View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Crossref

Kim et al. *Journal of Inequalities and Applications* 2014, 2014:376
http://www.journalofinequalitiesandapplications.com/content/2014/1/376
Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Barnes-type Narumi of the second kind and Barnes-type Peters of the second kind hybrid polynomials

Dae San Kim[1], Taekyun Kim[2*], Takao Komatsu[3] and Hyuck In Kwon[2]

*Correspondence: tkkim@kw.ac.kr
[2]Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea
Full list of author information is available at the end of the article

Abstract
In this paper, by considering Barnes-type Narumi polynomials of the second kind and Barnes-type Peters polynomials of the second kind, we define and investigate the hybrid polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.
MSC: 05A15; 05A40; 11B68; 11B75; 11B83

1 Introduction

In this paper, we consider the polynomials

$$\widehat{\mathrm{NS}}_n(x) = \widehat{\mathrm{NS}}_n(x|a;\lambda;\mu) = \widehat{\mathrm{NS}}_n(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)$$

called Barnes-type Narumi of the second kind and Barnes-type Peters of the second kind hybrid polynomials, whose generating function is given by

$$\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}(1+t)^x = \sum_{n=0}^{\infty}\widehat{\mathrm{NS}}_n(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\frac{t^n}{n!}, \qquad (1)$$

where $a_1,\ldots,a_r,\lambda_1,\ldots,\lambda_s,\mu_1,\ldots,\mu_s \in \mathbb{C}$ with $a_1,\ldots,a_r,\lambda_1,\ldots,\lambda_s \neq 0$. When $x = 0$, $\widehat{\mathrm{NS}}_n = \widehat{\mathrm{NS}}_n(0) = \widehat{\mathrm{NS}}_n(0|a;\lambda;\mu) = \widehat{\mathrm{NS}}_n(0|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)$ are called Barnes-type Narumi of the second kind and Barnes-type Peters of the second kind hybrid numbers.

Recall that the Barnes-type Narumi polynomials of the second kind, denoted by $\widehat{\mathrm{N}}_n(x|a_1,\ldots,a_r)$, are given by the generating function as

$$\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)(1+t)^x = \sum_{n=0}^{\infty}\widehat{\mathrm{N}}_n(x|a_1,\ldots,a_r)\frac{t^n}{n!}.$$

If $x = 0$, we write $\widehat{\mathrm{N}}_n(a_1,\ldots,a_r) = \widehat{\mathrm{N}}_n(0|a_1,\ldots,a_r)$. Narumi polynomials were mentioned in [1, p.127]. In addition, the Barnes-type Peters polynomials of the second kind, denoted by

Springer
©2014 Kim et al.; licensee Springer. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/2.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

$\hat{s}_n(x|\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)$, are given by the generating function as

$$\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}(1+t)^x=\sum_{n=0}^{\infty}\hat{s}_n(x|\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\frac{t^n}{n!}.$$

If $s=1$, then $\hat{s}_n(x|\lambda;\mu)$ are the Peters polynomials of the second kind. Peters polynomials were mentioned in [1, p.128] and have been investigated in *e.g.* [2].

In this paper, by considering Barnes-type Narumi polynomials of the second kind and Barnes-type Peters polynomials of the second kind, we define and investigate the hybrid polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

2 Umbral calculus

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F}=\left\{f(t)=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k\,\middle|\,a_k\in\mathbb{C}\right\}.\tag{2}$$

Let $\mathbb{P}=\mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ is the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L+M|p(x)\rangle=\langle L|p(x)\rangle+\langle M|p(x)\rangle$, $\langle cL|p(x)\rangle=c\langle L|p(x)\rangle$, where c is a complex constant in $\mathbb{C}$. For $f(t)\in\mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle=a_n\quad(n\geq 0).\tag{3}$$

In particular,

$$\langle t^k|x^n\rangle=n!\delta_{n,k}\quad(n,k\geq 0),\tag{4}$$

where $\delta_{n,k}$ is Kronecker's symbol.

For $f_L(t)=\sum_{k=0}^{\infty}\frac{\langle L|x^k\rangle}{k!}t^k$, we have $\langle f_L(t)|x^n\rangle=\langle L|x^n\rangle$. That is, $L=f_L(t)$. The map $L\mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the *umbral algebra* and the *umbral calculus* is the study of umbral algebra. The order $O(f(t))$ of a power series $f(t)$ ($\neq 0$) is the smallest integer k for which the coefficient of t^k does not vanish. If $O(f(t))=1$, then $f(t)$ is called a *delta series*; if $O(f(t))=0$, then $f(t)$ is called an *invertible series*. For $f(t),g(t)\in\mathcal{F}$ with $O(f(t))=1$ and $O(g(t))=0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x)=n$) such that $\langle g(t)f(t)^k|s_n(x)\rangle=n!\delta_{n,k}$, for $n,k\geq 0$. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $(g(t),f(t))$ which is denoted by $s_n(x)\sim(g(t),f(t))$.

For $f(t),g(t)\in\mathcal{F}$, and $p(x)\in\mathbb{P}$, we have

$$\langle f(t)g(t)|p(x)\rangle=\langle f(t)|g(t)p(x)\rangle=\langle g(t)|f(t)p(x)\rangle\tag{5}$$

and

$$f(t)=\sum_{k=0}^{\infty}\langle f(t)|x^k\rangle\frac{t^k}{k!},\qquad p(x)=\sum_{k=0}^{\infty}\langle t^k|p(x)\rangle\frac{x^k}{k!} \tag{6}$$

[1, Theorem 2.2.5]. Thus, by (6), we get

$$t^k p(x)=p^{(k)}(x)=\frac{d^k p(x)}{dx^k}\quad\text{and}\quad e^{yt}p(x)=p(x+y). \tag{7}$$

Sheffer sequences are characterized by the generating function of [1, Theorem 2.3.4].

Lemma 1 *The sequence $s_n(x)$ is Sheffer for $(g(t),f(t))$ if and only if*

$$\frac{1}{g(\bar{f}(t))}e^{y\bar{f}(t)}=\sum_{k=0}^{\infty}\frac{s_k(y)}{k!}t^k\quad(y\in\mathbb{C}),$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

For $s_n(x)\sim(g(t),f(t))$, we have the following equations [1, Theorem 2.3.7, Theorem 2.3.5, Theorem 2.3.9]:

$$f(t)s_n(x)=ns_{n-1}(x)\quad(n\geq 0), \tag{8}$$

$$s_n(x)=\sum_{j=0}^{n}\frac{1}{j!}\langle g(\bar{f}(t))^{-1}\bar{f}(t)^j|x^n\rangle x^j, \tag{9}$$

$$s_n(x+y)=\sum_{j=0}^{n}\binom{n}{j}s_j(x)p_{n-j}(y), \tag{10}$$

where $p_n(x)=g(t)s_n(x)$.

Assume that $p_n(x)\sim(1,f(t))$ and $q_n(x)\sim(1,g(t))$. Then the transfer formula [1, Corollary 3.8.2] is given by

$$q_n(x)=x\left(\frac{f(t)}{g(t)}\right)^n x^{-1}p_n(x)\quad(n\geq 1).$$

For $s_n(x)\sim(g(t),f(t))$, and $r_n(x)\sim(h(t),l(t))$, assume that

$$s_n(x)=\sum_{m=0}^{n}C_{n,m}r_m(x)\quad(n\geq 0).$$

Then we have [1, p.132]

$$C_{n,m}=\frac{1}{m!}\left\langle\frac{h(\bar{f}(t))}{g(\bar{f}(t))}l(\bar{f}(t))^m\middle|x^n\right\rangle. \tag{11}$$

3 Main results

For convenience, we introduce an Appell sequence of polynomials $F_n(a_1,\dots,a_r)$ $(a_1,\dots,a_r\neq 0)$, defined by

$$\prod_{i=1}^{r}\left(\frac{e^{a_it}-1}{t}\right)e^{xt}=\sum_{n=0}^{\infty}F_n(x|a_1,\dots,a_r)\frac{t^n}{n!}.$$

If $x=0$, we write $F_n(a_1,\dots,a_r)=F_n(0|a_1,\dots,a_r)$. By [1, Theorem 2.5.8] with $y=0$, we have

$$F_n(x|a_1,\dots,a_r)=\sum_{m=0}^{n}\binom{n}{m}F_{n-m}(a_1,\dots,a_r)x^m.$$

More precisely, one can show that

$$F_n(x|a_1,\dots,a_r)=\sum_{i=0}^{n}\sum_{l_1+\cdots+l_r=i}\frac{i!}{(i+r)!}\binom{n}{i}\binom{i+r}{l_1+1,\dots,l_r+1}a_1^{l_1+1}\cdots a_r^{l_r+1}x^{n-i},$$

where

$$\binom{i+r}{l_1+1,\dots,l_r+1}=\frac{(i+r)!}{(l_1+1)!\cdots(l_r+1)!}.$$

Remember that the generalized Barnes-type Euler polynomials $E_n(x|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)$ are defined by the generating function

$$\prod_{j=1}^{s}\left(\frac{2}{1+e^{\lambda_jt}}\right)^{\mu_j}e^{xt}=\sum_{n=0}^{\infty}E_n(x|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\frac{t^n}{n!}.$$

If $\mu_1=\cdots=\mu_s=1$, then $E_n(x|\lambda_1,\dots,\lambda_s)=E_n(x|\lambda_1,\dots,\lambda_s;1,\dots,1)$ are called the Barnes-type Euler polynomials. If further $\lambda_1=\cdots=\lambda_s=1$, then $E_n^{(r)}(x)=E_n(x|1,\dots,1;1,\dots,1)$ are called the Euler polynomials of order s. If $x=0$, we write $E_n(\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)=E_n(0|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)$. By [1, Theorem 2.5.8] with $y=0$, we have

$$E_n(x|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)=\sum_{m=0}^{n}\binom{n}{m}E_{n-m}(\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)x^m.$$

We note that

$$tF_n(x|a_1,\dots,a_r)=\frac{d}{dx}F_n(x|a_1,\dots,a_r)=nF_{n-1}(x|a_1,\dots,a_r),$$

$$\begin{aligned}tE_n(x|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)&=\frac{d}{dx}E_n(x|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\&=nE_{n-1}(x|\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s).\end{aligned}$$

From the definition (1), $\widehat{\mathrm{NS}}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)$ is the Sheffer sequence for the pair

$$g(t)=\prod_{i=1}^{r}\left(\frac{te^{a_it}}{e^{a_it}-1}\right)\prod_{j=1}^{s}\left(\frac{1+e^{\lambda_jt}}{e^{\lambda_jt}}\right)^{\mu_j}\quad\text{and}\quad f(t)=e^t-1.$$

So,

$$\widehat{\mathrm{NS}}_n(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s) \sim \left(\prod_{i=1}^{r}\left(\frac{te^{a_it}}{e^{a_it}-1}\right)\prod_{j=1}^{s}\left(\frac{1+e^{\lambda_jt}}{e^{\lambda_jt}}\right)^{\mu_j}, e^t-1\right). \quad (12)$$

3.1 Explicit expressions

Let $(n)_j = n(n-1)\cdots(n-j+1)$ $(j \geq 1)$ with $(n)_0 = 1$. The (signed) Stirling numbers of the first kind $S_1(n,m)$ are defined by

$$(x)_n = \sum_{m=0}^{n} S_1(n,m)x^m.$$

Theorem 1

$$\begin{aligned}
&\widehat{\mathrm{NS}}_n(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s) \\
&\quad = 2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}(-1)^m\binom{m}{l}S_1(n,m)E_{m-l}(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s) \\
&\qquad \times F_l(-x|a_1,\ldots,a_r) &(13)\\
&\quad = 2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m}{l}S_1(n,m)E_{m-l}(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s) \\
&\qquad \times F_l\left(x+\sum_{j=1}^{s}\lambda_j\mu_j-\sum_{i=1}^{r}a_i\,\middle|\,a_1,\ldots,a_r\right) &(14)\\
&\quad = 2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}(-1)^m\binom{m}{l}S_1(n,m)F_{m-l}(a_1,\ldots,a_r) \\
&\qquad \times E_l(-x|\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s) &(15)\\
&\quad = \sum_{j=0}^{n}\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\widehat{\mathrm{NS}}_{n-l}x^j &(16)\\
&\quad = \sum_{l=0}^{n}\binom{n}{l}\hat{s}_{n-l}(\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\widehat{N}_l(x|a_1,\ldots,a_r), &(17)\\
&\quad = \sum_{l=0}^{n}\binom{n}{l}\widehat{N}_{n-l}(a_1,\ldots,a_r)\hat{s}_l(x|\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s). &(18)
\end{aligned}$$

Proof Since

$$\prod_{i=1}^{r}\left(\frac{te^{a_it}}{e^{a_it}-1}\right)\prod_{j=1}^{s}\left(\frac{1+e^{\lambda_jt}}{e^{\lambda_jt}}\right)^{\mu_j}\widehat{\mathrm{NS}}_n(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s) \sim \left(1, e^t-1\right) \quad (19)$$

and

$$(x)_n \sim \left(1, e^t-1\right), \quad (20)$$

we have

$$\begin{aligned}
&\widehat{\mathrm{NS}}_n(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\\
&\quad=\prod_{i=1}^{r}\left(\frac{e^{a_it}-1}{te^{a_it}}\right)\prod_{j=1}^{s}\left(\frac{e^{\lambda_jt}}{1+e^{\lambda_jt}}\right)^{\mu_j}(x)_n\\
&\quad=\sum_{m=0}^{n}S_1(n,m)\prod_{i=1}^{r}\left(\frac{e^{a_it}-1}{te^{a_it}}\right)\prod_{j=1}^{s}\left(\frac{e^{\lambda_jt}}{1+e^{\lambda_jt}}\right)^{\mu_j}x^m\\
&\quad=2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}S_1(n,m)\prod_{i=1}^{r}\left(\frac{e^{-a_it}-1}{-t}\right)\prod_{j=1}^{s}\left(\frac{2}{1+e^{-\lambda_jt}}\right)^{\mu_j}x^m\\
&\quad=2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}S_1(n,m)\prod_{i=1}^{r}\left(\frac{e^{-a_it}-1}{-t}\right)(-1)^mE_m(-x|\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\\
&\quad=2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}S_1(n,m)\prod_{i=1}^{r}\left(\frac{e^{-a_it}-1}{-t}\right)(-1)^m\\
&\qquad\times\sum_{l=0}^{m}\binom{m}{l}E_{m-l}(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)(-x)^l\\
&\quad=2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}(-1)^mS_1(n,m)\\
&\qquad\times\sum_{l=0}^{m}(-1)^l\binom{m}{l}E_{m-l}(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\prod_{i=1}^{r}\left(\frac{e^{-a_it}-1}{-t}\right)x^l\\
&\quad=2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}(-1)^mS_1(n,m)\\
&\qquad\times\sum_{l=0}^{m}(-1)^l\binom{m}{l}E_{m-l}(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)(-1)^lF_l(-x|a_1,\ldots,a_r).
\end{aligned}$$

So, we get (13).

We can obtain an alternative expression (14) for $\widehat{\mathrm{NS}}_n(x)$ as follows:

$$\begin{aligned}
\widehat{\mathrm{NS}}_n(x)&=2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}S_1(n,m)e^{(\sum_{j=1}^{s}\lambda_j\mu_j-\sum_{i=1}^{r}a_i)t}\\
&\quad\times\prod_{i=1}^{r}\left(\frac{e^{a_it}-1}{t}\right)\prod_{j=1}^{s}\left(\frac{2}{1+e^{\lambda_jt}}\right)^{\mu_j}x^m\\
&=2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}S_1(n,m)e^{(\sum_{j=1}^{s}\lambda_j\mu_j-\sum_{i=1}^{r}a_i)t}\\
&\quad\times\prod_{i=1}^{r}\left(\frac{e^{a_it}-1}{t}\right)E_m(x|\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\\
&=2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}S_1(n,m)e^{(\sum_{j=1}^{s}\lambda_j\mu_j-\sum_{i=1}^{r}a_i)t}\\
&\quad\times\prod_{i=1}^{r}\left(\frac{e^{a_it}-1}{t}\right)\sum_{l=0}^{m}\binom{m}{l}E_{m-l}(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)x^l
\end{aligned}$$

$$
\begin{aligned}
&= 2^{-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n} \sum_{l=0}^{m} \binom{m}{l} S_1(n,m) E_{m-l}(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s) \\
&\quad \times e^{(\sum_{j=1}^{s}\lambda_j\mu_j - \sum_{i=1}^{r} a_i)t} \prod_{i=1}^{r}\left(\frac{e^{a_i t}-1}{t}\right) x^l \\
&= 2^{-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n} \sum_{l=0}^{m} \binom{m}{l} S_1(n,m) E_{m-l}(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s) \\
&\quad \times F_l\left(x + \sum_{j=1}^{s}\lambda_j\mu_j - \sum_{i=1}^{r} a_i \middle| a_1,\ldots,a_r\right).
\end{aligned}
$$

From the proof of (13),

$$
\begin{aligned}
&\widehat{\mathrm{NS}}_n(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s) \\
&\quad = 2^{-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n} S_1(n,m) \prod_{j=1}^{s}\left(\frac{2}{1+e^{-\lambda_j t}}\right)^{\mu_j} \prod_{i=1}^{r}\left(\frac{e^{-a_i t}-1}{-t}\right) x^m \\
&\quad = 2^{-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n} S_1(n,m) \prod_{j=1}^{s}\left(\frac{2}{1+e^{-\lambda_j t}}\right)^{\mu_j} (-1)^m F_m(-x|a_1,\ldots,a_r) \\
&\quad = 2^{-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n} (-1)^m S_1(n,m) \prod_{j=1}^{s}\left(\frac{2}{1+e^{-\lambda_j t}}\right)^{\mu_j} \sum_{l=0}^{m}\binom{m}{l} F_{m-l}(a_1,\ldots,a_r)(-x)^l \\
&\quad = 2^{-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n} (-1)^m S_1(n,m) \sum_{l=0}^{m}\binom{m}{l} F_{m-l}(a_1,\ldots,a_r)(-1)^l \prod_{j=1}^{s}\left(\frac{2}{1+e^{-\lambda_j t}}\right)^{\mu_j} x^l \\
&\quad = 2^{-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n} (-1)^m S_1(n,m) \sum_{l=0}^{m}\binom{m}{l} F_{m-l}(a_1,\ldots,a_r) E_l(-x|\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s) \\
&\quad = 2^{-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n} \sum_{l=0}^{m} (-1)^m \binom{m}{l} S_1(n,m) F_{m-l}(a_1,\ldots,a_r) E_l(-x|\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s),
\end{aligned}
$$

which is the identity (15).

By (12), we have

$$
\begin{aligned}
&\left\langle g(\bar{f}(t))^{-1}\bar{f}(t)^j \middle| x^n\right\rangle \\
&\quad = \left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right) \prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \left(\ln(1+t)\right)^j \middle| x^n \right\rangle \\
&\quad = \left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right) \prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \middle| j!\sum_{l=j}^{\infty} S_1(l,j)\frac{t^l}{l!} x^n \right\rangle \\
&\quad = j!\sum_{l=j}^{n}\binom{n}{l} S_1(l,j) \left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right) \prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \middle| x^{n-l} \right\rangle \\
&\quad = j!\sum_{l=j}^{n}\binom{n}{l} S_1(l,j) \left\langle \sum_{i=0}^{\infty} \widehat{\mathrm{NS}}_i \frac{t^i}{i!} \middle| x^{n-l} \right\rangle = j!\sum_{l=j}^{n}\binom{n}{l} S_1(l,j)\widehat{\mathrm{NS}}_{n-l}.
\end{aligned}
$$

By (9), we get the identity (16).

Next,

$$\begin{aligned}
&\widehat{\mathrm{NS}}_n(y|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s) \\
&\quad = \left\langle \sum_{i=0}^{\infty} \widehat{\mathrm{NS}}_i(y|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\frac{t^i}{i!} \middle| x^n \right\rangle \\
&\quad = \left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right) \prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}(1+t)^y \middle| x^n \right\rangle \\
&\quad = \left\langle \prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \middle| \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)(1+t)^y x^n \right\rangle \\
&\quad = \left\langle \prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \middle| \sum_{l=0}^{\infty} \widehat{N}_l(y|a_1,\ldots,a_r)\frac{t^l}{l!} x^n \right\rangle \\
&\quad = \sum_{l=0}^{n}\binom{n}{l}\widehat{N}_l(y|a_1,\ldots,a_r)\left\langle \prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \middle| x^{n-l} \right\rangle \\
&\quad = \sum_{l=0}^{n}\binom{n}{l}\widehat{N}_l(y|a_1,\ldots,a_r)\left\langle \sum_{i=0}^{\infty} \hat{s}_i(\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\frac{t^i}{i!} \middle| x^{n-l} \right\rangle \\
&\quad = \sum_{l=0}^{n}\binom{n}{l}\widehat{N}_l(y|a_1,\ldots,a_r)\hat{s}_{n-l}(\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r).
\end{aligned}$$

Thus, we obtain (17).

Finally, we obtain

$$\begin{aligned}
&\widehat{\mathrm{NS}}_n(y|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s) \\
&\quad = \left\langle \sum_{i=0}^{\infty} \widehat{\mathrm{NS}}_i(y|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\frac{t^i}{i!} \middle| x^n \right\rangle \\
&\quad = \left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right) \prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}(1+t)^y \middle| x^n \right\rangle \\
&\quad = \left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right) \middle| \prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}(1+t)^y x^n \right\rangle \\
&\quad = \left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right) \middle| \sum_{l=0}^{\infty} \hat{s}_l(y|\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\frac{t^l}{l!} x^n \right\rangle \\
&\quad = \sum_{l=0}^{n}\hat{s}_l(y|\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\binom{n}{l}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right) \middle| x^{n-l} \right\rangle \\
&\quad = \sum_{l=0}^{n}\hat{s}_l(y|\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\binom{n}{l}\left\langle \sum_{i=0}^{\infty} \widehat{N}_i(a_1,\ldots,a_r)\frac{t^i}{i!} \middle| x^{n-l} \right\rangle \\
&\quad = \sum_{l=0}^{n}\binom{n}{l}\hat{s}_l(y|\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\widehat{N}_{n-l}(a_1,\ldots,a_r).
\end{aligned}$$

Thus, we get the identity (18). □

3.2 Sheffer identity

Theorem 2

$$\begin{aligned}&\widehat{\mathrm{NS}}_n(x+y|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\\&\quad=\sum_{l=0}^{n}\binom{n}{l}\widehat{\mathrm{NS}}_l(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)(y)_{n-l}.\end{aligned}\tag{21}$$

Proof By (12) with

$$\begin{aligned}p_n(x)&=\prod_{i=1}^{r}\left(\frac{te^{a_it}}{e^{a_it}-1}\right)\prod_{j=1}^{s}\left(\frac{1+e^{\lambda_jt}}{e^{\lambda_jt}}\right)^{\mu_j}\widehat{\mathrm{NS}}_n(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\\&=(x)_n\sim\left(1,e^t-1\right),\end{aligned}$$

using (10), we have (21). □

3.3 Difference relations

Theorem 3

$$\begin{aligned}&\widehat{\mathrm{NS}}_n(x+1|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)-\widehat{\mathrm{NS}}_n(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\\&\quad=n\widehat{\mathrm{NS}}_{n-1}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s).\end{aligned}\tag{22}$$

Proof By (8) with (12), we get

$$\begin{aligned}&\left(e^t-1\right)\widehat{\mathrm{NS}}_n(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\\&\quad=n\widehat{\mathrm{NS}}_{n-1}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s).\end{aligned}$$

By (7), we have (22). □

3.4 Recurrence

Theorem 4

$$\begin{aligned}&\widehat{\mathrm{NS}}_{n+1}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\\&\quad=\left(x+\sum_{j=1}^{s}\mu_j\lambda_j-\sum_{i=1}^{r}a_i\right)\widehat{\mathrm{NS}}_n(x-1|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\\&\qquad+2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\frac{(-1)^{m+1}\binom{m}{l}}{l+1}S_1(n,m)E_{m-l}(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\\&\qquad\times\left(\sum_{i=1}^{r}a_iF_{l+1}(1-x|a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_r)-rF_{l+1}(1-x|a_1,\ldots,a_r)\right)\\&\qquad+2^{-1-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}(-1)^{m+1}\binom{m}{l}S_1(n,m)F_{m-l}(a_1,\ldots,a_r)\\&\qquad\times\sum_{i=1}^{s}\mu_i\lambda_iE_l(1-x|\lambda;\mu+e_i).\end{aligned}\tag{23}$$

Proof By applying

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x) \tag{24}$$

[1, Corollary 3.7.2] with (12), we get

$$\widehat{\mathrm{NS}}_{n+1}(x) = x\widehat{\mathrm{NS}}_n(x-1) - e^{-t}\frac{g'(t)}{g(t)}\widehat{\mathrm{NS}}_n(x).$$

Observe that

$$\begin{aligned}\frac{g'(t)}{g(t)} &= \bigl(\ln g(t)\bigr)' \\ &= \left(r\ln t + \left(\sum_{i=1}^{r} a_i\right)t - \sum_{i=1}^{r}\ln\bigl(e^{a_i t}-1\bigr) + \sum_{j=1}^{s}\mu_j\ln\bigl(1+e^{\lambda_j t}\bigr) - \left(\sum_{j=1}^{s}\mu_j\lambda_j\right)t\right)' \\ &= \frac{r}{t} + \sum_{i=1}^{r} a_i - \sum_{i=1}^{r}\frac{a_i e^{a_i t}}{e^{a_i t}-1} + \sum_{j=1}^{s}\frac{\mu_j\lambda_j e^{\lambda_j t}}{1+e^{\lambda_j t}} - \sum_{j=1}^{s}\mu_j\lambda_j \\ &= \frac{r - \sum_{i=1}^{r}\frac{a_i t e^{a_i t}}{e^{a_i t}-1}}{t} + \sum_{i=1}^{r} a_i + \sum_{j=1}^{s}\frac{\mu_j\lambda_j e^{\lambda_j t}}{1+e^{\lambda_j t}} - \sum_{j=1}^{s}\mu_j\lambda_j,\end{aligned}$$

where

$$r - \sum_{i=1}^{r}\frac{a_i t e^{a_i t}}{e^{a_i t}-1} = -\frac{1}{2}\left(\sum_{i=1}^{r} a_i\right)t + \cdots$$

has the order at least one. Since from the proofs of (13) and (15)

$$\begin{aligned}\widehat{\mathrm{NS}}_n(x) &= 2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}(-1)^m S_1(n,m) \\ &\quad\times\sum_{l=0}^{m}(-1)^l\binom{m}{l}E_{m-l}(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\prod_{i=1}^{r}\left(\frac{e^{-a_i t}-1}{-t}\right)x^l \\ &= 2^{-\sum_{j=1}^{s}\mu_j}\sum_{m=0}^{n}(-1)^m S_1(n,m) \\ &\quad\times\sum_{l=0}^{m}\binom{m}{l}F_{m-l}(a_1,\ldots,a_r)(-1)^l\prod_{j=1}^{s}\left(\frac{2}{1+e^{-\lambda_j t}}\right)^{\mu_j}x^l,\end{aligned}$$

we have

$$\begin{aligned}&\frac{g'(t)}{g(t)}\widehat{\mathrm{NS}}_n(x) \\ &\quad= \left(\sum_{i=1}^{r} a_i - \sum_{j=1}^{s}\mu_j\lambda_j\right)\widehat{\mathrm{NS}}_n(x)\end{aligned}$$

$$+ 2^{-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n} (-1)^m S_1(n,m) \sum_{l=0}^{m} (-1)^l \binom{m}{l} E_{m-l}(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)$$

$$\times \prod_{i=1}^{r} \left(\frac{e^{-a_i t}-1}{-t}\right) \frac{r-\sum_{i=1}^{r}\frac{a_i t}{1-e^{-a_i t}}}{t} x^l$$

$$+ 2^{-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n} (-1)^m S_1(n,m) \sum_{l=0}^{m} \binom{m}{l} F_{m-l}(a_1,\ldots,a_r)(-1)^l$$

$$\times \sum_{i=1}^{s} \frac{\mu_i\lambda_i}{1+e^{-\lambda_i t}} \prod_{j=1}^{s} \left(\frac{2}{1+e^{-\lambda_j t}}\right)^{\mu_j} x^l.$$

The third term is equal to

$$2^{-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n} \sum_{l=0}^{m} (-1)^m \binom{m}{l} S_1(n,m) F_{m-l}(a_1,\ldots,a_r)(-1)^l$$

$$\times \sum_{i=1}^{s} \frac{\mu_i\lambda_i}{2} (-1)^l E_l(-x|\lambda;\mu+e_i)$$

$$= 2^{-1-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n} \sum_{l=0}^{m} (-1)^m \binom{m}{l} S_1(n,m) F_{m-l}(a_1,\ldots,a_r)$$

$$\times \sum_{i=1}^{s} \mu_i\lambda_i E_l(-x|\lambda;\mu+e_i),$$

where $\lambda = (\lambda_1,\ldots,\lambda_s)$, $\mu = (\mu_1,\ldots,\mu_s)$, and $e_i = (\underbrace{0,\ldots,0}_{i-1},1,\underbrace{0,\ldots,0}_{s-i})$. The second term is

$$2^{-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n} (-1)^m S_1(n,m) \sum_{l=0}^{m} \frac{(-1)^l}{l+1} \binom{m}{l} E_{m-l}(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)$$

$$\times \prod_{i=1}^{r} \left(\frac{e^{-a_i t}-1}{-t}\right) \left(r - \sum_{i=1}^{r} \frac{a_i t}{1-e^{-a_i t}}\right) x^{l+1}$$

$$= r2^{-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n} (-1)^m S_1(n,m) \sum_{l=0}^{m} \frac{(-1)^l}{l+1} \binom{m}{l} E_{m-l}(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)$$

$$\times \prod_{i=1}^{r} \left(\frac{e^{-a_i t}-1}{-t}\right) x^{l+1}$$

$$- 2^{-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n} (-1)^m S_1(n,m) \sum_{l=0}^{m} \frac{(-1)^l}{l+1} \binom{m}{l} E_{m-l}(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)$$

$$\times \sum_{i=1}^{r} a_i \frac{-t}{e^{-a_i t}-1} \prod_{i=1}^{r} \left(\frac{e^{-a_i t}-1}{-t}\right) x^{l+1}$$

$$= r2^{-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n} (-1)^m S_1(n,m) \sum_{l=0}^{m} \frac{(-1)^l}{l+1} \binom{m}{l} E_{m-l}(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)$$

$$\times (-1)^{l+1} F_{l+1}(-x|a_1,\ldots,a_r)$$

$$- 2^{-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n} (-1)^m S_1(n,m) \sum_{l=0}^{m} \frac{(-1)^l}{l+1} \binom{m}{l} E_{m-l}(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)$$

$$\times \sum_{i=1}^{r} a_i(-1)^{l+1} F_{l+1}(-x|a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_r)$$
$$= 2^{-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n}\sum_{l=0}^{m} \frac{(-1)^{m+1}\binom{m}{l}}{l+1} S_1(n,m) E_{m-l}(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)$$
$$\times \left(rF_{l+1}(-x|a_1,\ldots,a_r) - \sum_{i=1}^{r} a_i F_{l+1}(-x|a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_r)\right).$$

Therefore, we obtain

$$\begin{aligned}
&\widehat{\mathrm{NS}}_{n+1}(x) \\
&= \left(x + \sum_{j=1}^{s}\mu_j\lambda_j - \sum_{i=1}^{r} a_i\right)\widehat{\mathrm{NS}}_n(x-1) \\
&\quad + 2^{-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n}\sum_{l=0}^{m} \frac{(-1)^{m+1}\binom{m}{l}}{l+1} S_1(n,m) E_{m-l}(\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s) \\
&\quad \times \left(\sum_{i=1}^{r} a_i F_{l+1}(1-x|a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_r) - rF_{l+1}(1-x|a_1,\ldots,a_r)\right) \\
&\quad + 2^{-1-\sum_{j=1}^{s}\mu_j} \sum_{m=0}^{n}\sum_{l=0}^{m} (-1)^{m+1}\binom{m}{l} S_1(n,m) F_{m-l}(a_1,\ldots,a_r) \\
&\quad \times \sum_{i=1}^{s} \mu_i\lambda_i E_l(1-x|\lambda;\mu+e_i),
\end{aligned}$$

which is the identity (23). □

3.5 Differentiation

Theorem 5

$$\begin{aligned}
&\frac{d}{dx}\widehat{\mathrm{NS}}_n(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s) \\
&= n!\sum_{l=0}^{n-1} \frac{(-1)^{n-l-1}}{l!(n-l)} \widehat{\mathrm{NS}}_l(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s). \qquad (25)
\end{aligned}$$

Proof We shall use

$$\frac{d}{dx}s_n(x) = \sum_{l=0}^{n-1}\binom{n}{l}\langle \bar{f}(t)|x^{n-l}\rangle s_l(x)$$

(*cf.* [1, Theorem 2.3.12]). Since

$$\begin{aligned}
\langle \bar{f}(t)|x^{n-l}\rangle &= \langle \ln(1+t)|x^{n-l}\rangle \\
&= \left\langle \sum_{m=1}^{\infty} \frac{(-1)^{m-1}t^m}{m} \middle| x^{n-l}\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=1}^{n-l} \frac{(-1)^{m-1}}{m} \langle t^m | x^{n-l} \rangle \\
&= \sum_{m=1}^{n-l} \frac{(-1)^{m-1}}{m} (n-l)! \delta_{m,n-l} \\
&= (-1)^{n-l-1}(n-l-1)!
\end{aligned}
$$

with (12), we have

$$
\begin{aligned}
&\frac{d}{dx} \widehat{\mathrm{NS}}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s) \\
&\quad = \sum_{l=0}^{n-1} \binom{n}{l} (-1)^{n-l-1}(n-l-1)! \widehat{\mathrm{NS}}_l(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s) \\
&\quad = n! \sum_{l=0}^{n-1} \frac{(-1)^{n-l-1}}{l!(n-l)} \widehat{\mathrm{NS}}_l(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s),
\end{aligned}
$$

which is the identity (25). □

3.6 One more relation

The classical Cauchy numbers of the first kind c_n are defined by

$$
\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}
$$

(see *e.g.* [3, 4]).

Theorem 6

$$
\begin{aligned}
&\widehat{\mathrm{NS}}_n(x|a;\lambda;\mu) \\
&\quad = \left(x - \sum_{i=1}^{r} a_i\right) \widehat{\mathrm{NS}}_{n-1}(x-1|a;\lambda;\mu) + \sum_{j=1}^{s} \lambda_j \mu_j \widehat{\mathrm{NS}}_{n-1}(x-\lambda_j-1|a;\lambda;\mu+e_j) \\
&\qquad + \frac{1}{n} \sum_{l=0}^{n} \binom{n}{l} c_l \left(\sum_{i=1}^{r} a_i \widehat{\mathrm{NS}}_{n-l}(x-1|a_1,\dots,a_{i-1},a_{i+1},\dots,a_r;\lambda;\mu) \right. \\
&\qquad \left. - r\widehat{\mathrm{NS}}_{n-l}(x-1|a;\lambda;\mu) \right). \qquad (26)
\end{aligned}
$$

Proof For $n \ge 1$, we have

$$
\begin{aligned}
&\widehat{\mathrm{NS}}_n(y|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s) \\
&\quad = \left\langle \sum_{l=0}^{\infty} \widehat{\mathrm{NS}}_l(y|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s) \frac{t^l}{l!} \middle| x^n \right\rangle \\
&\quad = \left\langle \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)} \right) \prod_{j=1}^{s} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^y \middle| x^n \right\rangle
\end{aligned}
$$

$$= \left\langle \partial_t \left(\prod_{i=1}^{r} \left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)} \right) \prod_{j=1}^{s} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^y \right) \middle| x^{n-1} \right\rangle$$

$$= \left\langle \left(\partial_t \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)} \right) \right) \prod_{j=1}^{s} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^y \middle| x^{n-1} \right\rangle$$

$$+ \left\langle \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)} \right) \left(\partial_t \prod_{j=1}^{s} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \right) (1+t)^y \middle| x^{n-1} \right\rangle$$

$$+ \left\langle \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)} \right) \prod_{j=1}^{s} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \left(\partial_t (1+t)^y \right) \middle| x^{n-1} \right\rangle.$$

The third term is

$$y \left\langle \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)} \right) \prod_{j=1}^{s} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^{y-1} \middle| x^{n-1} \right\rangle$$

$$= y\widehat{\mathrm{NS}}_{n-1}(y-1|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s).$$

Since

$$\partial_t \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)} \right)$$

$$= \frac{1}{1+t} \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)} \right) \frac{\sum_{v=1}^{r} \left(\frac{a_v t(1+t)^{a_v}}{(1+t)^{a_v}-1} - \frac{t}{\ln(1+t)} \right)}{t}$$

$$- \frac{1}{1+t} \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)} \right) \left(\sum_{l=1}^{r} a_l \right),$$

with

$$\sum_{v=1}^{r} \left(\frac{a_v t(1+t)^{a_v}}{(1+t)^{a_v}-1} - \frac{t}{\ln(1+t)} \right)$$

having order ≥ 1, the first term is

$$\left\langle \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)} \right) \prod_{j=1}^{s} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^{y-1} \middle| \frac{\sum_{v=1}^{r} \left(\frac{a_v t(1+t)^{a_v}}{(1+t)^{a_v}-1} - \frac{t}{\ln(1+t)} \right)}{t} x^{n-1} \right\rangle$$

$$- \left(\sum_{l=1}^{r} a_l \right) \left\langle \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)} \right) \prod_{j=1}^{s} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^{y-1} \middle| x^{n-1} \right\rangle$$

$$= -\left(\sum_{l=1}^{r} a_l \right) \widehat{\mathrm{NS}}_{n-1}(y-1) + \frac{1}{n} \left\langle \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)} \right) \prod_{j=1}^{s} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \right.$$

$$\left. \times (1+t)^{y-1} \middle| \sum_{v=1}^{r} \left(\frac{a_v t(1+t)^{a_v}}{(1+t)^{a_v}-1} - \frac{t}{\ln(1+t)} \right) x^n \right\rangle$$

$$= -\left(\sum_{l=1}^{r} a_l \right) \widehat{\mathrm{NS}}_{n-1}(y-1)$$

$$
\begin{aligned}
&+\frac{1}{n}\Bigg(\sum_{\nu=1}^{r}a_\nu\Bigg\langle\frac{(1+t)^{a_\nu}\ln(1+t)}{(1+t)^{a_\nu}-1}\prod_{i=1}^{r}\bigg(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\bigg)\prod_{j=1}^{s}\bigg(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\bigg)^{\mu_j}\\
&\times(1+t)^{y-1}\Bigg|\frac{t}{\ln(1+t)}x^n\Bigg\rangle\\
&-r\Bigg\langle\prod_{i=1}^{r}\bigg(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\bigg)\prod_{j=1}^{s}\bigg(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\bigg)^{\mu_j}(1+t)^{y-1}\Bigg|\frac{t}{\ln(1+t)}x^n\Bigg\rangle\Bigg)\\
={}&-\Bigg(\sum_{l=1}^{r}a_l\Bigg)\widehat{\mathrm{NS}}_{n-1}(y-1)\\
&+\frac{1}{n}\Bigg(\sum_{\nu=1}^{r}a_\nu\Bigg\langle\frac{(1+t)^{a_\nu}\ln(1+t)}{(1+t)^{a_\nu}-1}\prod_{i=1}^{r}\bigg(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\bigg)\prod_{j=1}^{s}\bigg(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\bigg)^{\mu_j}\\
&\times(1+t)^{y-1}\Bigg|\sum_{l=0}^{\infty}c_l\frac{t^l}{l!}x^n\Bigg\rangle\\
&-r\Bigg\langle\prod_{i=1}^{r}\bigg(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\bigg)\prod_{j=1}^{s}\bigg(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\bigg)^{\mu_j}(1+t)^{y-1}\Bigg|\sum_{l=0}^{\infty}c_l\frac{t^l}{l!}x^n\Bigg\rangle\Bigg)\\
={}&-\Bigg(\sum_{l=1}^{r}a_l\Bigg)\widehat{\mathrm{NS}}_{n-1}(y-1)\\
&+\frac{1}{n}\Bigg(\sum_{\nu=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_\nu c_l\widehat{\mathrm{NS}}_{n-l}(y-1|a_1,\ldots,a_{\nu-1},a_{\nu+1},\ldots,a_r;\lambda;\mu)\\
&-r\sum_{l=0}^{n}\binom{n}{l}c_l\widehat{\mathrm{NS}}_{n-l}(y-1|a;\lambda;\mu)\Bigg)\\
={}&-\Bigg(\sum_{\nu=1}^{r}a_\nu\Bigg)\widehat{\mathrm{NS}}_{n-1}(y-1)\\
&+\frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}c_l\Bigg(\sum_{\nu=1}^{r}a_\nu\widehat{\mathrm{NS}}_{n-l}(y-1|a_1,\ldots,a_{\nu-1},a_{\nu+1},\ldots,a_r;\lambda;\mu)\\
&-r\widehat{\mathrm{NS}}_{n-l}(y-1|a;\lambda;\mu)\Bigg).
\end{aligned}
$$

Since

$$
\partial_t\prod_{j=1}^{s}\bigg(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\bigg)^{\mu_j}=\sum_{l=1}^{s}\lambda_l\mu_l(1+t)^{-\lambda_l-1}\bigg(\frac{(1+t)^{\lambda_l}}{1+(1+t)^{\lambda_l}}\bigg)\prod_{j=1}^{s}\bigg(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\bigg)^{\mu_j},
$$

the second term is

$$
\begin{aligned}
&\sum_{l=1}^{s}\lambda_l\mu_l\Bigg\langle\prod_{i=1}^{r}\bigg(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\bigg)\bigg(\frac{(1+t)^{\lambda_l}}{1+(1+t)^{\lambda_l}}\bigg)\prod_{j=1}^{s}\bigg(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\bigg)^{\mu_j}(1+t)^{y-\lambda_l-1}\Bigg|x^{n-1}\Bigg\rangle\\
&=\sum_{l=1}^{s}\lambda_l\mu_l\widehat{\mathrm{NS}}_{n-1}(y-\lambda_l-1|a;\lambda;\mu+e_l).
\end{aligned}
$$

Therefore, we obtain

$$\begin{aligned}
&\widehat{\mathrm{NS}}_n(x|a;\lambda;\mu) \\
&= \left(x-\sum_{i=1}^{r}a_i\right)\widehat{\mathrm{NS}}_{n-1}(x-1|a;\lambda;\mu)+\sum_{j=1}^{s}\lambda_j\mu_j\widehat{\mathrm{NS}}_{n-1}(x-\lambda_j-1|a;\lambda;\mu+e_j) \\
&\quad+\frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}c_l\left(\sum_{i=1}^{r}a_i\widehat{\mathrm{NS}}_{n-l}(x-1|a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_r;\lambda;\mu)\right. \\
&\quad\left.-r\widehat{\mathrm{NS}}_{n-l}(x-1|a;\lambda;\mu)\right),
\end{aligned}$$

which is the identity (26). □

3.7 A relation involving the Stirling numbers of the first kind

Theorem 7 *For $n-1\geq m\geq 1$, we have*

$$\begin{aligned}
&\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{\mathrm{NS}}_l(a;\lambda;\mu) \\
&= -\left(\sum_{i=1}^{r}a_i\right)\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m)\widehat{\mathrm{NS}}_l(-1|a;\lambda;\mu) \\
&\quad+\frac{1}{n}\sum_{k=0}^{n-m}\sum_{l=k}^{n-m}\binom{n}{l}\binom{l}{k}S_1(n-l,m)c_{l-k} \\
&\quad\times\left(\sum_{i=1}^{r}a_i\widehat{\mathrm{NS}}_k(-1|a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_r;\lambda;\mu)-r\widehat{\mathrm{NS}}_k(-1|a;\lambda;\mu)\right) \\
&\quad+\sum_{j=1}^{s}\lambda_j\mu_j\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m)\widehat{\mathrm{NS}}_l(-\lambda_j-1|a;\lambda;\mu+e_j) \\
&\quad+\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{\mathrm{NS}}_l(-1|a;\lambda;\mu). \qquad (27)
\end{aligned}$$

Proof We shall compute

$$\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\left(\ln(1+t)\right)^m\middle|x^n\right\rangle$$

in two different ways. On the one hand, it is equal to

$$\begin{aligned}
&\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\middle|\left(\ln(1+t)\right)^m x^n\right\rangle \\
&= \left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\middle|m!\sum_{l=m}^{\infty}S_1(l,m)\frac{t^l}{l!}x^n\right\rangle \\
&= m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\middle|x^{n-l}\right\rangle
\end{aligned}$$

$$= m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\left\langle \sum_{i=0}^{\infty}\widehat{\mathrm{NS}}_i(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)\frac{t^i}{i!}\middle| x^{n-l}\right\rangle$$

$$= m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\widehat{\mathrm{NS}}_{n-l}(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)$$

$$= m!\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{\mathrm{NS}}_l(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s).$$

On the other hand, it is equal to

$$\left\langle \partial_t\left(\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\left(\ln(1+t)\right)^m\right)\middle| x^{n-1}\right\rangle$$

$$= \left\langle\left(\partial_t\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\left(\ln(1+t)\right)^m\middle| x^{n-1}\right\rangle$$

$$+ \left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\left(\partial_t\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\right)\left(\ln(1+t)\right)^m\middle| x^{n-1}\right\rangle$$

$$+ \left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\left(\partial_t\left(\ln(1+t)\right)^m\right)\middle| x^{n-1}\right\rangle. \tag{28}$$

The third term of (28) is equal to

$$m\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}(1+t)^{-1}\middle|\left(\ln(1+t)\right)^{m-1}x^{n-1}\right\rangle$$

$$= m\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\right.$$

$$\left.\times (1+t)^{-1}\middle|(m-1)!\sum_{l=m-1}^{\infty}S_1(l,m-1)\frac{t^l}{l!}x^{n-1}\right\rangle$$

$$= m!\sum_{l=m-1}^{n-1}\binom{n-1}{l}S_1(l,m-1)$$

$$\times\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}(1+t)^{-1}\middle| x^{n-1-l}\right\rangle$$

$$= m!\sum_{l=m-1}^{n-1}\binom{n-1}{l}S_1(l,m-1)$$

$$\times\widehat{\mathrm{NS}}_{n-1-l}(-1|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)$$

$$= m!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)$$

$$\times\widehat{\mathrm{NS}}_l(-1|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s).$$

Since

$$\begin{aligned}
&\partial_t \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right) \\
&= \frac{1}{1+t}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\frac{\sum_{\nu=1}^{r}\left(\frac{a_\nu t(1+t)^{a_\nu}}{(1+t)^{a_\nu}-1}-\frac{t}{\ln(1+t)}\right)}{t} \\
&\quad - \frac{1}{1+t}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\left(\sum_{\nu=1}^{r}a_\nu\right),
\end{aligned}$$

the first term of (28) is equal to

$$\begin{aligned}
&\frac{1}{n}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\right. \\
&\quad\left. \times (1+t)^{-1}\sum_{\nu=1}^{r}\left(\frac{a_\nu t(1+t)^{a_\nu}}{(1+t)^{a_\nu}-1}-\frac{t}{\ln(1+t)}\right)\middle| \big(\ln(1+t)\big)^m x^n\right\rangle \\
&\quad - \left(\sum_{\nu=1}^{r}a_\nu\right)\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\right. \\
&\quad\left. \times (1+t)^{-1}\middle| \big(\ln(1+t)\big)^m x^{n-1}\right\rangle \\
&= \frac{1}{n}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}(1+t)^{-1}\right. \\
&\quad\left. \times \sum_{\nu=1}^{r}\left(\frac{a_\nu t(1+t)^{a_\nu}}{(1+t)^{a_\nu}-1}-\frac{t}{\ln(1+t)}\right)\middle| m!\sum_{l=m}^{\infty}S_1(l,m)\frac{t^l}{l!}x^n\right\rangle \\
&\quad - \left(\sum_{\nu=1}^{r}a_\nu\right)\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\right. \\
&\quad\left. \times (1+t)^{-1}\middle| m!\sum_{l=m}^{\infty}S_1(l,m)\frac{t^l}{l!}x^{n-1}\right\rangle \\
&= \frac{m!}{n}\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\right. \\
&\quad\left. \times (1+t)^{-1}\sum_{\nu=1}^{r}\left(\frac{a_\nu t(1+t)^{a_\nu}}{(1+t)^{a_\nu}-1}-\frac{t}{\ln(1+t)}\right)\middle| x^{n-l}\right\rangle \\
&\quad - m!\left(\sum_{\nu=1}^{r}a_\nu\right)\sum_{l=m}^{n-1}\binom{n-1}{l}S_1(l,m) \\
&\quad \times \left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}(1+t)^{-1}\middle| x^{n-l-1}\right\rangle \\
&= -m!\left(\sum_{\nu=1}^{r}a_\nu\right)\sum_{l=m}^{n-1}\binom{n-1}{l}S_1(l,m)\widehat{\mathrm{NS}}_{n-l-1}(-1)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{m!}{n}\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\Biggl(\Biggl(\sum_{v=1}^{r}a_v\Biggr)\Biggl\langle\frac{(1+t)^{a_v}\ln(1+t)}{(1+t)^{a_v}-1}\prod_{i=1}^{r}\biggl(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\biggr) \\
&\times\prod_{j=1}^{s}\biggl(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\biggr)^{\mu_j}(1+t)^{-1}\biggl|\frac{t}{\ln(1+t)}x^{n-l}\Biggr\rangle \\
&-r\Biggl\langle\prod_{i=1}^{r}\biggl(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\biggr)\prod_{j=1}^{s}\biggl(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\biggr)^{\mu_j}(1+t)^{-1}\biggl|\frac{t}{\ln(1+t)}x^{n-l}\Biggr\rangle\Biggr) \\
={}&-m!\Biggl(\sum_{v=1}^{r}a_v\Biggr)\sum_{l=m}^{n-1}\binom{n-1}{l}S_1(l,m)\widehat{\mathrm{NS}}_{n-l-1}(-1) \\
&+\frac{m!}{n}\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\sum_{k=0}^{n-l}\binom{n-l}{k}c_k \\
&\times\Biggl(\Biggl(\sum_{v=1}^{r}a_v\Biggr)\widehat{\mathrm{NS}}_{n-l-k}(-1|a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_r;\lambda;\mu)-r\widehat{\mathrm{NS}}_{n-l-k}(-1|a;\lambda;\mu)\Biggr) \\
={}&-m!\Biggl(\sum_{v=1}^{r}a_v\Biggr)\sum_{l=0}^{n-1-m}\binom{n-1}{l}S_1(n-l-1,m)\widehat{\mathrm{NS}}_{l}(-1) \\
&+\frac{m!}{n}\sum_{k=0}^{n-m}\sum_{l=k}^{n-m}\binom{n}{l}\binom{l}{k}S_1(n-l,m)c_{l-k} \\
&\times\Biggl(\Biggl(\sum_{v=1}^{r}a_v\Biggr)\widehat{\mathrm{NS}}_{k}(-1|a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_r;\lambda;\mu)-r\widehat{\mathrm{NS}}_{k}(-1|a;\lambda;\mu)\Biggr).
\end{aligned}
$$

The second term of (28) is equal to

$$
\begin{aligned}
&\sum_{v=1}^{s}\lambda_v\mu_v\Biggl\langle\prod_{i=1}^{r}\biggl(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\biggr)\biggl(\frac{(1+t)^{\lambda_v}}{1+(1+t)^{\lambda_v}}\biggr) \\
&\quad\times\prod_{j=1}^{s}\biggl(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\biggr)^{\mu_j}(1+t)^{-\lambda_v-1}\biggl|\bigl(\ln(1+t)\bigr)^m x^{n-1}\Biggr\rangle \\
&=\sum_{v=1}^{s}\lambda_v\mu_v\Biggl\langle\prod_{i=1}^{r}\biggl(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\biggr)\biggl(\frac{(1+t)^{\lambda_v}}{1+(1+t)^{\lambda_v}}\biggr) \\
&\quad\times\prod_{j=1}^{s}\biggl(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\biggr)^{\mu_j}(1+t)^{-\lambda_v-1}\biggl|m!\sum_{l=m}^{\infty}S_1(l,m)\frac{t^l}{l!}x^{n-1}\Biggr\rangle \\
&=m!\sum_{v=1}^{s}\lambda_v\mu_v\sum_{l=m}^{n-1}\binom{n-1}{l}S_1(l,m)\Biggl\langle\prod_{i=1}^{r}\biggl(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\biggr)\biggl(\frac{(1+t)^{\lambda_v}}{1+(1+t)^{\lambda_v}}\biggr) \\
&\quad\times\prod_{j=1}^{s}\biggl(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\biggr)^{\mu_j}(1+t)^{-\lambda_v-1}\biggl|x^{n-l-1}\Biggr\rangle \\
&=m!\sum_{v=1}^{s}\lambda_v\mu_v\sum_{l=m}^{n-1}\binom{n-1}{l}S_1(l,m)\widehat{\mathrm{NS}}_{n-l-1}(-\lambda_v-1|a;\lambda;\mu+e_v) \\
&=m!\sum_{v=1}^{s}\lambda_v\mu_v\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m)\widehat{\mathrm{NS}}_{l}(-\lambda_v-1|a;\lambda;\mu+e_v).
\end{aligned}
$$

Therefore, we get, for $n-1 \geq m \geq 1$,

$$\begin{aligned}
&m!\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{\mathrm{NS}}_l(a;\lambda;\mu) \\
&= -m!\left(\sum_{i=1}^{r}a_i\right)\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m)\widehat{\mathrm{NS}}_l(-1|a;\lambda;\mu) \\
&\quad + \frac{m!}{n}\sum_{k=0}^{n-m}\sum_{l=k}^{n-m}\binom{n}{l}\binom{l}{k}S_1(n-l,m)c_{l-k} \\
&\quad \times \left(\sum_{i=1}^{r}a_i\widehat{\mathrm{NS}}_k(-1|a_1,\dots,a_{i-1},a_{i+1},\dots,a_r;\lambda;\mu) - r\widehat{\mathrm{NS}}_k(-1|a;\lambda;\mu)\right) \\
&\quad + m!\sum_{j=1}^{s}\lambda_j\mu_j\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m)\widehat{\mathrm{NS}}_l(-\lambda_j-1|a;\lambda;\mu+e_j) \\
&\quad + m!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{\mathrm{NS}}_l(-1|a;\lambda;\mu).
\end{aligned}$$

Dividing both sides by $m!$, we obtain, for $n-1 \geq m \geq 1$,

$$\begin{aligned}
&\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{\mathrm{NS}}_l(a;\lambda;\mu) \\
&= -\left(\sum_{i=1}^{r}a_i\right)\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m)\widehat{\mathrm{NS}}_l(-1|a;\lambda;\mu) \\
&\quad + \frac{1}{n}\sum_{k=0}^{n-m}\sum_{l=k}^{n-m}\binom{n}{l}\binom{l}{k}S_1(n-l,m)c_{l-k} \\
&\quad \times \left(\sum_{i=1}^{r}a_i\widehat{\mathrm{NS}}_k(-1|a_1,\dots,a_{i-1},a_{i+1},\dots,a_r;\lambda;\mu) - r\widehat{\mathrm{NS}}_k(-1|a;\lambda;\mu)\right) \\
&\quad + \sum_{j=1}^{s}\lambda_j\mu_j\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m)\widehat{\mathrm{NS}}_l(-\lambda_j-1|a;\lambda;\mu+e_j) \\
&\quad + \sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{\mathrm{NS}}_l(-1|a;\lambda;\mu).
\end{aligned}$$

Thus, we get (27). □

3.8 A relation with the falling factorials

Theorem 8

$$\begin{aligned}
&\widehat{\mathrm{NS}}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s) \\
&= \sum_{m=0}^{n}\binom{n}{m}\widehat{\mathrm{NS}}_{n-m}(a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)(x)_m. \qquad (29)
\end{aligned}$$

Proof For (12) and (20), assume that

$$\widehat{\mathrm{NS}}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)=\sum_{m=0}^{n}C_{n,m}(x)_m.$$

By (11), we have

$$\begin{aligned}
C_{n,m}&=\frac{1}{m!}\left\langle\frac{1}{\prod_{i=1}^{r}\bigl(\frac{\ln(1+t)e^{a_i\ln(1+t)}}{e^{a_i\ln(1+t)}-1}\bigr)\prod_{j=1}^{s}\bigl(\frac{1+e^{\lambda_j\ln(1+t)}}{e^{\lambda_j\ln(1+t)}}\bigr)^{\mu_j}}t^m\middle|x^n\right\rangle\\
&=\frac{1}{m!}\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\middle|t^mx^n\right\rangle\\
&=\binom{n}{m}\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\middle|x^{n-m}\right\rangle\\
&=\binom{n}{m}\widehat{\mathrm{NS}}_{n-m}.
\end{aligned}$$

Thus, we get the identity (29). □

3.9 A relation with higher-order Frobenius-Euler polynomials

For $\alpha\in\mathbb{C}$ with $\alpha\neq 1$, the Frobenius-Euler polynomials of order r, $H_n^{(r)}(x|\alpha)$ are defined by the generating function

$$\left(\frac{1-\alpha}{e^t-\alpha}\right)^r e^{xt}=\sum_{n=0}^{\infty}H_n^{(r)}(x|\alpha)\frac{t^n}{n!}$$

(see *e.g.* [5]).

Theorem 9

$$\begin{aligned}
&\widehat{\mathrm{NS}}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)\\
&\quad=\sum_{m=0}^{n}\left(\sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{\sigma}{j}\binom{n-j}{l}(n)_j(1-\alpha)^{-j}S_1(n-j-l,m)\widehat{\mathrm{NS}}_l\right)H_m^{(\sigma)}(x|\alpha).
\end{aligned}\tag{30}$$

Proof For (12) and

$$H_n^{(\sigma)}(x|\alpha)\sim\left(\left(\frac{e^t-\alpha}{1-\alpha}\right)^{\sigma},t\right),\tag{31}$$

assume that $\widehat{\mathrm{NS}}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)=\sum_{m=0}^{n}C_{n,m}H_m^{(\sigma)}(x|\alpha)$. By (11), similarly to the proof of (27), we have

$$\begin{aligned}
C_{n,m}&=\frac{1}{m!}\left\langle\frac{\bigl(\frac{e^{\ln(1+t)}-\alpha}{1-\alpha}\bigr)^{\sigma}}{\prod_{i=1}^{r}\bigl(\frac{\ln(1+t)e^{a_i\ln(1+t)}}{e^{a_i\ln(1+t)}-1}\bigr)\prod_{j=1}^{s}\bigl(\frac{1+e^{\lambda_j\ln(1+t)}}{e^{\lambda_j\ln(1+t)}}\bigr)^{\mu_j}}\bigl(\ln(1+t)\bigr)^m\middle|x^n\right\rangle\\
&=\frac{1}{m!(1-\alpha)^{\sigma}}
\end{aligned}$$

$$\times \left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\big(\ln(1+t)\big)^m(1-\alpha+t)^{\sigma}\,\middle|\, x^n\right\rangle$$

$$= \frac{1}{m!(1-\alpha)^{\sigma}}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\right.$$

$$\left.\times \big(\ln(1+t)\big)^m \,\middle|\, \sum_{\nu=0}^{\min\{\sigma,n\}}\binom{\sigma}{\nu}(1-\alpha)^{\sigma-\nu}t^{\nu}x^n\right\rangle$$

$$= \frac{1}{m!(1-\alpha)^{\sigma}}\sum_{\nu=0}^{n-m}\binom{\sigma}{\nu}(1-\alpha)^{\sigma-\nu}(n)_{\nu}$$

$$\times \left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\big(\ln(1+t)\big)^m\,\middle|\, x^{n-\nu}\right\rangle$$

$$= \frac{1}{m!(1-\alpha)^{\sigma}}\sum_{\nu=0}^{n-m}\binom{\sigma}{\nu}(1-\alpha)^{\sigma-\nu}(n)_{\nu}\sum_{l=0}^{n-m-\nu}m!\binom{n-\nu}{l}S_1(n-\nu-l,m)\widehat{\mathrm{NS}}_l$$

$$= \sum_{\nu=0}^{n-m}\sum_{l=0}^{n-m-\nu}\binom{\sigma}{\nu}\binom{n-\nu}{l}(n)_{\nu}(1-\alpha)^{-\nu}S_1(n-\nu-l,m)\widehat{\mathrm{NS}}_l.$$

Thus, we get the identity (30). □

3.10 A relation with higher-order Bernoulli polynomials

Bernoulli polynomials $\mathfrak{B}_n^{(r)}(x)$ of order r are defined by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty}\frac{\mathfrak{B}_n^{(r)}(x)}{n!}t^n$$

(see *e.g.* [1, Section 2.2]). In addition, the Cauchy numbers of the first kind $\mathfrak{C}_n^{(r)}$ of order r are defined by

$$\left(\frac{t}{\ln(1+t)}\right)^r = \sum_{n=0}^{\infty}\frac{\mathfrak{C}_n^{(r)}}{n!}t^n$$

(see *e.g.* [6, equation (2.1)], [7, equation (6)]).

Theorem 10

$$\widehat{\mathrm{NS}}_n(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)$$
$$= \sum_{m=0}^{n}\left(\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(\sigma)}S_1(n-i-l,m)\widehat{\mathrm{NS}}_l\right)\mathfrak{B}_m^{(\sigma)}(x). \tag{32}$$

Proof For (12) and

$$\mathfrak{B}_n^{(\sigma)}(x)\sim\left(\left(\frac{e^t-1}{t}\right)^{\sigma},t\right), \tag{33}$$

assume that $\widehat{\mathrm{NS}}_n(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s) = \sum_{m=0}^{n} C_{n,m}\mathfrak{B}_m^{(s)}(x)$. By (11), similarly to the proof of (27), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{\left(\frac{e^{\ln(1+t)}-1}{\ln(1+t)}\right)^{\sigma}}{\prod_{i=1}^{r}\left(\frac{\ln(1+t)e^{a_i\ln(1+t)}}{e^{a_i\ln(1+t)}-1}\right)\prod_{j=1}^{s}\left(\frac{1+e^{\lambda_j\ln(1+t)}}{e^{\lambda_j\ln(1+t)}}\right)^{\mu_j}}\bigl(\ln(1+t)\bigr)^m \middle| x^n \right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\bigl(\ln(1+t)\bigr)^m \middle| \left(\frac{t}{\ln(1+t)}\right)^{\sigma} x^n \right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\bigl(\ln(1+t)\bigr)^m \middle| \sum_{i=0}^{\infty}\mathfrak{C}_i^{(\sigma)}\frac{t^i}{i!}x^n \right\rangle \\
&= \frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(\sigma)}\binom{n}{i}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\prod_{j=1}^{s}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\bigl(\ln(1+t)\bigr)^m \middle| x^{n-i} \right\rangle \\
&= \frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(\sigma)}\binom{n}{i}\sum_{l=0}^{n-m-i} m!\binom{n-i}{l}S_1(n-i-l,m)\widehat{\mathrm{NS}}_l \\
&= \sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(\sigma)}S_1(n-i-l,m)\widehat{\mathrm{NS}}_l.
\end{aligned}$$

Thus, we get the identity (32). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Sogang University, Seoul, 121-741, Republic of Korea. [2]Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea. [3]Graduate School of Science and Technology, Hirosaki University, Hirosaki, 036-8561, Japan.

Acknowledgements
The work reported in this paper was conducted during the sabbatical year of Kwangwoon University in 2014.

Received: 23 July 2014 Accepted: 10 September 2014 Published: 29 Sep 2014

References
1. Roman, S: The Umbral Calculus. Dover, New York (2005)
2. Kim, DS, Kim, T: Poly-Cauchy and Peters mixed-type polynomials. Adv. Differ. Equ. **2014**, 4 (2014)
3. Comtet, L: Advanced Combinatorics. Reidel, Dordrecht (1974)
4. Kim, DS, Kim, T: Higher-order Cauchy of the first kind and poly-Cauchy of the first kind mixed type polynomials. Adv. Stud. Contemp. Math. **23**(4), 621-636 (2013)
5. Kim, DS, Kim, T: Some identities of Frobenius-Euler polynomials arising from umbral calculus. Adv. Differ. Equ. **2012**, 196 (2012)
6. Carlitz, L: A note on Bernoulli and Euler polynomials of the second kind. Scr. Math. **25**, 323-330 (1961)
7. Liang, H, Wuyungaowa: Identities involving generalized harmonic numbers and other special combinatorial sequences. J. Integer Seq. **15**, Article 12.9.6 (2012)

10.1186/1029-242X-2014-376
Cite this article as: Kim et al.: **Barnes-type Narumi of the second kind and Barnes-type Peters of the second kind hybrid polynomials.** *Journal of Inequalities and Applications* 2014, 2014:376