \equiv \psi(x)$. Obviously, according to this definition

$$K_n^\sigma(\psi)(r) = -\frac{1}{2n+1}\left(\left(r^{-2n-1} - R^{-2n-1}\right)\int_0^r t^{2n+2}q(t)K_n^{\sigma-1}(\psi)(t)dt\right.$$

$$\left. + \int_r^R t\left(1 - \left(\frac{t}{R}\right)^{2n+1}\right)q(t)K_n^{\sigma-1}(\psi)(t)dt\right), \quad \sigma = 1, 2, \ldots. \tag{14}$$

Lemma 1. Let relation (5) hold. If $n > \sigma(\alpha - 1) - \frac{1}{2}$, then

$$K_n^\sigma(\psi)(r) \le \frac{M_\sigma}{n^\sigma}r^{2\sigma(1-\alpha)}) \text{ on } (0, R), \tag{15}$$

where M_σ is some constant independent of n, and $K_n^\sigma(\psi)(R) = 0$ ($\sigma = 1, 2, \ldots$).

Proof. We prove relation (15) by induction.

Since $\psi(r)$ is bounded on $(0, R)$, inequality (15) holds for $\sigma = 0$ with some constant M_0. It follows from relation (5) that $0 < q(r) \le Mr^{-2\alpha} \, \forall r \in (0, R)$, where M is a positive constant. Then, $q(t)|\psi(t)| \le MM_0 t^{-2\alpha} \, \forall t \in (0, R)$. Assuming that $n > \alpha - \frac{3}{2}$, we obtain that

$$|K_n(\psi)(r)| \le \frac{M}{2n+1}\left(r^{-2n-1}\int_0^r t^{2(n+1-\alpha)}dt + \int_r^R t^{1-2\alpha}dt\right)$$

$$\le \frac{M}{2n+1}\left(\frac{1}{2n-2\alpha+3} + \frac{1}{2(\alpha-1)}\right)r^{2(1-\alpha)},$$

i.e., the estimate

$$|K_n(\psi)(r)| \le \frac{M_1}{n}r^{2(1-\alpha)}$$

with some constant M_1 independent of n holds. Thus, the validity of (15) is proved for $\sigma = 1$.

Let (15) be valid for $\sigma = k - 1$ under the condition $n > (k-1)(\alpha - 1) - \frac{1}{2}$. Then,

$$t^{2n+2} q(t) \left| K_n^{k-1}(\psi)(t) \right| \leq \frac{MM_{k-1}}{n^{k-1}} t^{2(n+k(1-\alpha))} \quad \forall t \in (0, R),$$

i.e., the first integral in expression (14) converges, if $n > k(\alpha - 1) - \frac{1}{2}$, and

$$\left| K_n^k(\psi)(r) \right| \leq \frac{MM_{k-1}}{n^{k-1}(2n+1)} \left(r^{-2n-1} \int_0^r t^{2(n+k(1-\alpha))} dt + \int_r^R t^{2k(1-\alpha)-1} dt \right)$$

$$\leq \frac{MM_{k-1}}{n^{k-1}(2n+1)} \left(\frac{1}{2n+2k(1-\alpha)+1} + \frac{1}{2k(\alpha - 1)} \right) r^{2k(1-\alpha)}.$$

Therefore, there exists a constant M_k such that (15) holds for $\sigma = k$ under the condition $n > k(\alpha - 1) - \frac{1}{2}$.

If $n > \sigma(\alpha - 1) - \frac{1}{2}$, then the first integral on the right-hand side of (14) converges as $r \in (0, R)$, and, evidently, $K_n^\sigma(\psi)(R) = 0$. ∎

It is easy to verify that

$$l_n \left(K_n^\sigma(\psi) \right) = q(r) K_n^{\sigma-1}(\psi)(r), \quad \sigma = 1, 2, \ldots, \tag{16}$$

under the conditions of Lemma 1.

Note that $w_1 \equiv 1$ and $w_2 = r^{-2n-1}$ are linearly independent solutions of the differential equation $l_n(w) = 0$. Thus, if w is the solution of this equation such that $w(r) = o(r^{-2n-1})$ as $r \to 0$, then $w(r) \equiv$ const.

Denoting, as usual, by $[a]$ the integer part of the real number a, we introduce the integer $\alpha_k = \left[k(\alpha - 1) + \frac{1}{2} \right]$, where k is a non-negative integer. (Note that $\alpha_0 = 0$.)

We use below denotation $K_n^{\sigma-1}(r) = K_n^{\sigma-1}(\psi)(r)$ in the case $\psi(x) \equiv 1$.

Theorem 1. Let relation (5) hold. If $n \geq \alpha_{s-1}$, then there exists a unique matrix solution $Q_n(r) = \{q_{nij}(r)\}$ of Equation 12 such that

$$q_{nij}(r) = o(r^{-2n-1}), \quad \text{as} \quad r \to 0, \quad i, j = \overline{1, s}, \tag{17}$$

and

$$Q_n(R) = E. \tag{18}$$

Proof. Let the condition $n \geq \alpha_{s-1}$ be valid. Then, according to Lemma 1, the functions $\sigma = \overline{0, s-1}$, $\sigma = \overline{0, s-1}$, are continuous on the interval $(0, R)$. Introduce the $s \times s$ matrix $Q_n(r) = \{q_{nij}(r)\}$ by the formula

$$q_{nij}(r) = \begin{cases} 0, & \text{if } i < j, \\ K_n^{i-j}(r), & \text{if } i \geq j. \end{cases} \tag{19}$$

Note that estimate (15) yields the relations

$$q_{nij}(r) = O\left(r^{2(i-j)(1-\alpha)} \right), \quad \text{as} \quad r \to 0, \quad 1 \leq j \leq i \leq s. \tag{20}$$

This implies the validity of condition (17), because $2n+1 > 2(s-1)(\alpha-1) \geq 2(i-j)(\alpha-1)$, for $i, j = \overline{1, s}$. Moreover,

$$l_n(Q_n) = q(r)L_s(0)Q_n \quad \text{on } (0, R)$$

because of (16), i.e., Q_n is the matrix solution of Equation 12. Evidently, equality (18) follows from (14).

It remains to prove the uniqueness of the solution of problems (12), (17), and (18). Let $\tilde{Q}_n(r) = \{\tilde{q}_{nij}\}$ be a matrix solution of system (12) continuous on $(0, R)$, and satisfying both conditions $\tilde{q}_{nij}(r) = o(r^{-2n-1})$, as $r \to 0$, and $\tilde{Q}_n(R) = \Theta$, where Θ is zero matrix. Then, the equalities

$$l_n(\tilde{q}_{n1j}) = 0,$$
$$l_n(\tilde{q}_{n,k+1,j}) = q(r)\tilde{q}_{nkj}(r), \quad k = \overline{1, s-1},$$

on the interval $(0, R)$ hold. Since $\tilde{q}_{n1j}(r) = o(r^{-2n-1})$ as $r \to 0$, we obtain that $\tilde{q}_{n1j}(r) = \text{const}$ on the interval $(0, R)$. Then, the condition $\tilde{q}_{n1j}(R) = 0$ yields the identity $\tilde{q}_{n1j}(r) \equiv 0$ because of the continuity of the function $\tilde{q}_{n1j}$ on $(0, R)$. In such a case, the elements of the second row of matrix $\tilde{Q}_n$ satisfy the equation $l_n(\tilde{q}_{n2j}) = 0$. For the same reason as above, we obtain that $\tilde{q}_{n2j}(r) \equiv 0$ $(j = \overline{1, s})$ on $(0, R)$. Further, continuing this process, we get that $\tilde{q}_{n3j}(r) \equiv 0, ..., \tilde{q}_{nsj}(r) \equiv 0$ $(j = \overline{1, s})$ on $(0, R)$. Hence, $\tilde{Q}_n(r) \equiv \Theta$ on $(0, R)$. This yields the uniqueness of the solution of problems (12), (17), and (18).

What is the structure of the solutions of system (12) that increase slower than r^{-2n-1} in the case where n does not satisfy the condition $n \geq \alpha_{s-1}$? In order to get the answer to this question, we introduce $s \times s$ matrices $E_k = \left\{ e_{ij}^{(k)} \right\}$ $(k = \overline{1, s})$ with entries $e_{ii}^{(k)} = 1$ for $s - k + 1 \leq i \leq s$, and $e_{ij}^{(k)} = 0$ for all rest i and j. (Note that $E_s = E$ according to this definition.) Let us compose the matrixes

$$Q_n^{(k)}(r) = Q_n(r)E_k, \quad k = \overline{1, s},$$

where Q_n is matrix elements of which are given by (19). It is easily seen that $Q_n^{(s)}(r) = Q_n(r)E = Q_n(r)$, and the elements $q_{nij}^{(k)}(r)$ of rest matrixes $Q_n^{(k)}(r)$ $(k = \overline{1, s-1})$ are defined by following formula

$$q_{nij}^{(k)}(r) = \begin{cases} K_n^{i-j}(r), & \text{if } s - k + 1 \leq j \leq i \leq s, \\ 0, & \text{if } 1 \leq j \leq s - k \text{ and } i > j. \end{cases}$$

If $n \geq \alpha_{k-1}$, then the powers $K_n^{i-j}(r)$ exist for all i and j such that $s - k + 1 \leq j \leq i \leq s$, and according to (20), the relation

$$q_{nij}^{(k)}(r) = O(r^{2(i-j)(1-\alpha)}) = o(r^{-2n-1}), \quad \text{as} \quad r \to 0 \tag{21}$$

holds. Moreover, we obtain by direct calculation that

$$l_n(Q_n^{(k)}) = q(r)L_s(0)Q_n^{(k)} \quad \text{on } (0, R),$$
$$Q_n^{(k)}(R) = E_k \tag{22}$$

for $\forall k = \overline{1, s-1}$ due to the definition of matrix $Q_n^{(k)}$. Hence, there holds the following

Theorem 2. Let relation (5) hold, and let natural k, $1 \leq k \leq s - 1$, be such that $\alpha_{k+1} \leq n \, \alpha_k$. Then, there exists a unique matrix solution $Q_n^{(k)}(r) = \left\{ q_{nij}^{(k)}(r) \right\}$ of Equation 12 such that relation (21) holds, and boundary value condition (22) is satisfied.

The uniqueness of the matrix solution $Q_n^{(k)}$ can be proved in the same way as that of the matrix solution Q_n. In this case, condition (21) is essential, just similar to condition (17) in Theorem 1.

Hence, we obtain to system (9) the following set of particular solutions (see (13)):

$$v_{nm}^{(k)} = \left(\frac{r}{R} \right)^n H_n^m(\omega) Q_n^{(k)}(r) c_{nm} \text{ for } n_{k-1} \leq n < \alpha_k, \qquad k = \overline{1, s-1},$$

$$v_{nm}^{(s)} = \left(\frac{r}{R} \right)^n H_n^m(\omega) Q_n(r) c_{nm} \text{ for } n \geq \alpha_{s-1},$$

where c_{nm} is arbitrary constant column vector.

3 Existence and uniqueness of the solutions of problems D_1 and D_2

Let us compose the superposition

$$
v = \sum_{k=1}^{s-1} \sum_{\alpha_{k-1} \leq n < \alpha_k} \left(\frac{r}{R} \right)^n Q_n^{(k)}(r) \sum_{|m| \leq n} H_n^m(\omega) c_{nm}
$$
$$
+ \sum_{n=\alpha_{s-1}}^{\infty} \left(\frac{r}{R} \right)^n Q_n(r) \sum_{|m| \leq n} H_n^m(\omega) c_{nm}
$$

(23)

of the particular solutions obtained above. Note, if $\alpha_{k_0} = 0$ for some k_0, $1 \leq k_0 \leq s - 1$, then $\alpha_k = 0$ for all natural $k \leq k_0 - 1$. (Such a situation can come to exist, if $\alpha < 2$.) Therefore, all the sums $\sum_{\alpha_{k-1} \leq n < \alpha_k}$ in (23), in which the inequality $\alpha_{k-1} < \alpha_k$ is impossible, are taken to be equal to zero.

Evidently, if the series (23) converges and its sum v is twice differentiable in the spherical layer $\Sigma_R^\delta = \{x : \delta < |x| < R\}$ with arbitrarily small δ, then this series satisfies system (9) in the ball Σ_R^0. Note that

$$
v|_{S_R} = \sum_{k=1}^{s-1} \sum_{\alpha_{k-1} \leq n < \alpha_k} \sum_{|m| \leq n} H_n^m \left(\frac{x}{R} \right) E_k c_{nm}
$$
$$
+ \sum_{n=\alpha_{s-1}}^{\infty} \sum_{|m| \leq n} H_n^m \left(\frac{x}{R} \right) c_{nm}
$$

(24)

due to both (18) and (22).

Assume that the boundary vector function $g = (g_1, g_2, ..., g_s)$ (see (10)) is twice differentiable on unit sphere S_1. Thus, it can be expressed on the sphere S_R by Laplace series [10]:

$$
g(x) = \sum_{n=0}^{\infty} \sum_{|m| \leq n} H_n^m(\omega) \, a_{nm}, \qquad x \in S_R,
$$

(25)

which converge (component-wise) uniformly and absolutely according to the assumed smoothness of the vector function g. The coefficients $a_{nm} = (a_{nm}^{(1)}, a_{nm}^{(2)}, ..., a_{nm}^{(s)})$ in (25) can be calculated as follows [b]:

$$
a_{nm}^{(i)} = \frac{2n+1}{4\pi R^2} \frac{(n-m)!}{(n+m)!} \iint\limits_{S_R} h_i(\varphi, \vartheta) Y_n^m(\varphi, \vartheta) \, d\varphi \, d\vartheta, \qquad |m| \leq n
$$

where $h_i\ (\phi, \vartheta) = g_i(x)$, for $|x| = R$ and $Y_n^m(\varphi, \vartheta) = H_n^m(\omega)$, ϕ, ϑ ($0 \leq \phi \leq 2\pi$, $0 \leq \vartheta \leq \pi$) are spherical coordinates which are introduced by the rule: $x_1 = r \sin \vartheta \cos \phi$, $x_2 = r \sin \vartheta \sin \phi$, and $x_3 = r \cos \vartheta$.

It is easily seen that series (24) coincides with series (25), if $c_{nm} = a_{nm}$ for $n \geq \alpha_{s-1}$, and $E_k c_{nm} = a_{nm}$ for $\alpha_{k-1} \leq n < \alpha_k$, $k = \overline{1, s-1}$, i.e., if components $h_1, h_2, ..., h_{s-1}$ of vector function h satisfy the following orthogonality conditions

$$\iint\limits_{S_R} h_k(\varphi, \vartheta) Y_n^m(\varphi, \vartheta) \mathrm{d}\varphi \mathrm{d}\vartheta = 0 \ \text{ for } 0 \leq n < \alpha_{s-k}, \quad k = \overline{1, s-1}, \tag{26}$$

on sphere S_R. Let us consider series (23), in which $c_{nm} = a_{nm}$:

$$v = \sum_{k=1}^{s-1} \sum_{\alpha_{k-1} \leq n < \alpha_k} \left(\frac{r}{R}\right)^n Q_n^{(k)}(r) \sum_{|m| \leq n} H_n^m(\omega) a_{nm} + \tag{27}$$
$$\sum_{n=\alpha_{s-1}}^{\infty} \left(\frac{r}{R}\right)^n Q_n(r) \sum_{|m| \leq n} H_n^m(\omega) a_{nm}.$$

Assume that condition (26) is fulfilled in addition to the smoothness of g. Then, $v|_{S_R} = g$, i.e., series (27) converges (component-wise) uniformly and absolutely on the sphere S_R.

We shall prove that series (27) converges uniformly and absolutely in the spherical layer Σ_R^δ with arbitrarily small δ. Note that components v_i $(i = \overline{1, s})$ of the vector function $v = (v_1, v_2, ..., v_s)$ in (27) can be formally represented in the form

$$v_1 = w_1(x), \quad v_k = w_k(x) + W_k(x), \quad k = \overline{2, s}, \tag{28}$$

where

$$w_k(x) = \sum_{n=\alpha_{s-k}}^{\infty} \left(\frac{r}{R}\right)^n \sum_{|m| \leq n} a_{nm}^{(k)} H_n^m(\omega), \quad k = \overline{1, s}, \tag{29}$$

$$W_k(x) = \sum_{l=1}^{k-1} \sum_{n=\alpha_{s-l}}^{\infty} \left(\frac{r}{R}\right)^n K_n^{k-l}(r) \sum_{|m| \leq n} a_{nm}^{(l)} H_n^m(\omega), \quad k = \overline{2, s}. \tag{30}$$

The terms

$$\left(\frac{r}{R}\right)^n \sum_{|m| \leq n} a_{nm}^{(k)} H_n^m(\omega)$$

of the series on the right-hand side of (29) are harmonic functions in Σ_R. Since these series converge uniformly on the sphere S_R, they also converge uniformly in Σ_R, and their sums $w_k(r, \omega)$, $k = \overline{1, s}$, are harmonic functions in Σ_R because of Harnack's theorem [11].

Further, according to Lemma 1 estimates

$$K_n^{k-l}(r) \leq \frac{M_{k-l}}{n^{k-l}} r^{2(k-l)(1-\alpha)}, \quad l = \overline{k-1, s+k-1},$$

hold, where $n \geq \alpha_{k-l}$ and M_{k-l} is a constant independent of n. Consequently,

$$\left| \left(\frac{r}{R}\right)^n K_n^{k-l}(r) \sum_{|m| \leq n} H_n^m(\omega) a_{nm}^{(l)} \right| < \frac{M_{k-l}}{n^{k-l}} r^{2(k-l)(1-\alpha)} \sum_{|m| \leq n} \left| H_n^m(\omega) a_{nm}^{(l)} \right|$$

in Σ_R^δ for $\forall n \geq \alpha_{k-l}$. Note that the constants M_k, $k = \overline{1, s-1}$, do not depend on n as well as on δ. Evidently, they yield the uniform and absolute convergence of series (30) in $\overline{\Sigma_R^\delta} = \Sigma_R^\delta \cup S_R \cup S_\delta$.

Let $G_\delta(x, \xi)$ be the Green function of the Dirichlet problem to Laplace equation in Σ_R^δ, and let $w_{kn}(x)$ and $W_{kn}(x)$ be the nth partial sum of corresponding series (29) and (30). Since

$$\Delta W_{2n}(x) \equiv q(r)w_{1n}(x),$$

in Σ_R^0, relations

$$W_{2n}(x) = \int\limits_{\Sigma_R^\delta} G_\delta(x, \xi)w_{1n}(\xi)\, d\sigma_\xi,$$

$$W_{kn}(x) = \int\limits_{\Sigma_R^\delta} G_\delta(x, \xi)\left(w_{(k-1)n}(\xi) + W_{(k-1)n}(\xi)\right) d\sigma_\xi, \quad k = \overline{3, s},$$

hold, where $d\sigma_\xi$ is a volume element of Σ_R^δ. These yield the equalities

$$\frac{\partial^2 W_{2n}(x)}{\partial x_i^2} = \int\limits_{\Sigma_R^\delta} \frac{\partial^2 G_\delta(x, \xi)}{\partial x_i^2} w_{1n}(\xi)\, d\sigma_\xi, \tag{31}$$

$$\frac{\partial^2 W_{kn}(x)}{\partial x_i^2} = \int\limits_{\Sigma_R^\delta} \frac{\partial^2 G_\delta(x, \xi)}{\partial x_i^2}\left(w_{(k-1)n}(\xi) + W_{(k-1)n}(\xi)\right) d\sigma_\xi, \quad k = \overline{3, s}. \tag{32}$$

Owing to the uniform and absolute convergence in $\overline{\Sigma_R^\delta}$ of sequences $\{w_{kn}(r, \omega)\}$ and $\{W_{kn}(r, \omega)\}$, as $n \to \infty$, we obtain, from (31) and (32), coherently, that the functions $w_k(r, \omega)$ and $W_k(r, \omega)$, defined by (29) and (30), are twice differentiable and

$$\frac{\partial^2 w_k(x)}{\partial x_i^2} = \sum_{n=\alpha_{s-k}}^{\infty} \frac{\partial^2}{\partial x_i^2}\left(\left(\frac{r}{R}\right)^n \sum_{|m|\leq n} a_{nm}^{(1)} H_n^m(\omega)\right), \quad k = \overline{1, s},$$

$$\frac{\partial^2 W_k(x)}{\partial x_i^2} = \sum_{l=1}^{k-1} \sum_{n=\alpha_{s-l}}^{\infty} \frac{\partial^2}{\partial x_i^2}\left(\left(\frac{r}{R}\right)^n K_n^{k-l}(r) \sum_{|m|\leq n} a_{nm}^{(l)} H_n^m(\omega)\right), \quad k = \overline{2, s},$$

in Σ_R^δ ($i = 1, 2,$ *and* 3).

Hence, the vector function $v = (v_1, v_2, ..., v_s)$ with the components v_i defined by (28)-(30) is from class $C^2(\Sigma_R^0 \cup S_R)$, and it satisfies system (9) in Σ_R^0 and the Dirichlet condition $v|_{S_R} = g$, only if orthogonality conditions (26) hold. Besides, it follows from Lemma 1 that

$$v_k(x) = O(r^{\alpha_{s-1}-2(k-1)(\alpha-1)}), \quad \text{as } x \to 0, \quad k = \overline{1, s}.$$

Therefore, $r\|v(x)\| = o(1)$ as $x \to 0$, if

$$\alpha_{s-1} - 2(s-1)(\alpha-1) > -1. \tag{33}$$

Note that this inequality holds, if, for instance,

$$1 < \alpha < \frac{2s-1}{2(s-1)}.$$

We prove thereby the existence of the solution of problem D_1, if both α and s are related by (33).

If the coefficients $a_{nm}^{(k)}(k = \overline{1, s-1})$ in (28) and (29) are such that

$$a_{nm}^{(k)} = 0 \quad \text{for } 0 \le n < 2(s-k)(\alpha-1),$$

i.e., the components $h_k(k = \overline{1, s-1})$ of the vector function h satisfy the orthogonality conditions

$$\iint\limits_{S_R} h_k(\varphi, \vartheta) Y_n^m(\varphi, \vartheta) \mathrm{d}\varphi \mathrm{d}\vartheta = 0 \quad \text{for } 0 \le n < 2(s-k)(\alpha-1), \tag{34}$$

then the solution v of system (8), given by (28)-(30), is bounded in Σ_R^0 and continuous in $\overline{\Sigma}_R$. Thus, under ortogonality conditions (34), we obtain the solution $v = (v_1, v_2, ..., v_s)$ of problem D_2 of the shape

$$v_k(x) = \sum_{l=1}^{k} \sum_{n=n_{s-l}}^{\infty} \left(\frac{r}{R}\right)^n K_n^{k-l}(r) \sum_{|m|\le n} a_{nm}^{(l)} H_n^m(\omega), \quad k = \overline{1, s}, \tag{35}$$

where

$$n_k = \begin{cases} 2k(\alpha-1), & \text{if } 2k(\alpha-1) \text{ is an integer,} \\ [2k(\alpha-1)] + 1 & \text{in the opposite case.} \end{cases}$$

The uniqueness of the solutions of both the problems D_1 and D_2 yields the following lemma.

Lemma 2. Let $v = (v_1, v_2, ..., v_s)$ be a solution of problem D_1 or problem D_2 with the homogeneous Dirichlet condition $v|_{S_R} = 0$. If relation (33) holds, then $v_i = 0$ in $\Sigma_R^0 (i = \overline{1, n})$.

Proof. Assume that $v = (v_1, v_2, ..., v_s)$ is a solution of problem D_1. Since $\Delta v_1 = 0$ in Σ_R^0 and $v_1|_{S_R} = 0$, we get that $v_1 \equiv 0$ in Σ_R^0 because of the relation $v_1(x) = o(r^{-1})$, as $x \to 0$, which holds because of the validity of condition (11). Then, it follows from system (9) that $\Delta v_2 = 0$ in Σ_R^0. Both the conditions $v_2|_{S_R} = 0$ and $v_2(x) = o(r^{-1})$, as $x \to 0$, yield the identity $v_2 \equiv 0$ in Σ_R^0 to (11). Continuing this process, we obtain that all the components $v_i \equiv 0 (i = \overline{1, n})$ in Σ_R^0.

If $v = (v_1, v_2, ..., v_s)$ is a solution of problem D_2, then it satisfies (11), too. This implies the identity $v \equiv 0$ in Σ_R^0, without doubt.

One can summarize the reasoning given above as follows:

Theorem 3. Let $g \in C^2(S_R)$, and let relation (5) hold. If orthogonality conditions (26) are fulfilled, and the parameters α and s satisfy inequality (33), then there exists a unique solution v of problem D_1, which can be represented by formulas (28)-(30). If orthogonality conditions (34) hold, then there exists a unique solution v of problem D_2 with the components v_i of the shape (35).

Endnotes
[a]One can express the spherical function $H_n^m(x)$ in Cartesian coordinates $x = (x_1, x_2, x_3)$ by formula [12]:

$$H_n^m(x) = r^n (r^2 - x_3^2)^{-\frac{m}{2}} P_n^{|m|}(x_3/r) \times \begin{cases} \mathrm{Re}\,(x_1 - ix_2), & \text{if } 0 \le m \le n, \\ \mathrm{Im}\,(x_1 - ix_2), & \text{if } -n \le m < 0, \end{cases}$$

where P_n^m is adjoint Legendre's function, and i is the imaginary unit. [b]Our opinion is that spherical coordinates are more convenient than Cartesian in the calculation of the coefficients a_{nm} of series (25). The matter is such that spherical functions $Y_n^m(\varphi, \vartheta)$ have quite a simple expresion:

$$Y_n^m(\varphi, \vartheta) = P_n^m(\cos \vartheta) \times \begin{cases} \cos m\varphi, -n \le m \le 0, \\ \sin m\varphi, 0 < m \le n. \end{cases}$$

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