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Strong convergence theorem for total quasi-ϕ-asymptotically nonexpansive mappings in a Banach space

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Abstract

In this paper, we prove strong convergence theorems to a point which is a fixed point of multi-valued mappings, a zero of an α-inverse-strongly monotone operator and a solution of the equilibrium problem. Next, we obtain strong convergence theorems to a solution of the variational inequality problem, a fixed point of multi-valued mappings and a solution of the equilibrium problem. The results presented in this paper are improvement and generalization of the previously known results.

Keywords: total quasi-ϕ-asymptotically nonexpansive multi-valued mappings; hybrid scheme; equilibrium problem; variational inequality problems; inverse-strongly monotone operator

1 Introduction

Let E be a real Banach space with dual E^*, and let C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be an operator. A is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

α*-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C;$$

L*-Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

If A is α-inverse strongly monotone, then it is $\frac{1}{\alpha}$-Lipschitz continuous, *i.e.*,

$$\|Ax - Ay\| \leq \frac{1}{\alpha}\|x - y\|, \quad \forall x, y \in C.$$

A monotone operator A is said to be *maximal* if its graph $G(A) = \{(x, x^*) : x^* \in Ax\}$ is not properly contained in the graph of any other monotone operator.

Let A be a monotone operator. We consider the problem of finding $x \in E$ such that

$$0 \in Ax, \tag{1.1}$$

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a point $x \in E$ is called a *zero point* of A. Denote by $A^{-1}0$ the set of all points $x \in E$ such that $0 \in Ax$. This problem is very important in optimization theory and related fields.

Let A be a monotone operator. *The classical variational inequality problem* for an operator A is to find $\hat{z} \in C$ such that

$$\langle A\hat{z}, y - \hat{z}\rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $VI(A, C)$. This problem is connected with the convex minimization problem, the complementary problem, the problem of finding a point $x \in E$ satisfying $Ax = 0$.

The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^*\rangle$ or $x^*(x)$. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\right\}$$

for all $x \in E$. In particular, $J = J_2$ is called the *normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping.

Consider the functional defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2 \quad \text{for } x, y \in E, \tag{1.3}$$

where J is the normalized duality mapping. It is obvious from the definition of ϕ that

$$\left(\|y\| - \|x\|\right)^2 \leq \phi(y, x) \leq \left(\|y\| + \|x\|\right)^2, \quad \forall x, y \in E. \tag{1.4}$$

Alber [1] introduced that the *generalized projection* $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution of the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x), \tag{1.5}$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J.

Iiduka and Takahashi [2] introduced the following iterative scheme for finding a solution of the variational inequality problem for an inverse-strongly monotone operator A in a 2-uniformly convex and uniformly smooth Banach space E: $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \quad \forall n \geq 1, \tag{1.6}$$

where Π_C is the generalized projection from E onto C, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.6) converges weakly to some element of $VI(A, C)$. In connection, Iiduka and Takahashi [3] studied the following iterative scheme for finding a zero point of a monotone operator A in a 2-uniformly convex and uniformly smooth Banach space E:

$$\begin{cases} x_1 = x \in E \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(Jx_n - \lambda_n Ax_n), \\ X_n = \{z \in E : \phi(z, y_n) \le \phi(z, x_n)\}, \\ Y_{n+1} = \{z \in E : \langle x_n - z, Jx - Jx_n \rangle \ge 0\}, \\ x_{n+1} = \Pi_{X_n \cap Y_n}(x), \end{cases} \tag{1.7}$$

where $\Pi_{X_n \cap Y_n}$ is the generalized projection from E onto $X_n \cap Y_n$, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ converges strongly to an element of $A^{-1}0$. Moreover, under the additional suitable assumption they proved that the sequence $\{x_n\}$ converges strongly to some element of $VI(A, C)$. Some solution methods have been proposed to solve the variational inequality problem; see, for instance, [4–6].

A mapping $T : C \to C$ is said to be *ϕ-nonexpansive* [7, 8] if

$$\phi(Tx, Ty) \le \phi(x, y), \quad \forall x, y \in C.$$

T is said to be *quasi-ϕ-nonexpansive* [7, 8] if $F(T) \ne \emptyset$ and

$$\phi(p, Tx) \le \phi(p, x), \quad \forall x \in C, p \in F(T).$$

T is said to be *total quasi-ϕ-asymptotically nonexpansive*, if $F(T) \ne \emptyset$ and there exist nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that

$$\phi\big(p, T^n x\big) \le \phi(p, x) + \nu_n \varphi\big(\phi(p, x)\big) + \mu_n, \quad \forall n \ge 1, \forall x \in C, p \in F(T).$$

Let 2^C be the family of all nonempty subsets of C, and let $S : C \to 2^C$ be a multi-valued mapping. For a point $q \in C$, $n \ge 1$ define an iterative sequence as follows:

$$\begin{aligned} &Sq := \{q_1 : q_1 \in Sq\}, \\ &S^2 q = SSq := \bigcup_{q_1 \in Sq} Sq_1, \\ &S^3 q = SS^2 q := \bigcup_{q_2 \in T^2 q} Sq_2, \\ &\vdots \\ &S^n q = SS^{n-1} q := \bigcup_{q_{n-1} \in S^{n-1} q} Sq_{n-1}. \end{aligned}$$

A point $p \in C$ is said to be an *asymptotic fixed point* of S if there exists a sequence $\{x_n\}$ in C such that $\{x_n\}$ converges weakly to p and

$$\lim_{n \to \infty} d(x_n, Sx_n) := \lim_{n \to \infty} \inf_{x \in Sx_n} \|x_n - x\| = 0.$$

The asymptotic fixed point set of S is denoted by $\widehat{F}(S)$.

A multi-valued mapping S is said to be *total quasi-ϕ-asymptotically nonexpansive* if $F(S) \neq \emptyset$ and there exist nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that for all $x \in C$, $p \in F(S)$,

$$\phi(p, w_n) \leq \phi(p, x) + \nu_n \varphi\big(\phi(p, x)\big) + \mu_n, \quad \forall n \geq 1, w_n \in S^n x.$$

S is said to be *closed* if for any sequence $\{x_n\}$ and $\{w_n\}$ in C with $w_n \in Sx_n$ if $x_n \to x$ and $w_n \to w$, then $w \in Sx$.

A multi-valued mapping S is said to be *uniformly asymptotically regular* on C if

$$\lim_{n\to\infty} \Big(\sup_{x\in C} \|s_{n+1} - s_n\|\Big) = 0, \quad s_n \in S^n x.$$

Every quasi-ϕ-asymptotically nonexpansive multi-valued mapping implies a quasi-ϕ-asymptotically nonexpansive mapping but the converse is not true.

In 2012, Chang *et al.* [9] introduced the concept of total quasi-ϕ-asymptotically nonexpansive multi-valued mapping and then proved some strong convergence theorem by using the hybrid shrinking projection method.

Let $f : C \times C \to \mathbb{R}$ be a bifunction, the *equilibrium problem* is to find $x \in C$ such that

$$f(x, y) \geq 0, \quad \forall y \in C. \tag{1.8}$$

The set of solutions of (1.8) is denoted by $EP(f)$. The equilibrium problem is very general in the sense that it includes, as special cases, optimization problems, variational inequality problems, min-max problems, saddle point problem, fixed point problem, Nash EP. In 2008, Takahashi and Zembayashi [10, 11] introduced iterative sequences for finding a common solution of an equilibrium problem and a fixed point problem. Some solution methods have been proposed to solve the equilibrium problem; see, for instance, [12–21].

For a mapping $A : C \to E^*$, let $f(x, y) = \langle Ax, y - x\rangle$ for all $x, y \in C$. Then $x \in EP(f)$ if and only if $\langle Tx, y - x\rangle \geq 0$ for all $y \in C$; *i.e.*, x is a solution of the variational inequality.

Motivated and inspired by the work mentioned above, in this paper, we introduce and prove strong convergence of a new hybrid projection algorithm for a fixed point of total quasi-ϕ-asymptotically nonexpansive multi-valued mappings, the solution of the equilibrium problem, a zero point of monotone operators. Moreover, we prove strong convergence to the solution of the variation inequality in a uniformly smooth and 2-uniformly convex Banach space.

2 Preliminaries

A Banach space E with the norm $\|\cdot\|$ is called *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is called *smooth* if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U$. It is also called *uniformly smooth* if the limit exists uniformly for all $x, y \in U$. The *modulus of convexity* of E is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\right\}.$$

A Banach space E is *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0,2]$. Observe that every p-uniform convex is uniformly convex. Every uniformly convex Banach space E has the *Kadec-Klee property*, that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

Let E be a real Banach space with dual E^*, E is uniformly smooth if and only if E^* is a uniformly convex Banach space. If E is a uniformly smooth Banach space, then E is a smooth and reflexive Banach space.

Remark 2.1

- If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.
- If E is reflexive smooth and strictly convex, then the normalized duality mapping J is single-valued, one-to-one and onto.
- If E is a reflexive strictly convex and smooth Banach space and J is the duality mapping from E into E^*, then J^{-1} is also single-valued, bijective and is also the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$.

See [22] for more details.

Remark 2.2 If E is a reflexive, strictly convex and smooth Banach space, then $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x,y) = 0$, then $x = y$. From (1.3) we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$ (see [22, 23] for more details).

Lemma 2.3 (Beauzamy [24] and Xu [25]) *If E is a 2-uniformly convex Banach space, then, for all $x,y \in E$, we have*

$$\|x - y\| \leq \frac{2}{c^2}\|Jx - Jy\|,$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

The best constant $\frac{1}{c}$ in the lemma is called the *p-uniformly convex constant* of E.

Lemma 2.4 (Beauzamy [24] and Zalinescu [26]) *If E is a p-uniformly convex Banach space, and let p be a given real number with $p \geq 2$, then, for all $x,y \in E$, $J_x \in J_p(x)$ and $J_y \in J_p(y)$,*

$$\langle x - y, J_x - J_y\rangle \geq \frac{c^p}{2^{p-2}p}\|x - y\|^p,$$

where J_p is the generalized duality mapping of E and $\frac{1}{c}$ is the p-uniformly convex constant of E.

Lemma 2.5 (Kamimura and Takahashi [27]) *Let E be a uniformly convex and smooth Banach space, and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n,y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.6 (Alber [1]) *Let C be a nonempty closed convex subset of a smooth Banach space E, and let $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.7 (Alber [1]) *Let E be a reflexive strictly convex and smooth Banach space, C be a nonempty closed convex subset of E, and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 2.8 (Chang *et al.* [9]) *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let $S : C \to 2^C$ be a closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequence ν_n and μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$. If $\mu_1 = 0$, then the fixed point set $F(S)$ is a closed convex subset of C.*

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, *i.e.*, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} f\bigl(tz + (1-t)x, y\bigr) \leq f(x, y);$$

(A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semi-continuous.

Lemma 2.9 (Blum and Oettli [28]) *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4), *and let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.10 (Takahashi and Zembayashi [11]) *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions* (A1)-(A4). *For all $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows*:

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}.$$

Then the following hold:

(1) *T_r is single-valued*;

(2) *T_r is a firmly nonexpansive-type mapping* [29], *that is, for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle;$$

(3) $F(T_r) = EP(f)$;
(4) $EP(f)$ *is closed and convex.*

Lemma 2.11 (Takahashi and Zembayashi [11]) *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4), *and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Let A be an inverse-strongly monotone mapping of C into E^* which is said to be *hemicontinuous* if for all $x, y \in C$, the mapping h of $[0,1]$ into E^*, defined by $h(t) = A(tx + (1-t)y)$, is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$ *the normal cone* for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \forall y \in C\}. \tag{2.1}$$

Theorem 2.12 (Rockafellar [30]) *Let C be a nonempty, closed convex subset of a Banach space E, and let A be a monotone, hemicontinuous operator of C into E^*. Let $B \subset E \times E^*$ be an operator defined as follows*:

$$Bv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & \textit{otherwise}. \end{cases} \tag{2.2}$$

Then B is maximal monotone and $B^{-1}0 = VI(A, C)$.

Theorem 2.13 (Takahashi [31]) *Let C be a nonempty subset of a Banach space E, and let A be a monotone, hemicontinuous operator of C into E^* with $C = D(A)$. Then*

$$VI(A, C) = \{u \in C : \langle v - u, Av \rangle \geq 0, \forall v \in C\}. \tag{2.3}$$

It is obvious that the set $VI(A, C)$ is a closed and convex subset of C and the set $A^{-1}0 = VI(A, E)$ is a closed and convex subset of E.

Theorem 2.14 (Takahashi [31]) *Let C be a nonempty compact convex subset of a Banach space E, and let A be a monotone, hemicontinuous operator of C into E^* with $C = D(A)$. Then $VI(A, C)$ is nonempty.*

We make use of the following mapping V studied in Alber [1]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2, \quad \forall x \in E, x^* \in E^*, \tag{2.4}$$

that is, $V(x, x^*) = \phi(x, J^{-1}(x^*))$.

Lemma 2.15 (Alber [1]) *Let E be a reflexive strictly convex smooth Banach space, and let V be as in* (2.4). *Then we have*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*), \quad \forall x \in E, x^*, y^* \in E^*.$$

Lemma 2.16 (Beauzamy [24] and Xu [25]) *If E is a 2-uniformly convex Banach space, then, for all $x, y \in E$, we have*

$$\|x-y\| \leq \frac{2}{c^2}\|Jx - Jy\|,$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

Lemma 2.17 (Cho *et al.* [32]) *Let E be a uniformly convex Banach space, and let $B_r(0) = \{x \in E : \|x\| \leq r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \|\lambda x\|^2 + \|\mu y\|^2 + \|\gamma z\|^2 - \lambda\mu g\bigl(\|x-y\|\bigr)$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0,1]$ with $\lambda + \mu + \gamma = 1$.

Lemma 2.18 (Pascali and Sburlan [33]) *Let E be a real smooth Banach space, and let $A : E \to 2^{E^*}$ be a maximal monotone mapping. Then $A^{-1}0$ is a closed and convex subset of E and the graph $G(A)$ of A is demiclosed in the following sense: if $\{x_n\} \subset D(A)$ with $x_n \rightharpoonup x \in E$ and $y_n \in Ax_n$ with $y_n \to y \in E^*$, then $x \in D(A)$ and $y \in Ax$.*

3 Main results

Theorem 3.1 *Let C be a nonempty closed and convex subset of a uniformly smooth and 2-uniformly convex Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions (A1)-(A4), and let A be an α-inverse-strongly monotone mapping of E into E^*. Let $S : C \to 2^C$ be a closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$. Assume that S is uniformly asymptotically regular on C with $\mu_1 = 0$ and $F := F(S) \cap EP(f) \cap A^{-1}0 \neq \emptyset$. For arbitrary $x_1 \in C$, $C_1 = C$, generate a sequence $\{x_n\}$ by*

$$\begin{cases} z_n = J^{-1}(Jx_n - \lambda_n Ax_n), \\ u_n = T_{r_n} z_n, \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n Jw_n + \gamma_n Ju_n), \quad w_n \in S^n x_n, \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, z_n) \leq \phi(v, x_n) + K_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $K_n = \nu_n \sup_{q\in F} \psi(\phi(q, x_n)) + \mu_n$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the following conditions:

1. *$\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$,* $\liminf_{n\to\infty} \alpha_n\beta_n > 0$,
2. *$\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < \frac{c^2\alpha}{2}$ and $\frac{1}{c}$ is the 2-uniformly convex constant of E,*
3. *$\{r_n\} \subset [d,\infty)$ for some $d > 0$,*

then $\{x_n\}$ converges strongly to $\Pi_F x_1$.

Proof We will show that C_n is closed and convex for all $n \in \mathbb{N}$. Since $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for all $n \in \mathbb{N}$. For any $v \in C_n$, we know that $\phi(v, y_n) \leq \phi(v, x_n) + K_n$ is equivalent to

$$2\langle v, Jx_n - Jy_n\rangle \leq \|x_n\|^2 - \|y_n\|^2 + K_n.$$

That is, C_{n+1} is closed and convex, hence C_n is closed and convex for all $n \in \mathbb{N}$.

We show by induction that $F \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that $F \subset C = C_1$. Suppose that $F \subset C_n$ where $n \in \mathbb{N}$. Let $q \in F$, we have

$$\begin{aligned}
\phi(q, z_n) &= \phi\bigl(q, J^{-1}(Jx_n - \lambda_n Ax_n)\bigr) \\
&= V(q, Jx_n - \lambda_n Ax_n) \\
&\leq V\bigl(q, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n\bigr) - 2\bigl\langle J^{-1}(Jx_n - \lambda_n Ax_n) - q, \lambda_n Ax_n\bigr\rangle \\
&= V(q, Jx_n) - 2\lambda_n\bigl\langle J^{-1}(Jx_n - \lambda_n Ax_n) - q, Ax_n\bigr\rangle \\
&= \phi(q, x_n) - 2\lambda_n\langle x_n - q, Ax_n\rangle + 2\bigl\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n\bigr\rangle.
\end{aligned} \tag{3.2}$$

Since A is an α-inverse-strongly monotone mapping, we get

$$\begin{aligned}
-2\lambda_n\langle x_n - q, Ax_n\rangle &= -2\lambda_n\langle x_n - q, Ax_n - Aq\rangle - 2\lambda_n\langle x_n - q, Aq\rangle \\
&\leq -2\lambda_n\langle x_n - q, Ax_n - Aq\rangle \\
&= -2\alpha\lambda_n\|Ax_n - Aq\|^2.
\end{aligned} \tag{3.3}$$

It follows from Lemma 2.17 that

$$\begin{aligned}
2\bigl\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n\bigr\rangle &= 2\bigl\langle J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n), -\lambda_n Ax_n\bigr\rangle \\
&\leq 2\bigl\|J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n)\bigr\|\|\lambda_n Ax_n\| \\
&\leq \frac{4}{c^2}\bigl\|JJ^{-1}(Jx_n - \lambda_n Ax_n) - JJ^{-1}(Jx_n)\bigr\|\|\lambda_n Ax_n\| \\
&= \frac{4}{c^2}\|Jx_n - \lambda_n Ax_n - Jx_n\|\|\lambda_n Ax_n\| \\
&= \frac{4}{c^2}\|\lambda_n Ax_n\|^2 \\
&= \frac{4}{c^2}\lambda_n^2\|Ax_n\|^2 \\
&\leq \frac{4}{c^2}\lambda_n^2\|Ax_n - Aq\|^2.
\end{aligned} \tag{3.4}$$

Replacing (3.2) by (3.3) and (3.4), we get

$$\begin{aligned}
\phi(q, z_n) &\leq \phi(q, x_n) - 2\alpha\lambda_n\|Ax_n - Aq\|^2 + \frac{4}{c^2}\lambda_n^2\|Ax_n - Aq\|^2 \\
&= \phi(q, x_n) + 2\lambda_n\left(\frac{2}{c^2}\lambda_n - \alpha\right)\|Ax_n - Aq\|^2 \\
&\leq \phi(q, x_n).
\end{aligned} \tag{3.5}$$

From Lemma 2.11, we know that

$$\phi(q,u_n) = \phi(q,T_{r_n}z_n) \le \phi(q,z_n) \le \phi(q,x_n). \tag{3.6}$$

Since S is a total quasi-ϕ-asymptotically nonexpansive multi-valued mapping and $w_n \in S^n x_n$, it follows that

$$\begin{aligned}
\phi(q,y_n) &= \phi\big(q,J^{-1}(\alpha_n Jx_n + \beta_n Jw_n + \gamma_n Ju_n)\big) \\
&= \|q\|^2 - 2\langle q,\alpha_n Jx_n + \beta_n Jw_n + \gamma_n Ju_n\rangle + \|\alpha_n Jx_n + \beta_n Jw_n + \gamma_n Ju_n\|^2 \\
&\le \alpha_n\phi(q,x_n) + \beta_n\phi(q,w_n) + \gamma_n\phi(q,u_n) \\
&\le \alpha_n\phi(q,x_n) + \beta_n\phi(q,x_n) + \beta_n\nu_n\psi\big(\phi(q,x_n)\big) + \beta_n\mu_n + \gamma_n\phi(q,u_n) \\
&\le \alpha_n\phi(q,x_n) + \beta_n\phi(q,x_n) + \nu_n\sup_{q\in F}\psi\big(\phi(q,x_n)\big) + \mu_n + \gamma_n\phi(q,u_n) \\
&= \alpha_n\phi(q,x_n) + \beta_n\phi(q,x_n) + \gamma_n\phi(q,u_n) + K_n \\
&\le \alpha_n\phi(q,x_n) + \beta_n\phi(q,x_n) + \gamma_n\phi(q,T_{r_n}z_n) + K_n \\
&\le \alpha_n\phi(q,x_n) + \beta_n\phi(q,x_n) + \gamma_n\phi(q,z_n) + K_n \\
&\le \alpha_n\phi(q,x_n) + \beta_n\phi(q,x_n) + \gamma_n\phi(q,x_n) + K_n \\
&\le \phi(q,x_n) + K_n,
\end{aligned} \tag{3.7}$$

where $K_n = \nu_n \sup_{q\in F}\psi(\phi(q,x_n)) + \mu_n$.

This shows that $q \in C_{n+1}$, which implies that $F \subset C_{n+1}$. Hence $F \subset C_n$ for all $n \in \mathbb{N}$ and the sequence $\{x_n\}$ is well defined.

From the definition of C_{n+1} with $x_n = \Pi_{C_n}x_1$ and $x_{n+1} = \Pi_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, it follows that

$$\phi(x_n,x_1) \le \phi(x_{n+1},x_1), \quad \forall n \ge 1, \tag{3.8}$$

that is, $\{\phi(x_n,x_1)\}$ is nondecreasing. By Lemma 2.7, we get

$$\begin{aligned}
\phi(x_n,x_1) &= \phi(\Pi_{C_n}x_1,x_1) \\
&\le \phi(q,x_1) - \phi(q,x_n) \\
&\le \phi(q,x_1), \quad \forall q \in F.
\end{aligned} \tag{3.9}$$

This implies that $\{\phi(x_n,x_1)\}$ is bounded and so $\lim_{n\to\infty}\phi(x_n,x_1)$ exists. In particular, by (1.4), the sequence $\{(\|x_n\| - \|x_1\|)^2\}$ is bounded. This implies $\{x_n\}$ is also bounded. So, we have $\{u_n\}$, $\{z_n\}$ and $\{y_n\}$ are also bounded.

Since $x_m = \Pi_{C_m}x_1 \in C_m \subset C_n$ for all $m,n \ge 1$ with $m > n$, by Lemma 2.7, we have

$$\begin{aligned}
\phi(x_m,x_n) &= \phi(x_m,\Pi_{C_n}x_1) \\
&\le \phi(x_m,x_1) - \phi(\Pi_{C_n}x_1,x_1) \\
&= \phi(x_m,x_1) - \phi(x_n,x_1),
\end{aligned}$$

taking $m,n \to \infty$, we have $\phi(x_m,x_n) \to 0$. This implies that $\{x_n\}$ is a Cauchy sequence. From Lemma 2.5, it follows that $\|x_n - x_m\| \to 0$ and $\{x_n\}$ is a Cauchy sequence. By the completeness of E and the closedness of C, we can assume that there exists $p \in C$ such that

$$\lim_{n\to\infty} x_n = p, \tag{3.10}$$

we also get that

$$\lim_{n\to\infty} K_n = \lim_{n\to\infty} \nu_n \sup_{q\in F} \psi\bigl(\phi(q,x_n)\bigr) + \mu_n = 0. \tag{3.11}$$

Next, we show that $p \in F := F(S) \cap A^{-1}0 \cap EP(f)$.

(a) We show that $p \in F(S)$. By the definition of $\Pi_{C_n}x_1$, we have

$$\begin{aligned}
\phi(x_{n+1},x_n) &= \phi(x_{n+1},\Pi_{C_n}x_1) \\
&\le \phi(x_{n+1},x_1) - \phi(\Pi_{C_n}x_1,x_1) \\
&= \phi(x_{n+1},x_1) - \phi(x_n,x_1).
\end{aligned}$$

Since $\lim_{n\to\infty}\phi(x_n,x_1)$ exists, we get

$$\lim_{n\to\infty} \phi(x_{n+1},x_n) = 0. \tag{3.12}$$

It follows from Lemma 2.5 that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.13}$$

From the definition of C_{n+1} and $x_{n+1} = \Pi_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, we have $\phi(x_{n+1},y_n) \le \phi(x_{n+1}, x_n) + K_n \to 0$ as $n \to \infty$. By Lemma 2.5, it follows that

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.14}$$

From $\lim_{n\to\infty} x_n = p$, we also have

$$\lim_{n\to\infty} y_n = p. \tag{3.15}$$

By using the triangle inequality, we get $\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. Since J is uniformly norm-to-norm continuous, we obtain $\|Jx_n - Jy_n\| \to 0$ as $n \to \infty$. On the other hand, we note that

$$\begin{aligned}
\phi(q,x_n) - \phi(q,y_n) &= \|x_n\|^2 - \|y_n\|^2 - 2\langle q, Jx_n - Jy_n\rangle \\
&\le \|x_n - y_n\|\bigl(\|x_n + y_n\|\bigr) + 2\|q\|\|Jx_n - Jy_n\|.
\end{aligned}$$

In view of $\|x_n - y_n\| \to 0$ and $\|Jx_n - Jy_n\| \to 0$ as $n \to \infty$, we obtain that

$$\phi(q,x_n) - \phi(q,y_n) \to 0 \quad \text{as } n \to \infty. \tag{3.16}$$

From Lemma 2.17, we have

$$\begin{aligned}
\phi(q,y_n) &= \phi\bigl(q,J^{-1}[\alpha_n Jx_n + \beta_n Jw_n + \gamma_n Ju_n]\bigr) \\
&\le \|q\|^2 - 2\langle q,\alpha_n Jx_n + \beta_n Jw_n + \gamma_n Ju_n\rangle + \|\alpha_n Jx_n + \beta_n Jw_n + \gamma_n Ju_n\|^2 \\
&\quad - \alpha_n\beta_n g\bigl(\|Jx_n - Jw_n\|\bigr) \\
&= \alpha_n\phi(q,x_n) + \beta_n\phi(q,w_n) + \gamma_n\phi(q,u_n) - \alpha_n\beta_n g\bigl(\|Jx_n - Jw_n\|\bigr) \\
&\le \phi(q,x_n) + K_n - \alpha_n\beta_n g\bigl(\|Jx_n - Jw_n\|\bigr).
\end{aligned} \tag{3.17}$$

It follows from $\liminf_{n\to\infty}\alpha_n\beta_n > 0$, (3.16), (3.11) and the property of g that

$$\lim_{n\to\infty}\|Jx_n - Jw_n\| = 0.$$

Since J^{-1} is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty}\|x_n - w_n\| = 0. \tag{3.18}$$

From (3.10) it follows that

$$\lim_{n\to\infty}\|w_n - p\| = 0. \tag{3.19}$$

For $w_n \in S^n x_n$, generate a sequence $\{s_n\}$ by

$$\begin{aligned}
&s_2 \in Sw_1 \subset S^2 x_1, \\
&s_3 \in Sw_2 \subset S^3 x_2, \\
&s_4 \in Sw_3 \subset S^4 x_3, \\
&\vdots \\
&s_{n+1} \in Sw_n \subset S^{n+1} x_n.
\end{aligned}$$

On the other hand, we have $\|s_{n+1} - p\| \le \|s_{n+1} - w_n\| + \|w_n - p\|$. Since S is uniformly asymptotically regular, it follows that

$$\lim_{n\to\infty}\|s_{n+1} - p\| = 0, \tag{3.20}$$

we have

$$\lim_{n\to\infty}\bigl\|S^{n+1}x_n - p\bigr\| = 0, \tag{3.21}$$

that is, $SS^n x_n \to p$ as $n \to \infty$. From the closedness of S, we have $p \in F(S)$.

(b) We show that $p \in A^{-1}0$.

From the definition of C_{n+1} and $x_{n+1} = \Pi_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, we have $\phi(x_{n+1},z_n) \le \phi(x_{n+1},x_n) + K_n \to 0$ as $n \to \infty$. By Lemma 2.5, it follows that $\lim_{n\to\infty}\|x_{n+1} - z_n\| = 0$. By the triangle inequality, we get $\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \to 0$ as $n \to \infty$. From $\lim_{n\to\infty} \|z_n - x_n\| = 0$ and from (3.10), it follows that

$$\lim_{n\to\infty} z_n = p. \tag{3.22}$$

Since J is uniformly norm-to-norm continuous, we also have

$$\lim_{n\to\infty} \|Jz_n - Jx_n\| = 0. \tag{3.23}$$

Hence, from the definition of the sequence $\{z_n\}$, it follows that

$$\|Ax_n\| = \frac{\|Jz_n - Jx_n\|}{\lambda_n}. \tag{3.24}$$

From (3.23) and the definition of the sequence $\{\lambda_n\}$, we have

$$\lim_{n\to\infty} \|Ax_n\| = 0, \tag{3.25}$$

that is,

$$\lim_{n\to\infty} Ax_n = 0. \tag{3.26}$$

Since A is Lipschitz continuous, it follows from (3.10) that

$$Ap = 0. \tag{3.27}$$

Again, since A is Lipschitz continuous and monotone so it is maximal monotone. It follows from Lemma 2.18 that $p \in A^{-1}0$.

(c) We show that $p \in EP(f)$.

From $x_n, y_n \to 0$ and $K_n \to 0$ as $n \to \infty$ and applying (3.7) for any $q \in F$, we get $\lim_{n\to\infty} \phi(q, u_n) \to \phi(q, p)$, it follows that

$$\begin{aligned}
\phi(u_n, x_n) &= \phi(T_{r_n}, x_n) \\
&\leq \phi(q, x_n) - \phi(q, T_{r_n} x_n) \\
&= \phi(q, x_n) - \phi(q, u_n).
\end{aligned}$$

Taking limit as $n \to \infty$ on the both sides of the inequality, we have $\lim_{n\to\infty} \phi(u_n, x_n) = 0$. From Lemma 2.5, it follows that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0 \tag{3.28}$$

and

$$\lim_{n\to\infty} u_n = p. \tag{3.29}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty} \|Ju_n - Jz_n\| = 0.$$

Since $r_n > 0$ for all $n \geq 1$, we have $\frac{\|Ju_n - Jz_n\|}{r_n} \to 0$ as $n \to \infty$ and

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle \geq 0, \quad \forall y \in C.$$

From (A2), the fact that

$$\begin{aligned}\|y - u_n\| \frac{\|Ju_n - Jz_n\|}{r_n} &\geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle \\ &\geq -f(u_n, y) \\ &\geq f(y, u_n), \quad \forall y \in C,\end{aligned}$$

taking the limit as $n \to \infty$ in the above inequality and from the fact that $u_n \to p$ as $n \to \infty$, it follows that $f(y, p) \leq 0$ for all $y \in C$. For any $0 < t < 1$, define $y_t = ty + (1 - t)p$. Then $y_t \in C$, which implies that $f(y_t, p) \leq 0$. Thus it follows from (A1) that

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1 - t)\theta(y_t, p) \leq tf(y_t, y),$$

and so $f(y_t, y) \geq 0$. From (A3) we have $f(p, y) \geq 0$ for all $y \in C$ and so $p \in EP(f)$. Hence, by (a), (b) and (c), that is, $p \in F(S) \cap A^{-1}0 \cap EP(f)$.

Finally, we show that $p = \Pi_F x_1$. From $x_n = \Pi_{C_n} x_1$, we have $\langle Jx_1 - Jx_n, x_n - z\rangle \geq 0$ for all $z \in C_n$. Since $F \subset C_n$, we also have

$$\langle Jx_1 - Jx_n, x_n - \hat{p}\rangle \geq 0, \quad \forall \hat{p} \in F.$$

Taking limit $n \to \infty$, we obtain

$$\langle Jx_1 - Jp, p - \hat{p}\rangle \geq 0, \quad \forall \hat{p} \in F.$$

By Lemma 2.6, we can conclude that $p = \Pi_F x_1$ and $x_n \to p$ as $n \to \infty$. The proof is completed. □

Next, we define $z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n)$ and assume that $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(A, C) \neq \emptyset$. We can prove the strong convergence theorem for finding the set of solutions of the variational inequality problem in a real uniformly smooth and 2-uniformly convex Banach space.

Remark 3.2 (Qin *et al.* [7]) Let Π_C be the generalized projection from a smooth strictly convex and reflexive Banach space E onto a nonempty closed convex subset C of E. Then Π_C is a closed quasi-ϕ-nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

Corollary 3.3 *Let C be a nonempty closed and convex subset of a uniformly smooth and 2-uniformly convex Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions* (A1)-(A4), *and let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(A, C) \neq \emptyset$. Let $S : C \to 2^C$ be a closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous* *function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\psi(0) = 0$. *Assume that* S *is uniformly asymptotically regular on* C *with* $\mu_1 = 0$ *and* $F := F(S) \cap EP(f) \cap VI(A, C) \neq \emptyset$. *For arbitrary* $x_1 \in C$, $C_1 = C$, *generate a sequence* $\{x_n\}$ *by*

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ u_n = T_{r_n} x_n, \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n Jw_n + \gamma_n Ju_n), \quad w_n \in S^n x_n, \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, z_n) \leq \phi(v, x_n) + K_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \end{cases} \tag{3.30}$$

where $K_n = \nu_n \sup_{q \in F} \psi(\phi(q, x_n)) + \mu_n$. *Assume that the control sequences* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ *and* $\{r_n\}$ *satisfy the following conditions*:

1. $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0, 1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$, $\liminf_{n\to\infty} \alpha_n \beta_n > 0$,
2. $\{\lambda_n\} \subset [a, b]$ *for some* a, b *with* $0 < a < b < \frac{c^2\alpha}{2}$ *and* $\frac{1}{c}$ *is the* 2-*uniformly convex constant of* E,
3. $\{r_n\} \subset [d, \infty)$ *for some* $d > 0$,

then $\{x_n\}$ *converges strongly to* $\Pi_F x_1$.

Proof For $q \in F$ and Π_C is quasi-ϕ-nonexpansive mapping, we have

$$\phi(q, z_n) = \phi\big(q, \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)\big) \leq \phi\big(q, J^{-1}(Jx_n - \lambda_n Ax_n)\big).$$

So, we can show that $p \in VI(A, C)$.

Define $B \subset E \times E^*$ by Theorem 2.14, B is maximal monotone and $B^{-1}0 = VI(A, C)$. Let $(z, w) \in G(B)$. Since $w \in Bz = Az + N_C(z)$, we get $w - Az \in N_C(z)$.

From $z_n \in C$, we have

$$\langle z - z_n, w - Az \rangle \geq 0. \tag{3.31}$$

On the other hand, since $z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$. Then, by Lemma 2.6, we have

$$\big\langle z - z_n, Jz_n - (Jx_n - \lambda_n Ax_n)\big\rangle \geq 0,$$

and thus

$$\left\langle z - z_n, \frac{Jx_n - Jz_n}{\lambda_n} - Ax_n \right\rangle \leq 0. \tag{3.32}$$

It follows from (3.31) and (3.32) that

$$\begin{aligned} \langle z - z_n, w \rangle &\geq \langle z - z_n, Az \rangle \\ &\geq \langle z - z_n, Az \rangle + \left\langle z - z_n, \frac{Jx_n - Jz_n}{\lambda_n} - Ax_n \right\rangle \\ &= \langle z - z_n, Az - Ax_n \rangle + \left\langle z - z_n, \frac{Jx_n - Jz_n}{\lambda_n} \right\rangle \end{aligned}$$

$$
\begin{aligned}
&= \langle z - z_n, Az - Az_n \rangle + \langle z - z_n, Az_n - Ax_n \rangle + \left\langle z - z_n, \frac{Jx_n - Jz_n}{\lambda_n} \right\rangle \\
&\geq -\|z - z_n\| \frac{\|z_n - x_n\|}{\alpha} - \|z - z_n\| \frac{\|Jx_n - Jz_n\|}{a} \\
&\geq -M \left(\frac{\|z_n - x_n\|}{\alpha} + \frac{\|Jx_n - Jz_n\|}{a} \right),
\end{aligned}
$$

where $M = \sup_{n\geq 1} \|z - z_n\|$. From $\|x_n - z_n\| \to 0$ as $n \to \infty$ and (3.23), taking $\lim_{n\to\infty}$ on the both sides of the equality above, we have $\langle z - p, w \rangle \geq 0$. By the maximality of B, we have $p \in B^{-1}0$, that is, $p \in VI(A, C)$. From Theorem 3.1, we have $p \in F(S) \cap EP(f) \cap VI(A, C)$. The proof is completed. □

Let A be a strongly monotone mapping with constant k, Lipschitz with constant $L > 0$, that is,

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in D(A),$$

which implies that

$$\frac{1}{L}\|Ax - Ay\| \leq \|x - y\|, \quad \forall x, y \in D(A).$$

It follows that

$$\langle Ax - Ay, x - y \rangle \geq k\|x - y\|^2 \geq \frac{k}{L}\|Ax - Ay\|^2$$

hence A is α-inverse-strongly monotone with $\alpha = \frac{k}{L}$. Therefore, we have the following corollaries.

Corollary 3.4 *Let C be a nonempty closed and convex subset of a uniformly smooth and 2-uniformly convex Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions* (A1)-(A4), *and let $A : E \to E^*$ be a strongly monotone mapping with constant k, Lipschitz with constant $L > 0$. Let $S : C \to 2^C$ be a closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$. Assume that S is uniformly asymptotically regular on C with $\mu_1 = 0$ and $F := F(S) \cap EP(f) \cap A^{-1}0 \neq \emptyset$. For arbitrary $x_1 \in C$, $C_1 = C$, a sequence $\{x_n\}$ is generated by*

$$
\begin{cases}
z_n = J^{-1}(Jx_n - \lambda_n Ax_n), \\
u_n = T_{r_n} z_n, \\
y_n = J^{-1}(\alpha_n Jx_n + \beta_n Jw_n + \gamma_n Ju_n), \quad w_n \in S^n x_n, \\
C_{n+1} = \{\nu \in C_n : \phi(\nu, y_n) \leq \phi(\nu, z_n) \leq \phi(\nu, x_n) + K_n\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N},
\end{cases}
\tag{3.33}
$$

where $K_n = \nu_n \sup_{q\in F} \psi(\phi(q, x_n)) + \mu_n$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the following conditions:

1. $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$, $\liminf_{n\to\infty} \alpha_n\beta_n > 0$,
2. $\{\lambda_n\} \subset [a,b]$ *for some* a, b *with* $0 < a < b < \frac{c^2k}{2L}$ *and* $\frac{1}{c}$ *is the* 2*-uniformly convex constant of* E,
3. $\{r_n\} \subset [d,\infty)$ *for some* $d > 0$,

then $\{x_n\}$ *converges strongly to* $\Pi_F x_1$.

Corollary 3.5 *Let* C *be a nonempty closed and convex subset of a uniformly smooth and* 2*-uniformly convex Banach space* E. *Let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying conditions* (A1)-(A4), *and let* $A : C \to E^*$ *be a strongly monotone mapping with constant* k, *Lipschitz with constant* $L > 0$ *satisfying* $\|Ay\| \le \|Ay - Au\|$ *for all* $y \in C$ *and* $u \in VI(A,C) \neq \emptyset$. *Let* $S : C \to 2^C$ *be a closed and total quasi-*ϕ*-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences* ν_n, μ_n *with* $\nu_n \to 0$, $\mu_n \to 0$ *as* $n \to \infty$ *and a strictly increasing continuous function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\psi(0) = 0$. *Assume that* S *is uniformly asymptotically regular on* C *with* $\mu_1 = 0$ *and* $F := F(S) \cap EP(f) \cap VI(A,C) \neq \emptyset$. *For arbitrary* $x_1 \in C$, $C_1 = C$, *generate a sequence* $\{x_n\}$ *by*

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ u_n = T_{r_n} x_n, \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n Jw_n + \gamma_n Ju_n), \quad w_n \in S^n x_n, \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, z_n) \le \phi(v, x_n) + K_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \end{cases} \tag{3.34}$$

where $K_n = \nu_n \sup_{q\in F} \psi(\phi(q, x_n)) + \mu_n$. *Assume that the control sequences* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ *and* $\{r_n\}$ *satisfy the following conditions*:

1. $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$, $\liminf_{n\to\infty} \alpha_n\beta_n > 0$,
2. $\{\lambda_n\} \subset [a,b]$ *for some* a, b *with* $0 < a < b < \frac{c^2k}{2L}$ *and* $\frac{1}{c}$ *is the* 2*-uniformly convex constant of* E,
3. $\{r_n\} \subset [d,\infty)$ *for some* $d > 0$,

then $\{x_n\}$ *converges strongly to* $\Pi_F x_1$.

Let F be a Fréchet differentiable functional in a Banach space E and ∇F be the gradient of F, denote $(\nabla F)^{-1}0 = \{x \in E : F(x) = \min_{y\in E} F(y)\}$. Baillon and Haddad [34] proved the following lemma.

Lemma 3.6 (Baillon and Haddad [34]) *Let* E *be a Banach space. Let* F *be a continuously Fréchet differentiable convex functional on* E *and* ∇F *be the gradient of* F. *If* ∇F *is* $\frac{1}{\alpha}$*-Lipschitz continuous, then* ∇F *is an* α*-inverse strongly monotone mapping.*

We replace A in Theorem 3.1 by ∇F, then we can obtain the following corollary.

Corollary 3.7 *Let* C *be a nonempty closed and convex subset of a uniformly smooth and* 2*-uniformly convex Banach space* E. *Let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying conditions* (A1)-(A4). *Let* F *be a continuously Fréchet differentiable convex functional* *on E and ∇F be $\frac{1}{\alpha}$-Lipschitz continuous. Let $S : C \to 2^C$ be a closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$. Assume that S is uniformly asymptotically regular on C with $\mu_1 = 0$ and $F := F(S) \cap F(T) \cap EP(f) \cap A^{-1}0 \neq \emptyset$. For an initial point $x_1 \in E$, $C_1 = C$, define the sequence $\{x_n\}$ by*

$$\begin{cases} z_n = J^{-1}(Jx_n - \lambda_n \nabla F x_n), \\ u_n = T_{r_n} x_n, \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n Jw_n + \gamma_n Ju_n), \quad w_n \in S^n x_n, \\ C_{n+1} = \{\nu \in C_n : \phi(\nu, y_n) \le \phi(\nu, z_n) \le \phi(\nu, x_n) + K_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \end{cases} \tag{3.35}$$

where $\mu_n = \sup\{\mu_n^S, \mu_n^T\}$, $\nu_n = \sup\{\nu_n^S, \nu_n^T\}$, $\psi = \sup\{\psi^S, \psi^T\}$, $k_n = \nu_n \sup_{q \in \mathcal{F}} \psi(\phi(q, x_n)) + \mu_n$.

Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the following conditions:

1. *$\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\liminf_{n\to\infty} \alpha_n \beta_n > 0$ and $\liminf_{n\to\infty} \alpha_n \gamma_n > 0$,*
2. *$\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < \frac{c^2\alpha}{2}$ and the 2-uniformly convex constant $\frac{1}{c}$ of E,*
3. *$\{r_n\} \subset [d,\infty)$ for some $d > 0$,*

then $\{x_n\}$ converges strongly to $\Pi_F x_1$.

Competing interests
The author declares that they have no competing interests.

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