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Research Article

Threshold Dynamics of a Stochastic Chemostat Model with Two Nutrients and One Microorganism

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A new stochastic chemostat model with two substitutable nutrients and one microorganism is proposed and investigated. Firstly, for the corresponding deterministic model, the threshold for extinction and permanence of the microorganism is obtained by analyzing the stability of the equilibria. Then, for the stochastic model, the threshold of the stochastic chemostat for extinction and permanence of the microorganism is explored. Difference of the threshold of the deterministic model and the stochastic model shows that a large stochastic disturbance can affect the persistence of the microorganism and is harmful to the cultivation of the microorganism. To illustrate this phenomenon, we give some computer simulations with different intensity of stochastic noise disturbance.

1. Introduction

Chemostat is commonly used to describe the dynamics of a microbial population in a continuous bioreactor in which microorganisms grow on a substrate and has attracted great interest of many scholars [1–8], since it was first introduced by Monod [9]. A single simple species chemostat model with Michaelis-Menten-Monod functional response was proposed by [9] as follows:

$$\begin{aligned} \frac{\mathrm{d}S(t)}{dt} &= D\left(S^0 - S(t)\right) - \frac{r}{\delta}\frac{S(t)X(t)}{K+S(t)}, \\ \frac{\mathrm{d}X(t)}{dt} &= \frac{rS(t)X(t)}{K+S(t)} - DX(t), \end{aligned} \tag{1}$$

where $S(t)$ is the concentration of the nutrient, $X(t)$ is the concentration of the organism, D is the dilution (or washout) rate, r is the maximal growth rate, K is the Michaelis-Menten (or half-saturation) constant with units of concentration, and δ is a "yield" constant reflecting the conversion of nutrient to organism.

However, experimental results have indicated that the microorganisms depend on a variety of nutrition substances such as carbon, nitrogen, energy, growth factors, inorganic salts, and water. Then the model of microorganisms species growth in the chemostat on two nutrients is considered by [10–14]. A model of single-species growth in the chemostat on two substitutable resources with Michaelis-Menten-Monod functional response was proposed by [14] as follows:

$$\begin{aligned} \frac{\mathrm{d}S_1(t)}{dt} &= D\left(S_1^0 - S_1(t)\right) - \frac{r_1S_1(t)X(t)}{K_1+S_1(t)}, \\ \frac{\mathrm{d}S_2(t)}{dt} &= D\left(S_2^0 - S_2(t)\right) - \frac{r_2S_2(t)X(t)}{K_2+S_2(t)}, \\ \frac{\mathrm{d}X(t)}{dt} &= \frac{r_1S_1(t)X(t)}{K_1+S_1(t)} + \frac{r_2S_2(t)X(t)}{K_2+S_2(t)} - DX(t). \end{aligned} \tag{2}$$

However, it is now well known that stochastic noise is widely present in biological systems and so on [15–33] and microorganisms are inevitably influenced by some random factors in the process of cultivation. To better understand the dynamic behavior of the chemostat, a host of scholars proposed a slice of stochastic chemostat models and studied the effect of the random noise on the dynamic behavior of the stochastic models. As an example, Imhof and Walcher [34] proposed a stochastic chemostat model for a single microorganism species consuming a single nutrient. They found that random effects may lead to extinction in scenarios where the deterministic model predicts persistence. Recently, Xu and Yuan [35] established a stochastic chemostat model in which the maximal growth rate is influenced by the white noise in environment as follows:

$$\begin{aligned} \mathrm{d}S(t) &= \left(D\left(S^0 - S(t)\right) - \frac{mS(t)X(t)}{a+S(t)}\right)\mathrm{d}t \\ &\quad - \frac{\alpha S(t)X(t)\,\mathrm{d}B(t)}{a+S(t)}, \\ \mathrm{d}X(t) &= \left(\frac{mS(t)X(t)}{a+S(t)} - DX(t)\right)\mathrm{d}t \\ &\quad + \frac{\alpha S(t)X(t)\,\mathrm{d}B(t)}{a+S(t)}. \end{aligned} \tag{3}$$

They got an analogue break-even concentration involving the white noise which can determine the exclusion and persistence of the microorganism. And more stochastic chemostat models can be found in [36–39].

Motivated by the papers mentioned above, in this paper, we further consider a model of single-species growth in the chemostat on two supplementary resources with Michaelis-Menten-Monod functional response and environmental noise. We assume that the maximal growth rate r_i $(i = 1, 2)$ is perturbed by white noises so that

$$r_i \longrightarrow r_i + \sigma_i \dot{B}_i(t), \tag{4}$$

where $B_i(t)$ is a standard Brownian motion with intensity $\sigma_i > 0$. Then the resultant model takes the following form:

$$\begin{aligned} \mathrm{d}S_1(t) &= \left(D\left(S_1^0 - S_1(t)\right) - \frac{r_1 S_1(t)X(t)}{K_1+S_1(t)}\right)\mathrm{d}t \\ &\quad - \frac{\sigma_1 S_1(t)X(t)\,\mathrm{d}B_1(t)}{K_1+S_1(t)}, \\ \mathrm{d}S_2(t) &= \left(D\left(S_2^0 - S_2(t)\right) - \frac{r_2 S_2(t)X(t)}{K_2+S_2(t)}\right)\mathrm{d}t \\ &\quad - \frac{\sigma_2 S_2(t)X(t)\,\mathrm{d}B_2(t)}{K_2+S_2(t)}, \\ \mathrm{d}X(t) &= \left(\frac{r_1 S_1(t)X(t)}{K_1+S_1(t)} + \frac{r_2 S_2(t)X(t)}{K_2+S_2(t)} - DX(t)\right)\mathrm{d}t \\ &\quad + \frac{\sigma_1 S_1(t)X(t)\,\mathrm{d}B_1(t)}{K_1+S_1(t)} \\ &\quad + \frac{\sigma_2 S_2(t)X(t)\,\mathrm{d}B_2(t)}{K_2+S_2(t)}. \end{aligned} \tag{5}$$

Our main objective in the rest of this paper is to investigate the threshold dynamics of stochastic chemostat model (5) and explore the conditions under which microorganisms will die out or exist.

2. Preliminaries

In this section, we will give some notations, definitions, and lemmas which will be used for analyzing our main results. To this end, throughout this paper, we let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions: it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets; we use $B(t)$ to represent a scalar Brownian motion defined on the complete probability space Ω; also let $R_+^2 = \{x_i > 0,\ i = 1, 2\}$. If for an integrable function f on $[0, +\infty)$, define

$$\langle f(t)\rangle = \frac{1}{t}\int_0^t f(\theta)\,\mathrm{d}\theta. \tag{6}$$

Then we have the following.

Definition 1. For system (5),

(i) the microorganism $X(t)$ is said to be extinctive if $\lim_{t\to+\infty} X(t) = 0$,

(ii) the microorganism $X(t)$ is said to be permanent in mean if there exists a positive constant λ such that $\liminf_{t\to+\infty}\langle X(t)\rangle \geq \lambda$.

Then, one can show the following lemmas.

Lemma 2. *The solution $(S_1(t), S_2(t), X(t))$ of model (2) or (5) with the initial condition $(S_1(0), S_2(0), X(0)) \in R_+^3$ is ultimately bounded; that is,*

$$\begin{aligned} \limsup_{t\to\infty} S_1(t) &\leq S_1^0, \\ \limsup_{t\to\infty} S_2(t) &\leq S_2^0, \\ \limsup_{t\to\infty} M(t) &\leq S_1^0 + S_2^0, \end{aligned} \tag{7}$$

where $M(t) = S_1(t) + S_2(t) + X(t)$.

Proof. Letting $M(t) = S_1(t) + S_2(t) + X(t)$, from system (2) or system (5), we have

$$\frac{\mathrm{d}M(t)}{\mathrm{d}t} = D\left(S_1^0 + S_2^0\right) - DM(t). \tag{8}$$

This implies that

$$\lim_{t\to+\infty} M(t) = S_1^0 + S_2^0. \tag{9}$$

Thus, we have

$$\begin{aligned} \limsup_{t\to\infty} S_1(t) &\leq S_1^0, \\ \limsup_{t\to\infty} S_2(t) &\leq S_2^0, \\ \limsup_{t\to\infty} M(t) &= S_1^0 + S_2^0. \end{aligned} \tag{10}$$

This completes the proof of Lemma 2. □

By Lemma 2 and the strong law of large numbers for martingales [40], we can obtain the following lemma.

Lemma 3. *Letting* $(S_1(t), S_2(t), X(t))$ *be a solution of system (5) with initial value* $(S_1(0), S_2(0), X(0)) \in R_+^3$, *then*

$$\begin{aligned} \lim_{t\to+\infty} \frac{1}{t}\int_0^t \frac{\sigma_1 S_1(\tau)}{K_1 + S_1(\tau)} \mathrm{d}B(\tau) &= 0, \\ \lim_{t\to+\infty} \frac{1}{t}\int_0^t \frac{\sigma_2 S_2(\tau)}{K_2 + S_2(\tau)} \mathrm{d}B(\tau) &= 0. \end{aligned} \tag{11}$$

3. Dynamics of Deterministic System (2)

In this section, we will focus on the deterministic system (2). It is easy to see that the equilibria point of (2) satisfy

$$\begin{aligned} D\left(S_1^0 - S_1(t)\right) - \frac{r_1 S_1(t) X(t)}{K_1 + S_1(t)} &= 0, \\ D\left(S_2^0 - S_2(t)\right) - \frac{r_2 S_2(t) X(t)}{K_2 + S_2(t)} &= 0, \\ \frac{r_1 S_1(t) X(t)}{K_1 + S_1(t)} + \frac{r_2 S_2(t) X(t)}{K_2 + S_2(t)} - DX(t) &= 0, \end{aligned} \tag{12}$$

and, obviously, model (2) has a microorganism extinction equilibrium $E_0(S_1^0, S_2^0, 0)$. Let $E^*(S_1^*, S_2^*, X^*)$ be the coexistence equilibrium of model (2), which satisfies

$$f(S_2) =: a(S_2)^3 + b(S_2)^2 + cS_2 + d = 0, \tag{13}$$

where

$$\begin{aligned} a &= (r_2 - D)(r_1 + r_2 - D), \\ b &= r_1 S_2^0 D + DS_1^0 r_2 - 2r_1 K_2 D - r_2^2 S_1^0 + r_1 K_2 r_2 - S_2^0 r_2^2 \\ &\quad - r_2^2 K_1 + 2D^2 K_2 - S_2^0 D^2 - S_1^0 r_2 r_1 - 2r_2 DK_2 \\ &\quad + 2S_2^0 r_2 D + DK_1 r_2 - r_1 S_2^0 r_2, \\ c &= -2S_2^0 D^2 K_2 + r_2 K_1 DK_2 - K_2^2 r_1 D + S_1^0 r_2 DK_2 \\ &\quad - K_2 S_2^0 r_1 r_2 + 2DS_2^0 K_2 r_1 + D^2 K_2^2 - S_1^0 r_2 r_1 K_2 \\ &\quad + 2S_2^0 r_2 DK_2, \\ d &= K_2^2 S_2^0 D(r_1 - D). \end{aligned} \tag{14}$$

Then we have that

$$\begin{aligned} f(0) &= K_2^2 S_2^0 D(r_1 - D), \\ f\left(S_2^0\right) &= DS_1^0 r_2 \left(S_2^0\right)^2 - r_2^2 S_1^0 \left(S_2^0\right)^2 - r_2^2 K_1 \left(S_2^0\right)^2 \\ &\quad - S_1^0 r_2 r_1 \left(S_2^0\right)^2 + DK_1 r_2 \left(S_2^0\right)^2 + r_2 K_1 DK_2 S_2^0 \\ &\quad + S_1^0 r_2 DK_2 S_2^0 - S_1^0 r_2 r_1 K_2 S_2^0 \\ &= r_2 S_2^0 \left(D\left(S_1^0 + K_1\right)\left(S_2^0 + K_2\right) - S_1^0 r_1 \left(K_2 + S_2^0\right)\right. \\ &\quad \left. - r_2 S_2^0 \left(S_1^0 + K_1\right)\right). \end{aligned} \tag{15}$$

Denote

$$\mathscr{R} = \frac{1}{D}\left(\frac{1}{\delta_1} + \frac{1}{\delta_2}\right), \tag{16}$$

where $\delta_1 = (K_1 + S_1^0)/r_1 S_1^0$, $\delta_2 = (K_2 + S_2^0)/r_2 S_2^0$.

Obviously, If $\mathscr{R} > 1$, we have $f(S_2^0) < 0$. If $r_1 > D$, we have $f(0) > 0$. Thus, equation has one positive root S_2 at least, and $S_2 \in (0, S_2^0)$.

From the second equation of (12), one gets

$$X = \frac{D\left(S_2^0 - S_2\right)(K_2 + S_2)}{r_2 S_2}. \tag{17}$$

Substituting (17) into the first equation of (12), we have

$$\begin{aligned} & r_2 S_2 S_1^2 + \left(r_1\left(S_2^0 - S_2\right)(K_2 + S_2) - r_2\left(S_1^0 - K_1\right)S_2\right)S_1 \\ & \quad - r_2 S_2 S_1^0 K_1 = 0. \end{aligned} \tag{18}$$

Let

$$\begin{aligned} & g(S_1) \\ & \quad = r_2 S_2 S_1^2 \\ & \qquad + \left(r_1\left(S_2^0 - S_2\right)(K_2 + S_2) - r_2\left(S_1^0 - K_1\right)S_2\right)S_1 \\ & \qquad - r_2 S_2 S_1^0 K_1. \end{aligned} \tag{19}$$

It is easy to see that

$$\begin{aligned} g(0) &= -r_2 S_2 S_1^0 K_1 < 0, \\ g\left(S_1^0\right) &= r_1 S_1^0 \left(S_2^0 - S_2\right)(K_2 + S_2) > 0. \end{aligned} \tag{20}$$

Thus, (18) has one positive root S_1 at least, and $S_1 \in (0, S_1^0)$.

From the third equation of (12), we have $0 < X < S_1^0 + S_2^0$. Then we have the following theorem.

Theorem 4. *If* $r_1 > D$ *and* $\mathscr{R} > 1$, *then system (2) has unique positive equilibrium* E^*.

Regarding the stability of these equilibria, we have the following theorem.

Theorem 5. *Then for system (2), one has the following.*

(i) *If* $\mathscr{R} < 1$, *microorganism extinction equilibrium* E_0 *is locally stable; if* $\mathscr{R} > 1$ *it is unstable.*

(ii) *If* $r_1 > D$ *and* $\mathscr{R} > 1$, *the coexistence equilibrium* E^* *is locally stable.*

Proof. Linearizing the system at the equilibrium $E(S_1^\circ, S_2^\circ, X^\circ)$ gives the Jacobian

$$J = \begin{pmatrix} -D - a_1 & 0 & -a_4 \\ 0 & -D - a_2 & -a_3 \\ a_1 & a_2 & a_3 + a_4 - D \end{pmatrix}, \tag{21}$$

where

$$a_1 = \frac{K_1 r_1 X^\circ}{(K_1 + S_1^\circ)^2}, \quad a_2 = \frac{K_2 r_2 X^\circ}{(K_2 + S_2^\circ)^2}, \quad a_3 = \frac{r_2 S_2^\circ}{K_2 + S_2^\circ}, \quad a_4 = \frac{r_1 S_1^\circ}{K_1 + S_1^\circ}. \tag{22}$$

The characteristic equation gives

$$(\lambda + D)\left(\lambda^2 + A\lambda + B\right) = 0, \tag{23}$$

where

$$A = 2D + a_1 + a_2 - a_3 - a_4, \quad B = D^2 + a_1 D + a_2 D + a_1 a_2 - a_3 D - a_4 D - a_1 a_3 - a_2 a_4. \tag{24}$$

Obviously, we have, at E_0,

$$a_1 = 0, \quad a_2 = 0, \quad a_3 = \frac{r_2 S_2^0}{K_2 + S_2^0}, \quad a_4 = \frac{r_1 S_1^0}{K_1 + S_1^0}. \tag{25}$$

Then we have

$$A = 2D - a_3 - a_4, \quad B = D\left(D - (a_3 + a_4)\right), \tag{26}$$

and thus if $\mathscr{R} < 1$, all the eigenvalues of (23) have negative real part; then, by the stability theory, E_0 is stable.

And, at E^*, we have

$$A = 2D + a_1 + a_2 - a_3 - a_4 = D + a_1 + a_2 > 0, \quad B = D^2 + a_1 D + a_2 D + a_1 a_2 - a_3 D - a_4 D - a_1 a_3 - a_2 a_4 = a_1 a_2 + a_1\left(D - a_3\right) + a_2\left(D - a_4\right) > 0; \tag{27}$$

here $a_3 + a_4 = D$ is used. Then all the eigenvalues of (23) have negative real part; thus, by the stability theory, the diseases equilibrium is stable as long as it exists. □

4. Dynamics of Stochastic System (5)

4.1. Extinction. In this section, we explore the conditions leading to the extinction of the two infectious diseases. Denote

$$\mathscr{R}^* = \mathscr{R} - \frac{1}{D}\left(\frac{\sigma_1^2}{2}\left(\frac{S_1^0}{K_1 + S_1^0}\right)^2 - \frac{\sigma_2^2}{2}\left(\frac{S_2^0}{K_2 + S_2^0}\right)^2\right), \tag{28}$$

where $\mathscr{R}$ is introduced in (16). Then we have the following.

Theorem 6. *For system (5), if one of the following holds,*

(i) $\sigma_1^2 > \delta_1$, $\sigma_2^2 > \delta_2$, *and* $r_1^2/2\sigma_1^2 + r_2^2/2\sigma_2^2 < D$,

(ii) $\sigma_1^2 > \delta_1$, $\sigma_2^2 < \delta_2$, *and* $-D + r_1^2/2\sigma_1^2 + r_2 S_2^0/(K_2 + S_2^0) - (\sigma_2^2/2)(S_2^0/(K_2 + S_2^0))^2 < 0$,

(iii) $\sigma_1^2 < \delta_1$, $\sigma_2^2 > \delta_2$, *and* $-D + r_2^2/2\sigma_2^2 + r_1 S_1^0/(K_1 + S_1^0) - (\sigma_1^2/2)(S_1^0/(K_1 + S_1^0))^2 < 0$,

(iv) $\sigma_1^2 < \delta_1$, $\sigma_2^2 < \delta_2$, *and* $\mathscr{R}^* < 1$,

then the microorganism $X(t)$ of system (5) goes to extinction almost surely. Moreover,

$$\lim_{t \to +\infty} S_1(t) = S_1^0, \quad \lim_{t \to +\infty} S_2(t) = S_2^0, \tag{29}$$

almost surely.

Proof. Let $(S_1(t), S_2(t), X(t))$ be a solution of system (5) with initial value $(S_1(0), S_2(0), X(0)) \in R_+^3$. Applying Itô's formula to system (5) results in

$$\mathrm{d} \ln X(t) = \left(r_1 \phi_1(t) + r_2 \phi_2(t) - D - \frac{\sigma_1^2}{2}\phi_1^2(t) - \frac{\sigma_2^2}{2}\phi_2^2(t)\right)\mathrm{d}t + \frac{\sigma_1 S_1(t)\,\mathrm{d}B(t)}{K_1 + S_1(t)} + \frac{\sigma_2 S_2(t)\,\mathrm{d}B(t)}{K_2 + S_2(t)}, \tag{30}$$

where $\phi_1(t) = S_1(t)X(t)/(K_1 + S_1(t))$, $\phi_2(t) = S_2(t)X(t)/(K_2 + S_2(t))$.

Integrating both sides of (30) from 0 to t gives

$$\ln X(t) = \int_0^t \left(r_1 \phi_1(\tau) + r_2 \phi_2(\tau) - \frac{\sigma_1^2}{2}\phi_1^2(\tau) - \frac{\sigma_2^2}{2}\phi_2^2(\tau)\right)\mathrm{d}\tau - Dt + M_1(t) + M_2(t) + \ln X(0), \tag{31}$$

where

$$M_1(t) = \int_0^t \frac{\sigma_1 S_1(t)\,\mathrm{d}B(\tau)}{K_1 + S_1(t)}, \quad M_2(t) = \int_0^t \frac{\sigma_2 S_2(t)\,\mathrm{d}B(\tau)}{K_2 + S_2(t)}, \tag{32}$$

known as the local continuous martingale, and $M(0) = 0$. Obviously, we need to estimate the maximum value of $r_1\phi_1(t) + r_2\phi_2(t) - (\sigma_1^2/2)\phi_1^2(t) - (\sigma_2^2/2)\phi_2^2(t)$.

Let us consider quadratic function

$$g(z) = az - \frac{\sigma^2}{2}z^2, \quad z \in \left[0, \frac{b}{K+b}\right], \; b > 0. \tag{33}$$

It is easy to verify that when $\sigma^2 > \delta = (K+b)/ab$, $g(z)$ reaches its maximum value $g_{\max} = a^2/2\sigma^2$ at $z = a/\sigma^2$; and when $\sigma^2 < \delta$, $g(z)$ achieve its maximum value $g_{\max} = \delta_3 = ab/(K+b) - (\sigma^2/2)(b/(K+b))^2$ at $z = b/(K+b)$. Then, in (31), we have four cases to be discussed, depending on whether $\sigma_1^2 > \delta_1$ or $\sigma_2^2 > \delta_2$, which are as follows: Case 1: $\sigma_1^2 > \delta_1$, $\sigma_2^2 > \delta_2$; Case 2: $\sigma_1^2 > \delta_1$, $\sigma_2^2 < \delta_2$; Case 3: $\sigma_1^2 < \delta_1$, $\sigma_2^2 > \delta_2$; and Case 4: $\sigma_1^2 < \delta_1$, $\sigma_2^2 < \delta_2$.

For Case 1, since $\sigma_1^2 > \delta_1$, $\sigma_2^2 > \delta_2$, then $r_1\phi_1(t) + r_2\phi_2(t) - (\sigma_1^2/2)\phi_1^2(t) - (\sigma_2^2/2)\phi_2^2(t)$ achieve the maximum value $r_1^2/2\sigma_1^2 + r_2^2/2\sigma_2^2$. Then we can easily see from (31) that

$$\begin{aligned}\ln X(t) \le{}& \left(\frac{r_1^2}{2\sigma_1^2} + \frac{r_2^2}{2\sigma_2^2}\right)t - Dt + M_1(t) + M_2(t) \\ &+ \ln X(0).\end{aligned} \tag{34}$$

Dividing both sides of (34) by $t > 0$, we have

$$\frac{\ln X(t)}{t} \le -\left(D - \frac{r_1^2}{2\sigma_1^2} - \frac{r_2^2}{2\sigma_2^2}\right) + \frac{M(t)}{t} + \frac{\ln X(0)}{t} \tag{35}$$

and, by Lemma 3, we have

$$\lim_{t\to+\infty} \frac{M(t)}{t} = 0. \tag{36}$$

Then, taking the limit superior on both sides of (35) leads to

$$\limsup_{t\to+\infty} \frac{\ln X(t)}{t} \le -\left(D - \frac{r_1^2}{2\sigma_1^2} - \frac{r_2^2}{2\sigma_2^2}\right) < 0, \tag{37}$$

which implies $\lim_{t\to+\infty} X(t) = 0$, and here $r_1^2/2\sigma_1^2 + r_2^2/2\sigma_2^2 < D$ is used.

Case 2. $\sigma_1^2 > \delta_1 = (K + S_1^0)/r_1S_1^0$, $\sigma_2^2 < \delta_2 = (K + S_2^0)/r_2S_2^0$.

In this case, we can easily see from (31) that

$$\begin{aligned}\ln X(t) \le{}& \left(\frac{r_1^2}{2\sigma_1^2} + \frac{r_2S_2^0}{K_2 + S_2^0} - \frac{\sigma_2^2}{2}\left(\frac{S_2^0}{K_2 + S_2^0}\right)^2\right)t \\ &- Dt + M_1(t) + M_2(t) + \ln X(0).\end{aligned} \tag{38}$$

Dividing both sides of (38) by $t > 0$, we have

$$\begin{aligned}&\frac{\ln X(t)}{t} \\ &\le -\left(D - \frac{r_1^2}{2\sigma_1^2} - \frac{r_2S_2^0}{K_2 + S_2^0} + \frac{\sigma_2^2}{2}\left(\frac{S_2^0}{K_2 + S_2^0}\right)^2\right) \\ &\quad + \frac{M(t)}{t} + \frac{\ln X(0)}{t}\end{aligned} \tag{39}$$

and, by Lemma 3, we have

$$\lim_{t\to+\infty} \frac{M(t)}{t} = 0. \tag{40}$$

Then, taking the limit superior on both sides of (38) leads to

$$\begin{aligned}&\limsup_{t\to+\infty} \frac{\ln X(t)}{t} \\ &\le -\left(D - \frac{r_1^2}{2\sigma_1^2} - \frac{r_2S_2^0}{K_2 + S_2^0} + \frac{\sigma_2^2}{2}\left(\frac{S_2^0}{K_2 + S_2^0}\right)^2\right) \\ &< 0\end{aligned} \tag{41}$$

which implies $\lim_{t\to+\infty} X(t) = 0$.

The same discussion can be used in Case 3; here we omit it.

Next, we consider Case 4: $\sigma_1^2 < \delta_1 = (K + S_1^0)/r_1S_1^0$, $\sigma_2^2 < \delta_2 = (K + S_2^0)/r_2S_2^0$. From (31), we have

$$\begin{aligned}\ln X(t) \le{}& \left(\frac{r_1S_1^0}{K_1 + S_1^0} - \frac{\sigma_1^2}{2}\left(\frac{S_1^0}{K_1 + S_1^0}\right)^2 + \frac{r_2S_2^0}{K_2 + S_2^0}\right. \\ &\left. - \frac{\sigma_2^2}{2}\left(\frac{S_2^0}{K_2 + S_2^0}\right)^2\right) - Dt + M_1(t) + M_2(t) \\ &+ \ln X(0).\end{aligned} \tag{42}$$

Dividing both sides of (42) by $t > 0$, we have

$$\begin{aligned}\frac{\ln X(t)}{t} \le{}& -\left(D - \frac{r_1S_1^0}{K_1 + S_1^0} + \frac{\sigma_1^2}{2}\left(\frac{S_1^0}{K_1 + S_1^0}\right)^2\right. \\ &\left. - \frac{r_2S_2^0}{K_2 + S_2^0} + \frac{\sigma_2^2}{2}\left(\frac{S_2^0}{K_2 + S_2^0}\right)^2\right) + \frac{M(t)}{t} \\ &+ \frac{\ln X(0)}{t}\end{aligned} \tag{43}$$

and, by Lemma 3, we have

$$\lim_{t\to+\infty} \frac{M(t)}{t} = 0. \tag{44}$$

Then, taking the limit superior on both sides of (43) leads to

$$\begin{aligned}\limsup_{t\to+\infty} \frac{\ln X(t)}{t} \le{}& -\left(D - \frac{r_1S_1^0}{K_1 + S_1^0}\right. \\ &\left. + \frac{\sigma_1^2}{2}\left(\frac{S_1^0}{K_1 + S_1^0}\right)^2 - \frac{r_2S_2^0}{K_2 + S_2^0} + \frac{\sigma_2^2}{2}\left(\frac{S_2^0}{K_2 + S_2^0}\right)^2\right) \\ &< 0\end{aligned} \tag{45}$$

which implies $\lim_{t\to+\infty} X(t) = 0$.

Next, we prove the last conclusion. Given $0 < \varepsilon \ll 1$, since $\lim_{t\to+\infty} X(t) = 0$, we have $0 < X(t) < \varepsilon$ for t large enough. By the first equation of system (5), we have

$$\begin{aligned} \mathrm{d}S_1(t) &\geq \left(D\left(S_1^0 - S_1(t)\right) - \frac{r_1 S_1(t)\,\varepsilon}{K_1 + S_1(t)}\right)\mathrm{d}t \\ &\quad - \frac{\sigma_1 S_1(t)\,\varepsilon \mathrm{d}B(t)}{K_1 + S_1(t)}, \\ \frac{\mathrm{d}S_1(t)}{\mathrm{d}t} &\geq DS_1^0 - \left(D + \frac{r_1\varepsilon}{K_1} + \frac{\sigma_1\varepsilon|B(t)|}{K_1}\right)S_1(t). \end{aligned} \tag{46}$$

Then when $\varepsilon \to 0$ we have

$$\liminf_{t\to+\infty} S_1(t) \geq S_1^0. \tag{47}$$

On the other hand from the proof of Lemma 2, we have

$$\lim_{t\to+\infty} S_1(t) \leq S_1^0 + \varepsilon. \tag{48}$$

Let $\varepsilon \to 0$. Then one has

$$\limsup_{t\to+\infty} S_1(t) \leq S_1^0. \tag{49}$$

From (47) and (49), we have

$$\lim_{t\to+\infty} S_1(t) = S_1^0 \tag{50}$$

almost surely.

By employing the method similar above, it then follows that

$$\lim_{t\to+\infty} S_2(t) = S_2^0 \tag{51}$$

almost surely. This completes the proof of Theorem 6. □

4.2. Permanence in Mean

Theorem 7. *If $\mathscr{R} > 1$, then the microorganism $X(t)$ is permanent in mean; moreover, $X(t)$ satisfies*

$$\liminf_{t\to+\infty} \langle X(t)\rangle \geq D\Delta\left(\mathscr{R}^* - 1\right), \tag{52}$$

where $\Delta = \min\{(K_1 + S_1^0)D/r_1, (K_2 + S_2^0)D/r_2\}$.

Proof. Integrating from 0 to t and dividing by t on both sides of system (5) yield

$$\begin{aligned} \Theta(t) &\triangleq \frac{S_1(t) - S_1(0)}{t} + \frac{S_2(t) - S_2(0)}{t} \\ &\quad + \frac{X(t) - X(0)}{t} \\ &= D\left(S_1^0 + S_2^0\right) - D\left(\langle S_1(t)\rangle + \langle S_2(t)\rangle\right) \\ &\quad - D\langle X(t)\rangle. \end{aligned} \tag{53}$$

Then one can get

$$\begin{aligned} D\langle X(t)\rangle &= D\left(S_1^0 + S_2^0\right) - D\left(\langle S_1(t)\rangle + \langle S_2(t)\rangle\right) \\ &\quad - \Theta(t). \end{aligned} \tag{54}$$

Applying Itô's formula gives

$$\begin{aligned} \mathrm{d}\ln X(t) &= \left(r_1\phi_1(t) + r_2\phi_2(t) - D - \frac{\sigma_1^2}{2}\phi_1^2(t)\right. \\ &\quad \left. - \frac{\sigma_2^2}{2}\phi_2^2(t)\right)\mathrm{d}t + \frac{\sigma_1 S_1(t)\,\mathrm{d}B(t)}{K_1 + S_1(t)} \\ &\quad + \frac{\sigma_2 S_2(t)\,\mathrm{d}B(t)}{K_2 + S_2(t)} \\ &\geq \left(r_1\phi_1(t) + r_2\phi_2(t) - D - \frac{\sigma_1^2}{2}\left(\frac{S_1^0}{K_1 + S_1^0}\right)^2\right. \\ &\quad \left. - \frac{\sigma_2^2}{2}\left(\frac{S_2^0}{K_2 + S_2^0}\right)^2\right)\mathrm{d}t + \frac{\sigma_1 S_1(t)\,\mathrm{d}B(t)}{K_1 + S_1(t)} \\ &\quad + \frac{\sigma_2 S_2(t)\,\mathrm{d}B(t)}{K_2 + S_2(t)}. \end{aligned} \tag{55}$$

Integrating from 0 to t and dividing by t on both sides of (55) yields

$$\begin{aligned} &\frac{\ln X(t) - \ln X(0)}{t} \\ &\quad \geq r_1\frac{1}{t}\int_0^t \phi_1(\theta)\,\mathrm{d}\theta + r_2\frac{1}{t}\int_0^t \phi_2(\theta)\,\mathrm{d}\theta \\ &\quad - \left[D + \frac{\sigma_1^2}{2}\left(\frac{S_1^0}{K_1 + S_1^0}\right)^2 + \frac{\sigma_2^2}{2}\left(\frac{S_2^0}{K_2 + S_2^0}\right)^2\right] \\ &\quad + \frac{M_1(t)}{t} + \frac{M_2(t)}{t}, \end{aligned} \tag{56}$$

where $M_1(t) = \int_0^t(\sigma_1 S_1(\tau)/(K_1 + S_1(\tau)))\mathrm{d}B(\tau)$, $M_2(t) = \int_0^t(\sigma_2 S_2(\tau)/(K_2 + S_2(\tau)))\mathrm{d}B(\tau)$. Noticing that

$$\begin{aligned} \int_0^t \phi_1(\theta)\,\mathrm{d}\theta &= \int_0^t \frac{S_1(\theta)}{K_1 + S_1(\theta)}\mathrm{d}\theta \geq \int_0^t \frac{S_1(\theta)}{K_1 + S_1^0}\mathrm{d}\theta, \\ \int_0^t \phi_2(\theta)\,\mathrm{d}\theta &= \int_0^t \frac{S_2(\theta)}{K_2 + S_2(\theta)}\mathrm{d}\theta \geq \int_0^t \frac{S_2(\theta)}{K_2 + S_2^0}\mathrm{d}\theta, \end{aligned} \tag{57}$$

then we have

$$\begin{aligned}
&\frac{\ln X(t)-\ln X(0)}{t} \\
&\geq -\left[D+\frac{\sigma_1^2}{2}\left(\frac{S_1^0}{K_1+S_1^0}\right)^2+\frac{\sigma_2^2}{2}\left(\frac{S_2^0}{K_2+S_2^0}\right)^2\right] \\
&\quad+\frac{r_1}{K_1+S_1^0}\langle S_1(t)\rangle+\frac{r_2}{K_2+S_2^0}\langle S_2(t)\rangle+\frac{M_1(t)}{t} \\
&\quad+\frac{M_2(t)}{t} \\
&=-\left[D+\frac{\sigma_1^2}{2}\left(\frac{S_1^0}{K_1+S_1^0}\right)^2+\frac{\sigma_2^2}{2}\left(\frac{S_2^0}{K_2+S_2^0}\right)^2\right] \\
&\quad+\frac{r_2}{(K_2+S_2^0)}\left(S_1^0+S_2^0\right) \\
&\quad+\left(\frac{r_1}{K_1+S_1^0}-\frac{r_2}{K_2+S_2^0}\right)\langle S_1(t)\rangle \\
&\quad-\frac{r_2}{K_2+S_2^0}\langle X(t)\rangle-\frac{r_2}{K_2+S_2^0}\frac{\Theta(t)}{D}+\frac{M_1(t)}{t} \\
&\quad+\frac{M_2(t)}{t}.
\end{aligned} \tag{58}$$

If $r_1/(K_1+S_1^0)\leq r_2/(K_2+S_2^0)$, we can get

$$\begin{aligned}
&\frac{\ln X(t)-\ln X(0)}{t} \\
&\geq -\left[D+\frac{\sigma_1^2}{2}\left(\frac{S_1^0}{K_1+S_1^0}\right)^2+\frac{\sigma_2^2}{2}\left(\frac{S_2^0}{K_2+S_2^0}\right)^2\right] \\
&\quad+\frac{r_2}{(K_2+S_2^0)}\left(S_1^0+S_2^0\right) \\
&\quad+\left(\frac{r_1}{K_1+S_1^0}-\frac{r_2}{K_2+S_2^0}\right)S_1^0-\frac{r_2}{K_2+S_2^0}\langle X(t)\rangle \\
&\quad-\frac{r_2}{K_2+S_2^0}\frac{\Theta(t)}{D}+\frac{M_1(t)}{t}+\frac{M_2(t)}{t} \\
&\geq -\left[D+\frac{\sigma_1^2}{2}\left(\frac{S_1^0}{K_1+S_1^0}\right)^2+\frac{\sigma_2^2}{2}\left(\frac{S_2^0}{K_2+S_2^0}\right)^2\right] \\
&\quad+\frac{r_2S_2^0}{K_2+S_2^0}+\frac{r_1S_1^0}{K_1+S_1^0}-\frac{r_2}{K_2+S_2^0}\langle X(t)\rangle \\
&\quad-\frac{r_2}{K_2+S_2^0}\frac{\Theta(t)}{D}+\frac{M_1(t)}{t}+\frac{M_2(t)}{t}.
\end{aligned} \tag{59}$$

By inequality (59), we have

$$\begin{aligned}
\langle X(t)\rangle &\geq \frac{K_2+S_2^0}{r_2}\left(\frac{r_1S_1^0}{K_1+S_1^0}+\frac{r_2S_2^0}{K_2+S_2^0}\right. \\
&\quad-\left.\frac{\sigma_1^2}{2}\left(\frac{S_1^0}{K_1+S_1^0}\right)^2-\frac{\sigma_2^2}{2}\left(\frac{S_2^0}{K_2+S_2^0}\right)^2-D\right) \\
&\quad-\frac{K_2+S_2^0}{r_2}\left(\frac{\ln X(t)-\ln X(0)}{t}+\frac{r_2}{K_2+S_2^0}\frac{\Theta(t)}{D}\right. \\
&\quad+\left.\frac{M_1(t)}{t}+\frac{M_2(t)}{t}\right).
\end{aligned} \tag{60}$$

By Lemma 3, we get that $\lim_{t\to+\infty}(M(t)/t)=0$. According to Lemma 2, one sees that $\limsup_{t\to+\infty}S_1(t)\leq S_1^0$, $\limsup_{t\to+\infty}S_2(t)\leq S_2^0$, and $\limsup_{t\to+\infty}X(t)\leq C_0+S_2^0$, and then one has $\lim_{t\to+\infty}(\ln X(t)/t)=0$ and $\lim_{t\to+\infty}\Theta(t)=0$. Thus taking the inferior limit of both sides of (60) yields

$$\liminf_{t\to+\infty}\langle X(t)\rangle \geq \frac{\left(K_2+S_2^0\right)D}{r_2}\left(\mathscr{R}^*-1\right). \tag{61}$$

And if $r_1/(K_1+S_1^0)\geq r_2/(K_2+S_2^0)$, we can get

$$\begin{aligned}
\frac{\ln X(t)-\ln X(0)}{t} &\geq -\left[D+\frac{\sigma_1^2}{2}\left(\frac{S_1^0}{K_1+S_1^0}\right)^2\right. \\
&\quad+\left.\frac{\sigma_2^2}{2}\left(\frac{S_2^0}{K_2+S_2^0}\right)^2\right]+\frac{r_1}{K_1+S_1^0}\left(\left(S_1^0+S_2^0\right)\right. \\
&\quad-\left.\langle S_2(t)\rangle-\langle X(t)\rangle-\frac{\Theta(t)}{D}\right)+\frac{r_2}{K_2+S_2^0}\langle S_2(t)\rangle \\
&\quad+\frac{M_1(t)}{t}+\frac{M_2(t)}{t}=-\left[D+\frac{\sigma_1^2}{2}\left(\frac{S_1^0}{K_1+S_1^0}\right)^2\right. \\
&\quad+\left.\frac{\sigma_2^2}{2}\left(\frac{S_2^0}{K_2+S_2^0}\right)^2\right]+\frac{r_1S_1^0}{K_1+S_1^0}+\frac{r_2S_2^0}{K_2+S_2^0} \\
&\quad-\frac{r_1}{K_1+S_1^0}\langle X(t)\rangle-\frac{r_2}{K_2+S_2^0}\frac{\Theta(t)}{D}+\frac{M_1(t)}{t} \\
&\quad+\frac{M_2(t)}{t},
\end{aligned} \tag{62}$$

where $M_1(t)=\int_0^t(\sigma_1S_1(\tau)/(K_1+S_1(\tau)))dB(\tau)$, $M_2(t)=\int_0^t(\sigma_2S_2(\tau)/(K_2+S_2(\tau)))dB(\tau)$.

Inequality (62) can be rewritten as

$$\langle X(t)\rangle \geq \frac{\left(K_2+S_2^0\right)}{r_2}\left(\frac{r_1 S_1^0}{K_1+S_1^0}+\frac{r_2 S_2^0}{K_2+S_2^0}\right.$$
$$-\frac{\sigma_1^2}{2}\left(\frac{S_1^0}{K_1+S_1^0}\right)^2-\frac{\sigma_2^2}{2}\left(\frac{S_2^0}{K_2+S_2^0}\right)^2-D\Bigg)$$
$$-\frac{\left(K_2+S_2^0\right)}{r_2}\left(\frac{\ln X(t)-\ln X(0)}{t}+\frac{r_1}{K_1+S_1^0}\frac{\Theta(t)}{D}\right.$$
$$\left.+\frac{M_1(t)}{t}+\frac{M_2(t)}{t}\right). \quad (63)$$

Taking the inferior limit of both sides of (63) yields

$$\liminf_{t\to+\infty}\langle X(t)\rangle \geq \frac{\left(K_1+S_1^0\right)D}{r_1}\left(\mathscr{R}^*-1\right). \quad (64)$$

Let $\Delta=\min\{(K_1+S_1^0)D/r_1,(K_2+S_2^0)D/r_2\}$, and we get from (61) and (64)

$$\liminf_{t\to+\infty}\langle X(t)\rangle \geq D\Delta\left(\mathscr{R}^*-1\right). \quad (65)$$

This completes the proof of Theorem 7. □

Remark 8. Theorems 6 and 7 show that the condition for the microorganism to go to extinction or permanence depends on the intensity of the noise disturbances completely. And small noise disturbances will be beneficial to the cultivation of the microorganism; conversely, large white noise disturbance is harmful to the cultivation of the microorganism.

5. Conclusion and Numerical Simulation

This paper proposes and investigates a new stochastic chemostat model with two substitutable nutrients and one microorganism. Then main objective in this paper is to investigate the threshold dynamics of stochastic chemostat model (5) and explore the conditions which can determine the extinction and permanence of the microorganism using two substitutable nutrients. Firstly, for the corresponding deterministic model, the threshold for extinction or existence of the microorganism is obtained by analyzing the stability of the equilibria. Then the threshold of the stochastic chemostat for the extinction and the permanence in mean of the microorganism is explored. The results show that there exists a significant difference between the threshold of the deterministic system and the stochastic system, which makes the persistent microorganism of a deterministic system become extinct due to large stochastic disturbance. That is, large stochastic disturbance is harmful to the cultivation of the

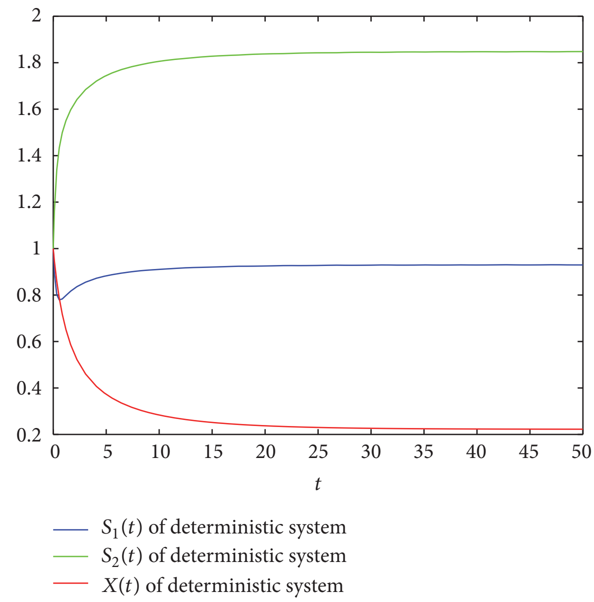


FIGURE 1: Time series for the paths $S_1(t)$, $S_2(t)$, $X(t)$ for the deterministic system with $D=3.8$, $S_1^0=1$, $S_2^0=2$, $r_1=2.5$, $r_2=4$, $K_1=1$, $K_2=1$, $\mathscr{R}=1.0307>1$.

microorganism. It is worth mentioning that this paper is a promotion of the work of Xu and Yuan [35].

Next, using the Euler Maruyama (EM) method [40], we give some numerical simulation to illustrate the extinction and persistence of the microorganism in stochastic system and corresponding deterministic system for comparison.

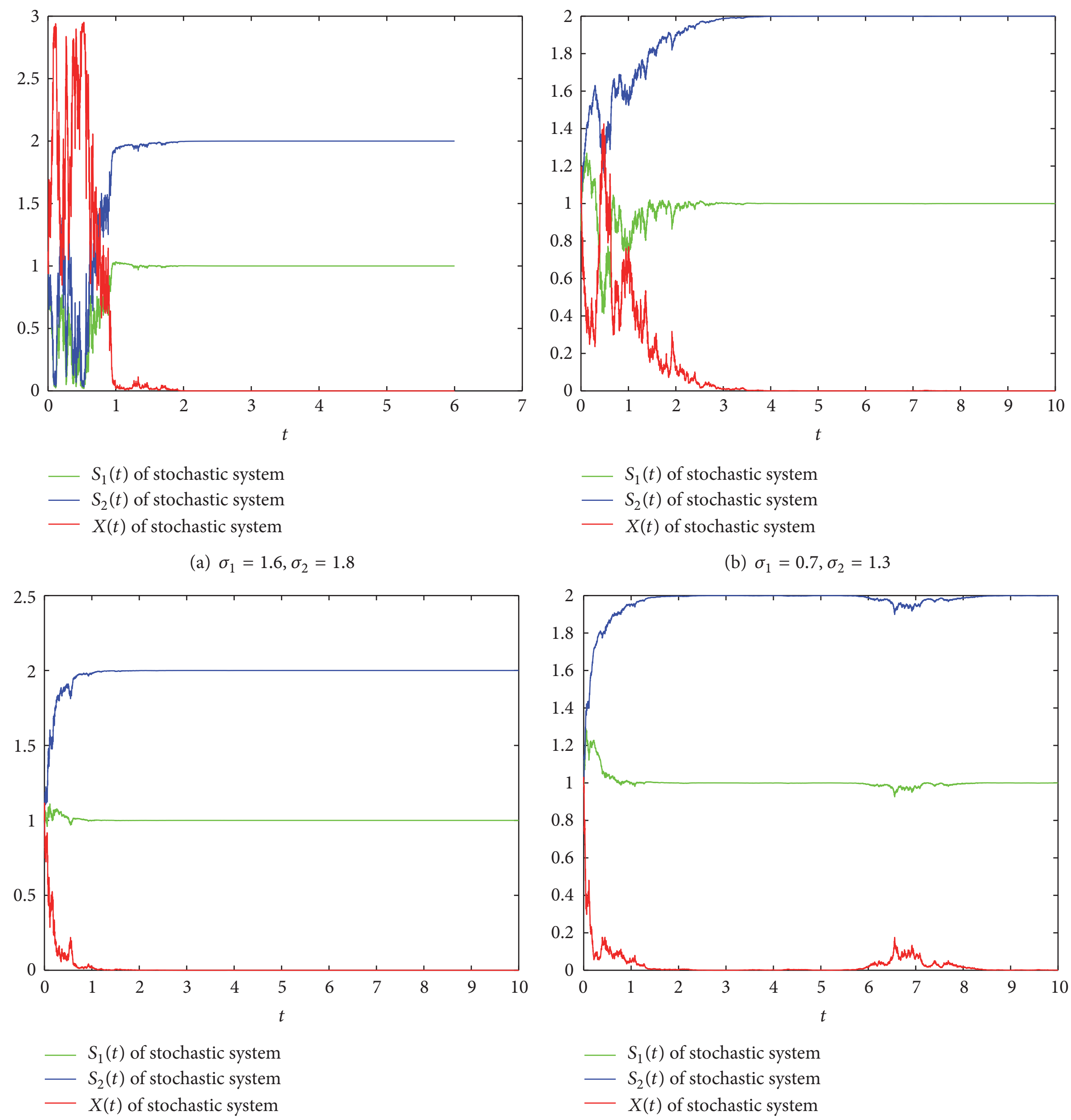
Firstly, we begin from a deterministic system; the basic parameters are set as $D=3.8$, $S_1^0=1$, $S_2^0=2$, $r_1=2.5$, $r_2=4$, $K_1=1$, and $K_2=1$. Direct calculation shows that $\delta_1=0.8$, $\delta_2=0.375$, and $\mathscr{R}=1.0307>1$. Then according to Theorems 4 and 5, the deterministic system has a unique stable positive equilibrium $E^*(0.9295785859, 1.848231322, 0.2221900926)$, which is locally stable and the deterministic system is permanent (see Figure 1).

Next, we consider the influence of stochastic disturbance on the above deterministic system. According to Theorem 6, different parameters are chosen to give insights into the reasonability of the results stated in Theorem 6.

We choose different value of parameters σ_1 and σ_2 and discuss below five different cases.

Case 1. Choose $\sigma_1=1.6$, $\sigma_2=1.8$, by direct calculation; we have $r_1^2/2\sigma_1^2+r_2^2/2\sigma_2^2=3.6898<D=3.8$. Then, by Theorem 6, the microorganism eventually tends to be extinct (see Figure 2(a)).

Case 2. Choose $\sigma_1=1$, $\sigma_2=0.5$, by direct calculation; we have $r_1^2/2\sigma_1^2+r_2S_2^0/(K_2+S_2^0)-(\sigma_2^2/2)(S_2^0/(K_2+S_2^0))^2=0.5139<D=3.8$. Then, by Theorem 6, the microorganism eventually tends to be extinct (see Figure 2(b)).



(a) $\sigma_1 = 1.6, \sigma_2 = 1.8$

(b) $\sigma_1 = 0.7, \sigma_2 = 1.3$

(c) $\sigma_1 = 0.7, \sigma_2 = 1.3$

(d) $\sigma_1 = 0.7, \sigma_2 = 0.6, \mathscr{R}' = 0.9935 < 1$

FIGURE 2: Time series for the paths $S_1(t)$, $S_2(t)$, $X(t)$ for the stochastic system.

Case 3. Choose $\sigma_1 = 0.7$, $\sigma_2 = 1.3$, by direct calculation; we have $r_2^2/2\sigma_2^2 + r_1 S_1^0/(K_1+S_1^0) - (\sigma_1^2/2)(S_1^0/(K_1+S_1^0))^2 = 3.5450 < D = 3.8$. Then, by Theorem 6, the microorganism eventually tends to be extinct (see Figure 2(c)).

Case 4. Choose $\sigma_1 = 0.7$, $\sigma_2 = 0.6$, by direct calculation; we have $\mathscr{R}' = 0.9935 < 1$. Then, by Theorem 6, the microorganism eventually tends to be extinct (see Figure 2(d)).

Case 5. Choose $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, by direct calculation; we have $\mathscr{R}' = 1.0241 > 1$. Then, by Theorem 7, the microorganism is persistent (see Figure 3).

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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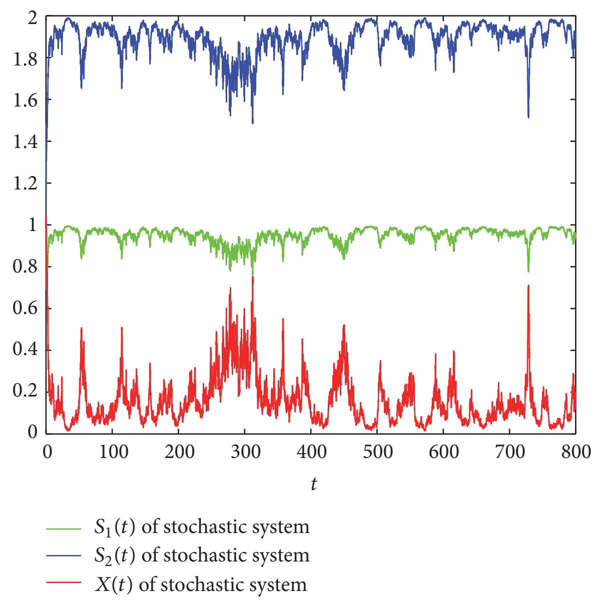

FIGURE 3: Time series for the paths $S_1(t), S_2(t), X(t)$ for the stochastic system with $\sigma_1 = 0.2, \sigma_2 = 0.3, \mathscr{R}' = 1.0241 > 1$.

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