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Existence of solutions to boundary value problem of a class of nonlinear fractional differential equations

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Abstract

In this paper, we study the existence of solutions for the boundary value problem of the following nonlinear fractional differential equation: $D_{0+}^{\alpha}[\frac{x(t)}{f(t,x(t))}] + g(t,x(t)) = 0$, $0 < t < 1$, $x(0) = x(1) = x'(0) = 0$, where $2 < \alpha \leq 3$ is a real number and D_{0+}^{α} is the Riemann-Liouville fractional derivative. By a fixed point theorem in Banach algebra, an existence theorem for the boundary value problem of the above fractional differential equation is proved under both Lipschitz and Carathéodory conditions. Two examples are presented to illustrate the main results.

MSC: 34A08; 34B18

Keywords: fractional differential equation; boundary value problem; fractional Green's function; fixed point theorem

1 Introduction

The theory of fractional derivatives goes back to Leibniz's note in his list to L'Hospital, dated 30 September 1695, in which the meaning of the derivative of order 1/2 is discussed. Leibniz's note led to the appearance of the theory of derivatives and integrals of arbitrary order, which by the end of the 19th century took a more or less final form due primarily to Liouville, Grünwald, Letnikov, and Riemann. Recently, there have been several books on the subject of fractional derivatives and fractional integrals; see [1–4]. For three centuries the theory of fractional derivatives developed mainly as a pure theoretical field of mathematics useful only for mathematicians. However, in the last few decades many authors pointed out that fractional derivatives and fractional integrals are very suitable for the description of properties of various real problems.

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications. Apart from diverse areas of mathematics, fractional differential equations arise in rheology, dynamical processes in self-similar and porous structures, fluid flows, electrical networks, viscoelasticity, chemical physics, and many other branches of science. There have appeared lots of works, in which fractional derivatives are used for a better description of considered material properties, mathematical modeling based on enhanced rheological models naturally leads to differential equations of fractional order and to the necessity of the formulation of initial conditions to such equations.

It should be noted that most of papers and books on fractional calculus are devoted to the solvability of linear fractional differential equations. Recently, there are some papers dealing with the existence of solutions (or positive solutions) of nonlinear initial (or boundary) fractional differential equation (or system) by the use of techniques of nonlinear analysis (fixed point theorems, Leray-Schauder theory, Adomian decomposition method, *etc.*); see [5–23]. In fact, there has the same requirements for boundary conditions. However, there exist some papers considered the boundary value problems of fractional differential equations; see [10–22].

Yu and Jiang [21] discussed the following two-point boundary value problem of fractional differential equations:

$$\begin{aligned} &D_{0^+}^{\alpha} u(t) + f\big(t, u(t)\big) = 0, \quad 0 < t < 1, \\ &u(0) = u(1) = u'(0) = 0, \end{aligned}$$

where $2 < \alpha \leq 3$ is a real number and $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative. By the properties of the Green function, they gave some results of multiple positive solutions for singular and nonsingular boundary value problems by means of the Leray-Schauder nonlinear alternative, the fixed-point theorem on cones, and a mixed monotone method.

In recent years, quadratic perturbations of nonlinear differential equations have attracted much attention. There have been many works on the theory of such differential equations, and we refer the readers to the articles [22–27].

Zhao *et al.* [23] studied fractional hybrid differential equations involving Riemann-Liouville differential operators of order $0 < q < 1$,

$$\begin{aligned} &D^q \left[\frac{x(t)}{f(t, x(t))}\right] = g\big(t, x(t)\big), \quad \text{a.e. } t \in [0, T], \\ &x(0) = 0, \end{aligned}$$

where $f \in C([0, T] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C([0, T] \times \mathbb{R}, \mathbb{R})$. They established the existence and uniqueness results and some fundamental differential inequalities for fractional hybrid differential equations, initiating the study of such systems, and proved, utilizing the theory of inequalities, the existence of extremal solutions and a comparison result.

From the above works, we consider the existence of solutions for the boundary value problem of the following nonlinear fractional differential equation:

$$D_{0^+}^{\alpha} \left[\frac{x(t)}{f(t, x(t))}\right] + g\big(t, x(t)\big) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$x(0) = x(1) = x'(0) = 0, \tag{1.2}$$

where $2 < \alpha \leq 3$ is a real number and $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative. Using the fixed point theorem, we give an existence theorem for the boundary value problem of the above nonlinear fractional differential equation under both Lipschitz and Carathéodory conditions. We present two examples to demonstrate our results.

Let $\mathbb{R}$ be the real line and $J = [0,1]$ be a bounded interval in $\mathbb{R}$. Let $C(J \times \mathbb{R}, \mathbb{R})$ denote the class of continuous functions $f : J \times \mathbb{R} \to \mathbb{R}$ and let $\mathrm{Car}(J \times \mathbb{R}, \mathbb{R})$ denote the class of functions $g : J \times \mathbb{R} \to \mathbb{R}$ such that

(i) the map $t \mapsto g(t,x)$ is measurable for each $x \in \mathbb{R}$, and

(ii) the map $x \mapsto g(t,x)$ is continuous for each $t \in J$.

The class $\mathrm{Car}(J \times \mathbb{R}, \mathbb{R})$ is called the Carathéodory class of functions on $J \times \mathbb{R}$ which are Lebesgue integrable when bounded by a Lebesgue integrable function on J.

In this paper, we assume $f \in C^1([0,1] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in \mathrm{Car}([0,1] \times \mathbb{R}, \mathbb{R})$.

The plan of this paper is as follows. In Section 2, we shall give some definitions and lemmas to prove our main results. In Section 3, we establish the existence of solutions for the boundary value problem (1.1) and (1.2) by the fixed point theorem. Two examples are presented to illustrate the main results in Section 4.

2 Preliminaries

For the convenience of the reader, we give some background materials from fractional calculus theory to facilitate the analysis of problem (1.1) and (1.2). These materials can be found in the recent literature; see [21, 28, 29].

Definition 2.1 ([28]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{(n)} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\, ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2.2 ([28]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided that the right side is pointwise defined on $(0, +\infty)$.

From the definition of the Riemann-Liouville derivative, we can obtain the following statement.

Lemma 2.1 ([28]) *Let $\alpha > 0$. If we assume $x \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$D_{0^+}^{\alpha} x(t) = 0$$

has $x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, as unique solutions, where n is the smallest integer greater than or equal to α.

Lemma 2.2 ([28]) *Assume that* $x \in C(0,1) \cap L(0,1)$ *with a fractional derivative of order* $\alpha > 0$ *that belongs to* $C(0,1) \cap L(0,1)$. *Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, *where* n *is the smallest integer greater than or equal to* α.

In the following, we present the Green function of the fractional differential equation boundary value problem.

Lemma 2.3 *Let* $y \in C[0,1]$ *and* $2 < \alpha \leq 3$. *The unique solution of the problem*

$$D_{0^+}^{\alpha}\left[\frac{x(t)}{f(t,x(t))}\right] + y(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$x(0) = x(1) = x'(0) = 0 \tag{2.2}$$

is

$$x(t) = f\bigl(t,x(t)\bigr)\int_0^1 G(t,s)y(s)\,ds,$$

where

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.3}$$

Here $G(t,s)$ *is called the Green function of the boundary value problem* (2.1) *and* (2.2).

Proof We may apply Lemma 2.2 to reduce (2.1) to an equivalent integral equation

$$\frac{x(t)}{f(t,x(t))} = -I_{0^+}^{\alpha} y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}, \tag{2.4}$$

for some $c_1, c_2, c_3 \in \mathbb{R}$. Consequently, the general solution of (2.1) is

$$x(t) = f\bigl(t,x(t)\bigr)\left(-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}\right).$$

By $x(0) = 0$, we get $c_3 = 0$. From (2.4), we have

$$\frac{x'(t)f(t,x(t)) - x(t)f_t(t,x(t))}{f^2(t,x(t))} = -I_{0^+}^{\alpha-1} y(t) + c_1(\alpha-1)t^{\alpha-2} + c_2(\alpha-2)t^{\alpha-3}.$$

By $x'(0) = 0$, we obtain $c_2 = 0$. From (2.2), we get

$$c_1 = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)\,ds.$$

Therefore, the unique solution of problem (2.1) and (2.2) is

$$\begin{aligned} x(t) &= f\big(t,x(t)\big)\left(-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}y(s)\,ds\right) \\ &= f\big(t,x(t)\big)\left(\frac{1}{\Gamma(\alpha)}\int_0^t \big\{t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}\big\}y(s)\,ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)}\int_t^1 t^{\alpha-1}(1-s)^{\alpha-1}y(s)\,ds\right) \\ &= f\big(t,x(t)\big)\int_0^1 G(t,s)y(s)\,ds. \end{aligned}$$

The proof is complete. □

Lemma 2.4 ([21]) *The function $G(t,s)$ defined by* (2.3) *satisfies the following conditions*:
(1) $G(t,s) = G(1-s,1-t)$, *for* $t,s \in (0,1)$;
(2) $t^{\alpha-1}(1-t)s(1-s)^{\alpha-1} \le \Gamma(\alpha)G(t,s) \le (\alpha-1)s(1-s)^{\alpha-1}$, *for* $t,s \in (0,1)$;
(3) $G(t,s) > 0$, *for* $t,s \in (0,1)$;
(4) $t^{\alpha-1}(1-t)s(1-s)^{\alpha-1} \le \Gamma(\alpha)G(t,s) \le (\alpha-1)(1-t)t^{\alpha-1}$, *for* $t,s \in (0,1)$.

Remark 2.1 Let $q(t) = t^{\alpha-1}(1-t)$, $k(s) = s(1-s)^{\alpha-1}$. Then we have

$$q(t)k(s) \le \Gamma(\alpha)G(t,s) \le (\alpha-1)k(s).$$

Define a supremum norm $\|\cdot\|$ in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J}\big|x(t)\big|$$

and a multiplication in $C(J,\mathbb{R})$ by

$$(xy)(t) = x(t)y(t)$$

for $x,y \in C(J,\mathbb{R})$. Clearly $C(J,\mathbb{R})$ is a Banach algebra with respect to above norm and multiplication in it. By $L^1(J,\mathbb{R})$ denote the space of Lebesgue integrable real-valued functions on J equipped with the norm $\|\cdot\|_{L^1}$ defined by

$$\|x\|_{L^1} = \int_0^1 \big|x(s)\big|\,ds.$$

The following fixed point theorem in Banach algebra due to Dhage [29] is fundamental in the proofs of our main results.

Lemma 2.5 ([29]) *Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and let $A : X \to X$ and $B : S \to X$ be two operators such that*
(a) *A is Lipschitzian with a Lipschitz constant β,*
(b) *B is completely continuous,*
(c) $x = AxBy \Rightarrow x \in S$ *for all* $y \in S$, *and*
(d) $\beta M < 1$, *where* $M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}$.
Then the operator equation $AxBx = x$ has a solution in S.

3 Existence result

In this section, we prove the existence results for fractional differential equation (1.1) on the closed and bounded interval $J = [0,1]$ under both Lipschitz and Carathéodory conditions on the nonlinearities involved in it. We place the fractional differential equation (1.1) in the space $C(J,\mathbb{R})$ of continuous real-valued functions defined on J.

For convenience, we denote

$$T = \frac{\alpha - 1}{\Gamma(\alpha)} \int_0^1 k(s)\,ds.$$

We consider the following hypotheses in what follows.

(H_1) There exists a constant $\beta > 0$ such that

$$\bigl|f(t,x) - f(t,y)\bigr| \leq \beta|x - y|$$

for all $t \in J$ and $x, y \in \mathbb{R}$.

(H_2) There exists a function $h \in L^1(J,\mathbb{R}^+)$ such that

$$\bigl|g(t,x)\bigr| \leq h(t), \quad t \in J,$$

for all $x \in \mathbb{R}$.

Theorem 3.1 *Assume that hypotheses* (H_1) *and* (H_2) *hold. Further, if*

$$\beta T\|h\|_{L^1} < 1, \tag{3.1}$$

then the boundary value problem (1.1) *and* (1.2) *has a solution defined on* J.

Proof Set $X = C(J,\mathbb{R})$ and define a subset S of X defined by

$$S = \bigl\{x \in X \mid \|x\| \leq N\bigr\}, \tag{3.2}$$

where $N = \frac{F_0 T\|h\|_{L^1}}{1-\beta T\|h\|_{L^1}}$ and $F_0 = \sup_{t\in J} |f(t,0)|$.

Clearly S is a closed, convex, and bounded subset of the Banach space X. By Lemma 2.3, the boundary value problem (1.1) and (1.2) is equivalent to the equation

$$x(t) = f\bigl(t,x(t)\bigr) \int_0^1 G(t,s)g\bigl(s,x(s)\bigr)\,ds, \quad t \in J. \tag{3.3}$$

Define two operators $A : X \to X$ and $B : S \to X$ by

$$Ax(t) = f\bigl(t,x(t)\bigr), \quad t \in J, \tag{3.4}$$

and

$$Bx(t) = \int_0^1 G(t,s)g\bigl(s,x(s)\bigr)\,ds, \quad t \in J. \tag{3.5}$$

Then (3.3) is transformed into the operator equation:

$$Ax(t)Bx(t) = x(t), \quad t \in J. \tag{3.6}$$

We shall show that the operators A and B satisfy all the conditions of Lemma 2.5.

First, we show that A is a Lipschitz operator on X with the Lipschitz constant β. Let $x, y \in X$. Then by hypothesis (H_1),

$$\left|Ax(t) - Ay(t)\right| = \left|f\bigl(t, x(t)\bigr) - f\bigl(t, y(t)\bigr)\right| \le \beta\left|x(t) - y(t)\right| \le \beta\|x - y\|,$$

for all $t \in J$. Taking the supremum over t, we obtain

$$\|Ax - Ay\| \le \beta\|x - y\|,$$

for all $x, y \in X$.

Next, we show that B is a compact and continuous operator on S into X. First we show that B is continuous on S. Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$. Then by the Lebesgue dominated convergence theorem,

$$\begin{aligned}\lim_{n\to\infty} Bx_n(t) &= \lim_{n\to\infty} \int_0^1 G(t,s)g\bigl(s, x_n(s)\bigr)\,ds = \int_0^1 G(t,s)\lim_{n\to\infty} g\bigl(s, x_n(s)\bigr)\,ds \\ &= \int_0^1 G(t,s)g\bigl(s, x(s)\bigr)\,ds = Bx(t),\end{aligned}$$

for all $t \in J$. This shows that B is a continuous operator on S.

Next we show that B is a compact operator on S. It is enough to show that $B(S)$ is a uniformly bounded and equicontinuous set in X. On the one hand, let $x \in S$ be arbitrary. By Lemma 2.4, we have

$$\left|Bx(t)\right| = \left|\int_0^1 G(t,s)g\bigl(s, x(s)\bigr)\,ds\right| \le \int_0^1 \frac{\alpha - 1}{\Gamma(\alpha)}k(s)h(s)\,ds \le T\|h\|_{L^1},$$

for all $t \in J$. Taking the supremum over t,

$$\|Bx\| \le T\|h\|_{L^1},$$

for all $x \in S$. This shows that B is uniformly bounded on S.

On the other hand, given $\epsilon > 0$, let

$$\delta < \min\left\{\frac{1}{2}, \frac{\Gamma(\alpha+1)\epsilon}{12\|h\|_{L^1}}\right\}.$$

Then for any $x \in S$, $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, $0 < t_2 - t_1 < \delta$, we have

$$\begin{aligned}\left|Bx(t_2) - Bx(t_1)\right| &= \left|\int_0^1 G(t_2,s)g\bigl(s, x(s)\bigr)\,ds - \int_0^1 G(t_1,s)g\bigl(s, x(s)\bigr)\,ds\right| \\ &\le \|h\|_{L^1}\left|\int_0^{t_2} \frac{[t_2(1-s)]^{\alpha-1} - (t_2 - s)^{\alpha-1}}{\Gamma(\alpha)}\,ds + \int_{t_2}^1 \frac{[t_2(1-s)]^{\alpha-1}}{\Gamma(\alpha)}\,ds\right.\end{aligned}$$

$$
\begin{aligned}
&\quad - \int_0^{t_1} \frac{[t_1(1-s)]^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds - \int_{t_1}^1 \frac{[t_1(1-s)]^{\alpha-1}}{\Gamma(\alpha)}\,ds \Bigg| \\
&\leq \|h\|_{L^1} \int_0^1 \frac{[{t_2}^{\alpha-1} - {t_1}^{\alpha-1}](1-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds \\
&\quad + \|h\|_{L^1} \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds \\
&\quad + \|h\|_{L^1} \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds \\
&\leq \frac{\|h\|_{L^1}}{\Gamma(\alpha+1)} \bigl({t_2}^{\alpha-1} - {t_1}^{\alpha-1} + {t_2}^{\alpha} - {t_1}^{\alpha} \bigr).
\end{aligned}
$$

In order to estimate ${t_2}^{\alpha-1} - {t_1}^{\alpha-1}$ and ${t_2}^{\alpha} - {t_1}^{\alpha}$, we divide the proof into three cases.

Case 1: $0 \leq t_1 < \delta$, $t_2 < 2\delta$.

$$
t_2^{\alpha-1} - t_1^{\alpha-1} \leq t_2^{\alpha-1} < (2\delta)^{\alpha-1} \leq 2^{\alpha-1}\delta \leq 4\delta, \qquad t_2^{\alpha} - t_1^{\alpha} \leq t_2^{\alpha} < (2\delta)^{\alpha} \leq 2^{\alpha}\delta \leq 8\delta.
$$

Case 2: $0 < t_1 < t_2 \leq \delta$.

$$
t_2^{\alpha-1} - t_1^{\alpha-1} \leq t_2^{\alpha-1} < \delta^{\alpha-1} \leq (\alpha-1)\delta < 4\delta, \qquad t_2^{\alpha} - t_1^{\alpha} \leq t_2^{\alpha} < \delta^{\alpha} \leq \alpha\delta < 8\delta.
$$

Case 3: $\delta \leq t_1 < t_2 \leq 1$.

$$
t_2^{\alpha-1} - t_1^{\alpha-1} \leq (\alpha-1)\delta < 4\delta, \qquad t_2^{\alpha} - t_1^{\alpha} \leq \alpha\delta < 8\delta.
$$

Thus, we obtain

$$
\bigl|Bx(t_2) - Bx(t_1)\bigr| < \epsilon,
$$

for all $t_1, t_2 \in J$ and for all $x \in S$. This shows that $B(S)$ is an equicontinuous set in X. Now the set $B(S)$ is uniformly bounded and equicontinuous set in X, so it is compact by the Arzela-Ascoli Theorem. As a result, B is a completely continuous operator on S.

Next, we show that hypothesis (c) of Lemma 2.5 is satisfied. Let $x \in X$ and $y \in S$ be arbitrary such that $x = AxBy$. Then, by assumption (H_1), we have

$$
\begin{aligned}
\bigl|x(t)\bigr| &= \bigl|Ax(t)\bigr|\bigl|By(t)\bigr| \\
&= \bigl|f\bigl(t, x(t)\bigr)\bigr| \left| \int_0^1 G(t,s) g\bigl(s, y(s)\bigr)\,ds \right| \\
&\leq \bigl[\bigl|f\bigl(t, x(t)\bigr) - f(t,0)\bigr| + \bigl|f(t,0)\bigr|\bigr] \left(\int_0^1 \frac{\alpha-1}{\Gamma(\alpha)} k(s) g\bigl(s, y(s)\bigr)\,ds \right) \\
&\leq \bigl[\beta\bigl|x(t)\bigr| + F_0\bigr] T \|h\|_{L^1}.
\end{aligned}
$$

Thus,

$$
\bigl|x(t)\bigr| \leq \frac{F_0 T \|h\|_{L^1}}{1 - \beta T \|h\|_{L^1}}.
$$

Taking the supremum over t,

$$\|x\| \le \frac{F_0 T\|h\|_{L^1}}{1 - \beta T\|h\|_{L^1}} = N.$$

This shows that hypothesis (c) of Lemma 2.5 is satisfied. Finally, we have

$$M = \|B(S)\| = \sup\{\|Bx\| : x \in S\} \le T\|h\|_{L^1}.$$

Therefore,

$$\beta M \le \beta T\|h\|_{L^1} < 1.$$

Thus, all the conditions of Lemma 2.5 are satisfied and hence the operator equation $AxBx = x$ has a solution in S. As a result, the boundary value problem (1.1) and (1.2) has a solution defined on J. This completes the proof. □

Remark 3.1 For the special case $f(t,x) \equiv 1$, we can find the corresponding existence results in Yu and Jiang (see [21]).

4 Examples

In this section, we will present two examples to illustrate the main results.

Example 4.1 Consider the boundary value problem

$$D_{0^+}^{\frac{5}{2}} x(t) + \cos x = 0, \quad 0 < t < 1, \tag{4.1}$$

$$x(0) = x(1) = x'(0) = 0, \tag{4.2}$$

where $\alpha = \frac{5}{2}$.

Let $f(t,x) \equiv 1$, $g(t,x) = \cos x$, $h(t) \equiv 1$. Then hypotheses (H_1) and (H_2) hold. Since

$$T = \frac{\alpha - 1}{\Gamma(\alpha)} \int_0^1 k(s)\,ds = \frac{\alpha - 1}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1}\,ds = \frac{\frac{5}{2} - 1}{\Gamma(\frac{5}{2})} \int_0^1 s(1-s)^{\frac{3}{2}}\,ds = \frac{8}{35\sqrt{\pi}},$$

choosing $\beta = 1$, then $\beta T\|h\|_{L^1} = 8/35\sqrt{\pi} < 1$. Therefore, the boundary value problem (4.1) and (4.2) has a solution.

Example 4.2 Consider the boundary value problem

$$D_{0^+}^{\frac{5}{2}} \left[\frac{x(t)}{\cos x + 2}\right] + \sin x = 0, \quad 0 < t < 1, \tag{4.3}$$

$$x(0) = x(1) = x'(0) = 0, \tag{4.4}$$

where $\alpha = \frac{5}{2}$.

Let $f(t,x) = \cos x + 2$, $g(t,x) = \sin x$, $h(t) \equiv 1$. Then hypotheses (H_1) and (H_2) hold. Since $T = 8/35\sqrt{\pi}$, choosing $\beta = 1$, then $\beta T\|h\|_{L^1} = 8/35\sqrt{\pi} < 1$. Therefore, the boundary value problem (4.3) and (4.4) has a solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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