

Research Article

The Least Squares Hermitian (Anti)reflexive Solution with the Least Norm to Matrix Equation $AXB = C$

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For a given generalized reflection matrix J , that is, $J^H = J$, $J^2 = I$, where J^H is the conjugate transpose matrix of J , a matrix $A \in C^{n \times n}$ is called a Hermitian (anti)reflexive matrix with respect to J if $A^H = A$ and $A = \pm JAJ$. By using the Kronecker product, we derive the explicit expression of least squares Hermitian (anti)reflexive solution with the least norm to matrix equation $AXB = C$ over complex field.

1. Introduction

Throughout this paper, we denote the set of all $m \times n$ matrices over the real field and complex field by $R^{m \times n}$ and $C^{m \times n}$, respectively. Let the symbols $HC^{n \times n}$, $SR^{n \times n}$, $ASR^{n \times n}$, I_n , and 0 stand for the set of all $n \times n$ complex Hermitian matrices, the set of all $n \times n$ real symmetric matrices, the set of all $n \times n$ real skew symmetric matrices, the $n \times n$ identity matrix, and the zero matrix with appropriate size, respectively. We also denote the transpose, the conjugate transpose, the Moore-Penrose inverse, and the Frobenius norm of A by A^T , A^H , A^\dagger , and $\|A\|$, respectively. Let $A \in C^{m \times n}$; we define $\text{vec}(A) = (a_1^T, a_2^T, \dots, a_n^T)^T$, where a_i is the i -column of A , $i = 1, 2, \dots, n$.

In this paper, we consider the Hermitian (anti)reflexive solution problem of matrix equation $AXB = C$. Let us first recall the definition of those two kinds of matrices.

Definition 1. Given a generalized reflection matrix J , that is, $J^H = J$, $J^2 = I$, a matrix $A \in C^{n \times n}$ is called a Hermitian reflexive matrix with respect to J if $A^H = A$ and $A = JAJ$. The set of all $n \times n$ Hermitian reflexive matrices with respect to J is denoted by $HC_r^{n \times n}(J)$.

Definition 2. Given a generalized reflection matrix J , that is, $J^H = J$, $J^2 = I$, a matrix $A \in C^{n \times n}$ is called a Hermitian anti-reflexive matrix with respect to J if $A^H = A$ and $A = -JAJ$. The set of all $n \times n$ Hermitian antireflexive matrices with respect to J is denoted by $HC_a^{n \times n}(J)$.

As an important kind of matrices, the reflexive matrices with different constraints were studied by many authors in recent years in literatures and references therein [1–29]. For instance, Zhao et al. [1] derived the general Hermitian reflexive solution to $AX = B$, while Yu and Shen [2] considered a more general case, that is, the Hermitian $\{J, k + 1\}$ -(anti)-reflexive solution to $AX = B$. As an extension of matrix equation $AX = B$, the matrix system $AX = B$, $XC = D$ was considered by Zhou and Yang [3]. They gave the necessary and sufficient conditions for the existence of a Hermitian reflexive solution to the above matrix system. As we can see, many important associated works were widely studied. But finding the least squares Hermitian (anti)reflexive solution with the least norm to matrix equation $AXB = C$ is still a problem. Therefore, we in this paper will fill this gap.

We consider the following two problems.

Problem 3. Given matrices $A \in C^{m \times n}$, $B \in C^{n \times k}$, and $C \in C^{m \times k}$, let

$$S = \left\{ X \mid X \in HC_r^{n \times n}(J), \|AXB - C\| \right. \\ \left. = \min_{\widehat{X} \in HC_r^{n \times n}(J)} \|A\widehat{X}B - C\| \right\}. \quad (1)$$

Find $X_0 \in S$ such that $\|X_0\| = \min_{X \in S} \|X\|$.

Problem 4. Given matrices $A \in C^{m \times n}$, $B \in C^{n \times k}$, and $C \in C^{m \times k}$, let

$$S_A = \left\{ X \mid X \in HC_a^{n \times n}(J), \|AXB - C\| \right. \\ \left. = \min_{\widehat{X} \in HC_a^{n \times n}(J)} \|A\widehat{X}B - C\| \right\}. \quad (2)$$

Find $X_0 \in S_A$ such that $\|X_0\| = \min_{X \in S_A} \|X\|$.

Now, we start with some useful and fundamental results in the following.

Lemma 5. *The least squares solution to the matrix equation $Ax = b$ with $A \in C^{m \times n}$ and $b \in C^m$ is given by $x = A^\dagger b + (I - A^\dagger A)y$, where $y \in C^n$ is an arbitrary vector. And the least squares solution with the least norm is $x = A^\dagger b$.*

Kronecker product method plays an important role in this paper. We use this method to get some important vectorizing formulas.

For any $A \in C^{m \times n}$, $A = A_1 + A_2i$, where $A_1, A_2 \in R^{m \times n}$. The real representation matrix of $A = A_1 + A_2i \in C^{m \times n}$ is given by $f(A) = \begin{bmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{bmatrix} \in C^{2m \times 2n}$, which has the following properties:

$$(1) f(A + B) = f(A) + f(B).$$

$$(2) f(AB) = f(A)f(B).$$

(3) For $A = A_1 + A_2i \in C^{m \times n}$, we define $g(A) = (A_1, A_2)$. Then

$$\|\text{vec}(A)\| = \|\text{vec}(g(A))\| = \left\| \begin{bmatrix} \text{vec}(A_1) \\ \text{vec}(A_2) \end{bmatrix} \right\|, \quad (3)$$

$$g(AB) = g(A)f(B).$$

Analogously to the method of Yuan et al. (see [30]), we can get a fundamental result as below.

Lemma 6. *Let A , B , and C be complex matrices with appropriate sizes; then $\text{vec}(g(ABC)) = f(C^H \otimes \bar{A}) \text{vec}(g(B))$.*

Proof. We denote $A = A_1 + A_2i$, $B = B_1 + B_2i$, and $C = C_1 + C_2i$. $g(ABC) = g(A)f(BC) = g(A)f(B)f(C)$. Then

$$g(A)f(B)f(C) = [A_1, A_2] \begin{bmatrix} B_1 & B_2 \\ -B_2 & B_1 \end{bmatrix} \begin{bmatrix} C_1 & C_2 \\ -C_2 & C_1 \end{bmatrix} \\ = [A_1B_1C_1 - A_2B_2C_1 - A_1B_2C_2 \\ -A_2B_1C_2, A_1B_1C_2 - A_2B_2C_2 + A_1B_2C_1 \\ + A_2B_1C_1]. \quad (4)$$

By the rule of

$$\text{vec}(XYZ) = (Z^T \otimes X) \text{vec}(Y), \quad (5)$$

we have

$$\text{vec}(g(ABC)) \\ = \begin{bmatrix} C_1^T \otimes A_1 - C_2^T \otimes A_2 & -C_2^T \otimes A_1 - C_1^T \otimes A_2 \\ C_2^T \otimes A_1 + C_1^T \otimes A_2 & C_1^T \otimes A_1 - C_2^T \otimes A_2 \end{bmatrix} \begin{bmatrix} \text{vec}(B_1) \\ \text{vec}(B_2) \end{bmatrix} \\ = f(C^H \otimes \bar{A}) \begin{bmatrix} \text{vec}(B_1) \\ \text{vec}(B_2) \end{bmatrix} = f(C^H \otimes \bar{A}) \text{vec}(g(B)). \quad (6)$$

□

The structure of Hermitian (anti)reflexive matrix promises the following decompositions, which would simplify our problems.

Lemma 7. *Let J be the $n \times n$ generalized reflection matrix. Then there exists an $n \times n$ unitary matrix U such that*

$$J = U \begin{bmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{bmatrix} U^H. \quad (7)$$

Lemma 8 (see [18]). *(1) Let $A \in C^{n \times n}$ and let the spectral decomposition of the $n \times n$ reflection matrix J be given as (7). Then $A \in HC_r^{n \times n}(J)$ if and only if*

$$A = U \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} U^H, \quad (8)$$

where $F_1 \in HC^{r \times r}$ and $F_2 \in HC^{(n-r) \times (n-r)}$.

(2) See [4]. Let $A \in C^{n \times n}$ and let the spectral decomposition of the $n \times n$ reflection matrix J be given as (7). Then $A \in HC_a^{n \times n}(J)$ if and only if

$$A = U \begin{bmatrix} 0 & F \\ F^H & 0 \end{bmatrix} U^H, \quad (9)$$

where $F \in C^{r \times (n-r)}$.

2. The Solution of Problem 3

In this section, we provide the least squares Hermitian reflexive solution with the least norm to matrix equation $AXB = C$. There are the vectorizing formulas of Hermitian matrix, which are very useful in what comes below.

Definition 9. Let $X = (x_{ij}) \in R^{n \times n}$, we define

$$\text{vec}_S(X) = (x_1, x_2, \dots, x_{n-1}, x_n)^T \in R^{n(n+1)/2}, \quad (10)$$

where $x_1 = (x_{11}, \sqrt{2}x_{21}, \dots, \sqrt{2}x_{n1})$, $x_2 = (x_{22}, \sqrt{2}x_{32}, \dots, \sqrt{2}x_{n2})$, \dots , $x_{n-1} = (x_{(n-1)(n-1)}, \sqrt{2}x_{n(n-1)})$, and $x_n = x_{nn}$.

Definition 10. Let $X = (x_{ij}) \in R^{n \times n}$, we define

$$\text{vec}_A(X) = \sqrt{2}(x_1, x_2, \dots, x_{n-2}, x_{n-1})^T \in R^{n(n-1)/2}, \quad (11)$$

where $x_1 = (x_{21}, x_{31}, \dots, x_{n1})$, $x_2 = (x_{32}, x_{42}, \dots, x_{n2})$, \dots , $x_{n-2} = (x_{(n-1)(n-2)}, x_{n(n-2)})$, and $x_{n-1} = x_{n(n-1)}$.

Lemma 11 (see [30]). Suppose that $X \in R^{n \times n}$. Then $X \in SR^{n \times n}$ if and only if $\text{vec}(X) = K_S \text{vec}_S(X)$, where $\text{vec}_S(X)$ is given by (10), where

$$K_S = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2}e_1 & e_2 & e_3 & \cdots & e_{n-1} & e_n & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & e_1 & 0 & \cdots & 0 & 0 & \sqrt{2}e_2 & e_3 & \cdots & e_{n-1} & e_n & \cdots & 0 & 0 & 0 \\ 0 & 0 & e_1 & \cdots & 0 & 0 & 0 & e_2 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e_1 & 0 & 0 & 0 & \cdots & e_2 & 0 & \cdots & \sqrt{2}e_{n-1} & e_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & e_1 & 0 & 0 & \cdots & 0 & e_2 & \cdots & 0 & e_{n-1} & \sqrt{2}e_n \end{bmatrix}. \quad (12)$$

Obviously, e_i is the i -column of I_n , $K_S \in R^{n^2 \times n(n+1)/2}$, and $K_S^T K_S = I_{n(n+1)/2}$.

Lemma 12. Suppose that $X \in R^{n \times n}$. Then $X \in ASR^{n \times n}$ if and only if $\text{vec}(X) = K_A \text{vec}_A(X)$, where $\text{vec}_A(X)$ is given by (11), where

$$K_A = \frac{1}{\sqrt{2}} \begin{bmatrix} e_2 & e_3 & \cdots & e_{n-1} & e_n & 0 & \cdots & 0 & 0 & \cdots & 0 \\ -e_1 & 0 & \cdots & 0 & 0 & e_3 & \cdots & e_{n-1} & e_n & \cdots & 0 \\ 0 & -e_1 & \cdots & 0 & 0 & -e_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & 0 & 0 & & 0 \\ 0 & 0 & \cdots & -e_1 & 0 & 0 & \cdots & -e_2 & 0 & \cdots & e_n \\ 0 & 0 & \cdots & 0 & -e_1 & 0 & \cdots & 0 & -e_2 & \cdots & -e_{n-1} \end{bmatrix}. \quad (13)$$

Obviously, e_i is the i -column of I_n , $K_A \in R^{n^2 \times n(n-1)/2}$, and $K_A^T K_A = I_{n(n-1)/2}$.

Let $X = X_1 + X_2 i \in HC^{n \times n}$, $X_1 \in R^{n \times n}$, $X_2 \in R^{n \times n}$. Then $X_1 = X_1^T$ and $X_2 = -X_2^T$. By Lemmas 11 and 12, one has

$$\begin{bmatrix} \text{vec}(X_1) \\ \text{vec}(X_2) \end{bmatrix} = \begin{bmatrix} K_S & 0 \\ 0 & K_A \end{bmatrix} \begin{bmatrix} \text{vec}_S(X_1) \\ \text{vec}_A(X_2) \end{bmatrix}. \quad (14)$$

If $X \in HC_r^{n \times n}(J)$, then, by Lemma 8, there exists a unitary matrix U such that

$$X = U \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix} U^H, \quad (15)$$

where $Y = Y_1 + Y_2 i \in HC^{r \times r}$, and $Z = Z_1 + Z_2 i \in HC^{(n-r) \times (n-r)}$.

Set

$$\begin{aligned} AU &= [D, E], \\ U^H B &= \begin{bmatrix} F \\ G \end{bmatrix}, \\ M &= [f(F^H \otimes \bar{D}), f(G^H \otimes \bar{E})] \end{aligned} \quad (16)$$

$$K = \begin{bmatrix} K_{S1} & 0 & 0 & 0 \\ 0 & K_{A1} & 0 & 0 \\ 0 & 0 & K_{S2} & 0 \\ 0 & 0 & 0 & K_{A2} \end{bmatrix}, \quad (17)$$

where U is given by (7). K_{S1} , K_{S2} are given by (12) with the sizes $r^2 \times r(r+1)/2$, $(n-r)^2 \times (n-r)(n-r+1)/2$, respectively. K_{A1} , K_{A2} are given by (13) with the sizes $r^2 \times r(r-1)/2$, $(n-r)^2 \times (n-r)(n-r-1)/2$, respectively. Then one has the following result.

Theorem 13. Let $A \in C^{m \times n}$, $B \in C^{n \times k}$, and $C \in C^{m \times k}$. Then the solution $X \in S$ to matrix equation $AXB = C$ is given by

$$X = U \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix} U^H, \quad (18)$$

$$\begin{aligned} \text{vec}(Y) &= [K_{S1}, K_{A1} i, 0, 0] [M^\dagger c + (I - M^\dagger M) y], \\ \text{vec}(Z) &= [0, 0, K_{S2}, K_{A2} i] [M^\dagger c + (I - M^\dagger M) y], \end{aligned} \quad (19)$$

where U is given by (7), $c = \text{vec}(g(C)) \in R^{2km}$, and $y \in R^{r^2 + (n-r)^2}$ is an arbitrary vector. And the solution $X \in S$ with

the least norm is given by

$$X = U \begin{bmatrix} Y_0 & 0 \\ 0 & Z_0 \end{bmatrix} U^H, \quad (20)$$

$$\text{vec}(Y_0) = [K_{S1}, K_{A1}i, 0, 0] M^\dagger c, \quad (21)$$

$$\text{vec}(Z_0) = [0, 0, K_{S2}, K_{A2}i] M^\dagger c.$$

Proof. Since

$$\begin{aligned} Y_1 &\in \text{SR}^{r \times r}, \\ Y_2 &\in \text{ASR}^{r \times r}, \\ Z_1 &\in \text{SR}^{(n-r) \times (n-r)}, \\ Z_2 &\in \text{ASR}^{(n-r) \times (n-r)}, \end{aligned} \quad (22)$$

it follows from (14) that

$$\begin{bmatrix} \text{vec}(Y_1) \\ \text{vec}(Y_2) \\ \text{vec}(Z_1) \\ \text{vec}(Z_2) \end{bmatrix} = \begin{bmatrix} K_{S1} & 0 & 0 & 0 \\ 0 & K_{A1} & 0 & 0 \\ 0 & 0 & K_{S2} & 0 \\ 0 & 0 & 0 & K_{A2} \end{bmatrix} \begin{bmatrix} \text{vec}_S(Y_1) \\ \text{vec}_A(Y_2) \\ \text{vec}_S(Z_1) \\ \text{vec}_A(Z_2) \end{bmatrix}. \quad (23)$$

Now, by Lemmas 6 and 11 and (23), we have

$$\begin{aligned} \|AXB - C\| &= \left\| AU \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix} U^H B - C \right\| = \|DYF \\ &+ EZG - C\| = \|g(DYF) + g(EZG) - g(C)\| \\ &= \|\text{vec}(g(DYF)) + \text{vec}(g(EZG)) - \text{vec}(g(C))\| \\ &= \left\| f(F^H \otimes \bar{D}) \begin{bmatrix} \text{vec}(Y_1) \\ \text{vec}(Y_2) \end{bmatrix} \right. \\ &+ \left. f(G^H \otimes \bar{E}) \begin{bmatrix} \text{vec}(Z_1) \\ \text{vec}(Z_2) \end{bmatrix} - \text{vec}(g(C)) \right\| \\ &= \left\| \begin{bmatrix} f(F^H \otimes \bar{D}), f(G^H \otimes \bar{E}) \end{bmatrix} \begin{bmatrix} \text{vec}(Y_1) \\ \text{vec}(Y_2) \\ \text{vec}(Z_1) \\ \text{vec}(Z_2) \end{bmatrix} \right. \\ &\left. - \text{vec}(g(C)) \right\| = \|Mx - c\|, \end{aligned} \quad (24)$$

where M is given by (17); $x = \begin{bmatrix} \text{vec}_S(Y_1) \\ \text{vec}_A(Y_2) \\ \text{vec}_S(Z_1) \\ \text{vec}_A(Z_2) \end{bmatrix}$. Then, by Lemma 5,

$$x = M^\dagger c + (I - M^\dagger M) y \quad (25)$$

is the required least squares solution, where $y \in R^{r^2+(n-r)^2}$ is an arbitrary vector. Recall that

$$\begin{bmatrix} \text{vec}(Y_1) \\ \text{vec}(Y_2) \\ \text{vec}(Z_1) \\ \text{vec}(Z_2) \end{bmatrix} = \begin{bmatrix} K_{S1} & 0 & 0 & 0 \\ 0 & K_{A1} & 0 & 0 \\ 0 & 0 & K_{S2} & 0 \\ 0 & 0 & 0 & K_{A2} \end{bmatrix} x. \quad (26)$$

Therefore

$$\begin{aligned} \text{vec}(Y) &= \text{vec}(Y_1) + \text{vec}(Y_2) i = [K_{S1} \ K_{A1}i \ 0 \ 0] x, \\ \text{vec}(Z) &= \text{vec}(Z_1) + \text{vec}(Z_2) i \\ &= [0 \ 0 \ K_{S2} \ K_{A2}i] x. \end{aligned} \quad (27)$$

Then the least squares Hermitian reflexive solution is given by (18). Obviously, the least squares Hermitian reflexive solution with the least norm is given by (20). \square

3. The Solution of Problem 4

In this section, we provide the least squares Hermitian antireflexive solution with the least norm to matrix equation $AXB = C$.

We first derive the relation between $\text{vec}(g(Y))$ and $\text{vec}(g(Y^H))$.

Lemma 14 (see [31]). *Let $Y \in R^{m \times n}$ be any matrix. Then*

$$\text{vec}(Y^T) = P_{(m,n)} \text{vec}(Y), \quad (28)$$

where

$$P_{(m,n)} = \begin{bmatrix} E_{11}^T & E_{12}^T & \cdots & E_{1n}^T \\ E_{21}^T & E_{22}^T & \cdots & E_{2n}^T \\ \vdots & \vdots & \ddots & \vdots \\ E_{m1}^T & E_{m2}^T & \cdots & E_{mn}^T \end{bmatrix} \in R^{mn \times mn}, \quad (29)$$

with E_{ij} , $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, being an $m \times n$ matrix with the element at position (i, j) being 1 and the others being 0. Moreover, $P_{(m,n)}$ is an orthogonal matrix, which is uniquely determined by the integers m, n .

Let $X \in HC_a^{n \times n}(J)$; then, by Lemma 8, there exists a unitary matrix U such that

$$X = U \begin{bmatrix} 0 & Y \\ Y^H & 0 \end{bmatrix} U^H, \quad (30)$$

where $Y \in C^{r \times (n-r)}$ and U is given by (7). Since $Y^H = Y_1^T - Y_2^T i$,

$$g(Y^H) = (Y_1^T, -Y_2^T). \quad (31)$$

By Lemma 14, one has

$$\begin{aligned} \text{vec}(g(Y^H)) &= \begin{bmatrix} \text{vec}(Y_1^T) \\ \text{vec}(-Y_2^T) \end{bmatrix} = \begin{bmatrix} P_{(r,n-r)} \text{vec}(Y_1) \\ -P_{(r,n-r)} \text{vec}(Y_2) \end{bmatrix} \\ &= \begin{bmatrix} P_{(r,n-r)} & 0 \\ 0 & -P_{(r,n-r)} \end{bmatrix} \text{vec}(g(Y)). \end{aligned} \quad (32)$$

Set

$$\begin{aligned} AU &= [D, E], \\ U^H B &= \begin{bmatrix} F \\ G \end{bmatrix}, \\ N &= f(F^H \otimes \bar{E}) \begin{bmatrix} P_{(r,n-r)} & 0 \\ 0 & -P_{(r,n-r)} \end{bmatrix} \\ &\quad + f(G^H \otimes \bar{D}), \end{aligned} \quad (33)$$

and $P_{(r,n-r)}$ is given by (29). Then one has the following results.

Theorem 15. Let $A \in C^{m \times n}$, $B \in C^{n \times k}$, and $C \in C^{m \times k}$. Then one has the following:

(1) The solution $X \in S_A$ to matrix equation $AXB = C$ is given by

$$X = U \begin{bmatrix} 0 & Y \\ Y^H & 0 \end{bmatrix} U^H, \quad (34)$$

where

$$\text{vec}(Y) = [I_{r(n-r)}, iI_{r(n-r)}] [N^\dagger c + (I - N^\dagger N) y], \quad (35)$$

U is given by (7), $c = \text{vec}(g(C)) \in R^{2km}$, and $y \in R^{2r(n-r)}$ is an arbitrary vector.

(2) The solution $X \in S_A$ with the least norm is given by

$$X = U \begin{bmatrix} 0 & Y_0 \\ Y_0^H & 0 \end{bmatrix} U^H, \quad (36)$$

where U is given by (7) and $\text{vec}(Y_0) = [I_{r(n-r)}, iI_{r(n-r)}] N^\dagger c$, $c = \text{vec}(g(C))$.

Proof. It follows from Lemma 6 that

$$\text{vec}(g(EY^H F)) = f(F^H \otimes \bar{E}) \text{vec}(g(Y^H)), \quad (37)$$

$$\text{vec}(g(DYG)) = f(G^H \otimes \bar{D}) \text{vec}(g(Y)). \quad (38)$$

Combining (32) and (37), we obtain

$$\begin{aligned} \text{vec}(g(EY^H F)) &= f(F^H \otimes \bar{E}) \text{vec}(g(Y^H)) \\ &= f(F^H \otimes \bar{E}) \begin{bmatrix} P_{(r,n-r)} & 0 \\ 0 & -P_{(r,n-r)} \end{bmatrix} \text{vec}(g(Y)). \end{aligned} \quad (39)$$

Then

$$\begin{aligned} \|AXB - C\| &= \left\| AU \begin{bmatrix} 0 & Y \\ Y^H & 0 \end{bmatrix} U^H B - C \right\| = \|EY^H F \\ &\quad + DY G - C\| = \|g(EY^H F) + g(DYG) - g(C)\| \\ &= \|\text{vec}(g(EY^H F)) + \text{vec}(g(DYG)) \\ &\quad - \text{vec}(g(C))\| \\ &= \left\| f(F^H \otimes \bar{E}) \begin{bmatrix} P_{(r,n-r)} & 0 \\ 0 & -P_{(r,n-r)} \end{bmatrix} \text{vec}(g(Y)) \right. \\ &\quad \left. + f(G^H \otimes \bar{D}) \text{vec}(g(Y)) - \text{vec}(g(C)) \right\| = \|Nx \\ &\quad - c\|, \end{aligned} \quad (40)$$

where $c = \text{vec}(g(C))$ and $x = \text{vec}(g(Y))$.

By Lemma 5, the least squares solution is given by

$$x = \begin{bmatrix} \text{vec}(Y_1) \\ \text{vec}(Y_2) \end{bmatrix} = N^\dagger c + (I - N^\dagger N) y, \quad (41)$$

where $y \in R^{2r(n-r)}$ is an arbitrary vector.

The least squares solution with least norm is $x_0 = N^\dagger c$.

So

$$\begin{aligned} \text{vec}(Y_1) &= [I_{r(n-r)}, 0] x, \\ \text{vec}(Y_2) &= [0, I_{r(n-r)}] x, \\ \text{vec}(Y_{10}) &= [I_{r(n-r)}, 0] x_0, \\ \text{vec}(Y_{20}) &= [0, I_{r(n-r)}] x_0. \end{aligned} \quad (42)$$

Now, it is easy to see that our required solutions are given by (34) and (36). \square

4. Conclusion

By exploiting the decomposition of Hermitian (anti)reflexive matrix and the Kronecker product, we derive the expression of the least squares Hermitian (anti)reflexive solution with the least norm to $AXB = C$. This method overcomes the difficulty of finding the structured least squares solution with the least norm but creates a large-scale linear system $\hat{A}x = \hat{b}$. Therefore, it fits well with the small-scale coefficient matrix, but, for the large-scale case, it may not work so well.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Both authors read and approved the final manuscript.

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