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POWERSUM FORMULA FOR DIFFERENTIAL RESOLVENTS

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We will prove that we can specialize the indeterminate α in a linear differential α-resolvent of a univariate polynomial over a differential field of characteristic zero to an integer q to obtain a q-resolvent. We use this idea to obtain a formula, known as the *powersum formula*, for the terms of the α-resolvent. Finally, we use the powersum formula to rediscover Cockle's differential resolvent of a cubic trinomial.

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1. Introduction. It was proved in [4, Theorem 37, page 67] that for any integer q, a polynomial $P(t) \equiv \sum_{k=0}^{N}(-1)^{N-k}e_{N-k}t^k$ of a single variable t whose coefficients $\{e_{N-k}\}_{k=0}^{N}$ lie in an ordinary differential ring $\mathbb{R}$ with derivation D possesses an ordinary linear differential α-resolvent and an ordinary linear differential q-resolvent, where α is a constant, transcendental over $\mathbb{R}$. With no loss of generality, we assume that P is monic and has no zero roots. Then the coefficient e_{N-k} is the $(N-k)$th elementary symmetric function of the roots of P. We assume that $P(t) = \prod_{k=1}^{n}(t-z_k)^{\pi_k}$ has $n \le N$ distinct roots and π_k is the multiplicity of the root z_k in P. It was proved in [2] that for each root z_k, there exists a nonzero solution y_k of the logarithmic differential equation $Dy_k/y_k = \alpha \cdot (Dz_k/z_k)$. Obviously, such solutions are unique only up to a constant multiple. We define the notation z_k^{α} to represent any such solution y_k. Hence, we will call y_k an *α-power* of z_k. From now on, we will drop the subscript k on z_k and y_k. It will be understood that a different z implies a different y.

In this paper, we present the powersum formula as a new method for computing resolvents, although it remains a conjecture whether the powersum formula always yields a (nonzero) resolvent rather than an identically zero equation. It was proved in [5, Theorem 4.1, page 726] that if all the distinct roots of a polynomial are differentially independent over constants, then the powersum formula yields a resolvent. It was shown in [6, Section 11, pages 344-345], how the solution of the Riccati nonlinear differential equation is related to the resolvent of a quadratic polynomial.

2. Notation. Let $\mathbb{N}$ denote the set of positive integers. Let $\mathbb{N}_0$ denote the set of nonnegative integers. Let $\mathbb{Z}^{\#}$ denote the set of nonzero integers. The following notation has been slightly modified from Kolchin's notation in [2] and Macdonald's notation in [3]. Let $\mathbb{Z}\{e\}$ denote the differential ring generated by the integers $\mathbb{Z}$ and the N coefficients $e \equiv \{e_k\}_{k=1}^{N}$ of t in P. Let $\mathbb{Q}\langle e\rangle$ denote the differential field generated by the rational numbers $\mathbb{Q}$ and e. For each $m \in \mathbb{N}$, let $\mathbb{Z}\{e\}_m$ denote the ordinary (nondifferential) ring generated by $\mathbb{Z}$, e, and the first m derivatives of e. (A differential ring must contain infinitely many derivatives of any of its elements.) Let $\mathbb{Q}\langle e\rangle_m(z) = \mathbb{Q}\langle e\rangle_m[z]$ denote the field generated by $\mathbb{Q}$, e, the first m derivatives of e, and the single root z. From this point on, we will write $\mathbb{Q}\langle e\rangle_m[z]$ instead of $\mathbb{Q}\langle e\rangle_m(z)$ for this field to emphasize the fact that elements in this field are polynomial in the root z. If $\mathbb{R}$ represents any of the rings or fields mentioned so far, then let $\mathbb{R}[t,\alpha]$ denote the polynomial ring in the indeterminates t and α over $\mathbb{R}$.

Let $\theta \equiv (n!)\cdot(\prod_{k=1}^{n}\pi_k)\cdot(\prod_{k=1}^{n} z_k)\cdot(\prod_{i<j}(z_i - z_j)^2)$. By our conditions on P, $\theta \neq 0$. It is also easy to show that $\theta \in \mathbb{Z}[e]$. For each $m \in \mathbb{N}$, it was proved in [4, Theorem 32, page 60] that there exists a polynomial $G_m(t,\alpha)$ in t and α satisfying the following definition.

Definition 2.1. Define $G_m(t,\alpha)$ to be the polynomial in t and α such that $G_m(z,\alpha) = D^m y/(\alpha\cdot y)$ for each root z of P and $\theta^m \cdot G_m(t,\alpha) \in \mathbb{Z}\{e\}_m[t,\alpha]$.

A *specialization* ϕ is a ring homomorphism $\phi : \mathbb{R} \to \hat{\mathbb{R}}$ from a ring $\mathbb{R}$ into an integral domain $\hat{\mathbb{R}}$. For any polynomial $P(t) = \sum_{k=0}^{N}(-1)^{N-k}e_{N-k}\cdot t^k \in \mathbb{R}[t]$, $\phi(P)$ is defined to be the polynomial $(\phi P)(t) = \sum_{k=0}^{N}(-1)^{N-k}\phi(e_{N-k})\cdot t^k \in \mathbb{R}[t]$. A *differential specialization* ϕ is a specialization $\phi : \mathbb{R} \to \hat{\mathbb{R}}$ from a differential ring $\mathbb{R}$ with derivation D into a differential integral domain $\hat{\mathbb{R}}$ with derivation $\hat{D}$ such that $\phi D = \hat{D}\phi$ on $\mathbb{R}$.

3. Specializing α.

Let $q \in \mathbb{N}$. Let $\phi_q : \mathbb{Q}\langle e\rangle[z,\alpha] \to \mathbb{Q}\langle e\rangle[z]$ be the ring specialization such that ϕ_q is the identity on $\mathbb{Q}\langle e\rangle[z]$ and $\phi_q(\alpha) = q$. We may compute $Dz^q/(q\cdot z^q)$. Since ϕ_q is not defined to act on y, we are not able to specialize y to z^q in Theorem 3.1. However, ϕ_q is defined to act on $D^m y/(\alpha\cdot y)$ since $D^m y/(\alpha\cdot y) = G_m(z,\alpha) \in \theta^{-m}\cdot \mathbb{Z}\{e\}_m[z,\alpha]$. Theorem 3.1 asserts that $G_m(z,q) = D^m z^q/(q\cdot z^q)$. Theorem 3.2 asserts that $\phi_q(D^m y/(\alpha\cdot y)) = D^m z^q/(q\cdot z^q)$.

Theorem 3.1. *Assume all the same definitions and notations as in the introduction. Then the mth derivative of z^q can be expressed as a product of $q\cdot z^q$ and an element in $\mathbb{Q}\langle e\rangle_m[z]$. More specifically, $G_m(z,q) = D^m z^q/(q\cdot z^q)$, where $G_m(z,q) \in \mathbb{Q}\langle e\rangle_m[z]$ and $G_m(t,\alpha)$ was given in Definition 2.1.*

Proof. For brevity, write $G_m = G_m(z,\alpha)$ for the particular root z. We emphasize that G_m is $G_m(t,\alpha)$ with t specialized to the particular root z. We find that $\theta\cdot(Dz^q/(q\cdot z^q)) = \theta\cdot(Dz/z) = \theta\cdot G_1 \in \mathbb{Z}\{e\}_1[z]$. Therefore,

$$\begin{aligned} Dz^q = q\cdot z^q\cdot G_1 \Longrightarrow D^2 z^q &= q\cdot(q\cdot z^{q-1}Dz\cdot G_1 + z^q\cdot DG_1)\\ &= q\cdot z^q\left(q\cdot\left(\frac{Dz}{z}\right)\cdot G_1 + DG_1\right) = q\cdot z^q(q\cdot G_1^2 + DG_1)\\ &= q\cdot z^q\cdot\phi_q(\alpha\cdot G_1^2 + DG_1) = q\cdot z^q\cdot\phi_q(G_2). \end{aligned} \tag{3.1}$$

So, $D^m z^q = q\cdot z^q\cdot\phi_q(G_m)$ is true for $m = 1$. Now assume that it is true for $m \geq 2$. Then $D^{m+1}z^q = q\cdot(q\cdot z^{q-1}(Dz)\cdot\phi_q(G_m) + z^q\cdot D(\phi_q(G_m)))$. But ϕ_q specializes α, whose derivative is 0, to an integer whose derivative is 0. Thus, $D(\phi_q(G_m)) = \phi_q(D(G_m))$.

Hence,

$$\begin{aligned}
D^{m+1}z^q &= q\cdot\left(q\cdot z^{q-1}\cdot(Dz)\cdot\phi_q(G_m)+z^q\cdot\phi_q(D(G_m))\right)\\
&= q\cdot z^q\cdot\left(q\cdot\frac{Dz}{z}\cdot\phi_q(G_m)+\phi_q(D(G_m))\right)\\
&= q\cdot z^q\cdot\left(\phi_q(\alpha)\cdot G_1\cdot\phi_q(G_m)+\phi_q(D(G_m))\right)\\
&= q\cdot z^q\cdot\phi_q(\alpha\cdot G_1\cdot G_m+D(G_m))\\
&= q\cdot z^q\cdot\phi_q(G_{m+1}).
\end{aligned}\tag{3.2}$$

Therefore, $D^{m+1}z^q = q\cdot z^q\cdot G_{m+1}(z,q)$ since ϕ_q affects only α. By the principle of mathematical induction, this equation is true for all positive integers m. □

Just because $Dy/(\alpha\cdot y) = Dz/z = Dz^q/(q\cdot z^q)$ implies that $Dy/(\alpha\cdot y)$ is independent of α, it does not follow that $D^m y/(\alpha\cdot y)$ is independent of α for $m\geq 2$. We can see this by observing that $D^m y/(\alpha\cdot y)\neq D^m z/z\neq D^m z^q/(q\cdot z^q)\neq D^m y/(\alpha\cdot y)$ for $m\geq 2$.

Theorem 3.2. *Assume all the same definitions and notations as in Theorem 3.1 and Section 2. Then, for each $m\in\mathbb{N}$, the specialization under ϕ_q of $D^m y/(\alpha\cdot y)$ is $D^m z^q/(q\cdot z^q)$. That is, $\phi_q(D^m y/(\alpha\cdot y)) = D^m z^q/(q\cdot z^q)$.*

Proof. By Definition 2.1, $D^m y/(\alpha\cdot y) = G_m(z,\alpha)$. By Theorem 3.1, $D^m z^q/(q\cdot z^q) = G_m(z,q)$. Putting these results together yields

$$\phi_q\left(\frac{D^m y}{\alpha\cdot y}\right) = \phi_q(G_m(z,\alpha)) = G_m(z,q) = \frac{D^m z^q}{q\cdot z^q}. \tag{3.3}$$ □

4. Powersum satisfaction theorem and formula. An *α-resolvent* of a polynomial $P(t)\equiv\sum_{i=0}^{N}(-1)^{n-i}e_{N-i}t^i\in\mathbb{F}[t]$ over a differential field $\mathbb{F}$ with derivation D is a linear ordinary differential equation $\sum_{m=0}^{o}B_m(\alpha)\cdot D^m y = 0$ of finite order o such that each of the *coefficient functions* $B_m(\alpha)$ lies in the field $\mathbb{Q}\langle e\rangle(\alpha)$ (or preferably in the ring $\mathbb{Z}\{e\}[\alpha]$) such that not all $B_m(\alpha)$ are identically zero, and which is satisfied by the α-power of every root z of P. In other words, the coefficient functions of the resolvent are independent of the choice of root and are not all zero. By [4, Theorem 37, page 67], resolvents for any polynomial are guaranteed to exist. We state this assertion in Theorem 4.1.

Theorem 4.1. *Let $P(t)\equiv\sum_{i=0}^{N}(-1)^{n-i}e_{N-i}t^i\in\mathbb{F}[t]$ be a polynomial of degree N in t over a d-field $\mathbb{F}$ with n distinct roots $\{z_i\}_{i=1}^{n}$. Then there exists an oth order differential resolvent $\sum_{m=0}^{o}B_m(\alpha)\cdot D^m y = 0$ with $B_m(\alpha)\in\mathbb{Z}\{e\}_n[\alpha]$, $B_0(0)=0$, and $\deg_\alpha B_m(\alpha)\leq o(o-1)/2-m+1$ for some $o\in[n]$. Furthermore, o may be chosen to equal the number of $\{y_j\}_{j=1}^{n}$ linearly independent over constants, and all solutions of this resolvent are linear combinations over constants of these o y_j's.*

Theorem 4.1 gives us an upper bound on the degree in α in an α-resolvent of P. Theorem 4.2 allows us to specialize the indeterminate α to an integer q (or any number) to obtain a q-resolvent.

Theorem 4.2 (*powersum satisfaction theorem*). *Let $P \in \mathbb{F}[t]$ be a monic polynomial with n distinct roots $z = \{z_i\}_{i=1}^n$, none of which is zero and not all of which are constants. Let $q \in \mathbb{Z}$. If $R_\alpha \equiv \sum_{m=0}^{o} B_m(\alpha) \cdot D^m y$ is an α-resolvent for P of arbitrary order o, where $B_m(\alpha) = \sum_{i \geq 0} b_{i,m}\alpha^i \in \mathbb{Z}\{e\}[\alpha]$, with $b_{i,m} \in \mathbb{Z}\{e\}$, then R_α specializes to the q-resolvent $R_q \equiv \sum_{m=0}^{o} B_m(q) \cdot D^m y$ for $q \in \mathbb{Z}^\#$ under $\phi_q(\alpha) = q$ and $\phi_q(u) = u$ for each $u \in \mathbb{Z}\{e\}$. Furthermore, the qth powersum p_q satisfies $\sum_{m=0}^{o} B_m(q) \cdot D^m p_q = 0$ for each $q \in \mathbb{Z}^\#$.*

Proof. By Definition 2.1 of $G_m(t,\alpha)$, we have

$$\begin{aligned}\sum_{m=0}^{o} B_m(\alpha) \cdot D^m y = 0 &\Longleftrightarrow \sum_{m=0}^{o} B_m(\alpha) \cdot \frac{D^m y}{\alpha \cdot y} = 0 \\ &\Longleftrightarrow \sum_{m=0}^{o} B_m(\alpha) \cdot G_m(z,\alpha) = 0.\end{aligned} \tag{4.1}$$

Now for each $q \in \mathbb{Z}^\#$, specialize this equation under ϕ_q to get $\sum_{m=0}^{o} B_m(q) \cdot G_m(z,q) = 0$ by Theorem 3.2, since $\phi_q|_{\mathbb{F}} = I$. For any of the roots of P, we have $G_m(z,q) = D^m z^q / (q \cdot z^q)$ by Theorem 3.1. Thus,

$$\sum_{m=0}^{o} B_m(q) \cdot \frac{D^m z^q}{q \cdot z^q} = 0 \Longleftrightarrow \sum_{m=0}^{o} B_m(q) \cdot D^m z^q = 0 \tag{4.2}$$

for each $q \in \mathbb{Z}^\#$. Therefore, an α-resolvent specializes to a q-resolvent for each $q \in \mathbb{Z}^\#$ under ϕ_q. Now sum over the N roots of P including their multiplicities to get $\sum_{m=0}^{o} B_m(q) \cdot D^m p_q = 0$ for each $q \in \mathbb{Z}^\#$. □

The powersum satisfaction theorem states that for any monic polynomial P, the coefficients $b_{i,m}$ of α in any α-resolvent $R_\alpha \equiv \sum_{(i,m)\in S} b_{i,m} \cdot \alpha^i D^m y$ of P satisfy an infinite system of homogeneous equations

$$[q^i D^m p_q]_{q \times (i,m)\ 1 \leq q < \infty\ (i,m) \in S} \cdot [b_{i,m}]_{(i,m)\in S} = [0_q]_{1 \leq q < \infty}. \tag{4.3}$$

Here, S denotes the set of pairs (i,m) consisting of a power of α, denoted by i, and an order of a derivative, denoted by m, such that $b_{i,m} \neq 0$. Let $|S|$ denote the size of S. We will be interested in proving that the rank

$$\operatorname{rk}[q^i D^m p_q]_{q \times (i,m)\ 1 \leq q < \infty\ (i,m) \in S} \tag{4.4}$$

of the matrix $[q^i D^m p_q]_{q \times (i,m)\ 1 \leq q < \infty\ (i,m)\in S}$ equals $|S| - 1$ under certain circumstances. Under those circumstances, one can solve this system of equations to get a nonzero solution for $b_{i,m}$. The solution is given by $b_{i,m} = F_{i,m} \equiv (-1)^{\operatorname{sgn}(i,m)} \cdot |q^{i'} D^{m'} p_q|_{(i',m') \neq (i,m) q \in \Gamma}$, where $\operatorname{sgn}(i,m)$ indicates the ordering of the term $b_{i,m}$ in the resolvent, and we take Γ to be the smallest possible set of positive integers that will guarantee a nonzero solution. In numerous examples, it has been found that $\Gamma \equiv \{k \in \mathbb{N} \ni 1 \leq k \leq |S| - 1\}$. We call this the *powersum formula* for a resolvent of P. We use the notation $F_{i,m}$ to denote the terms of the resolvent obtained by this method to suggest the word *formula*. We will denote the resolvent obtained by this formula by $\mathfrak{R}_\alpha$. So, $\mathfrak{R}_\alpha = \{F_{i,m}\}$.

If $\mathrm{rk}[q^i D^m p_q]_{q\times(i,m)\ 1\le q<\infty\ (i,m)\in S} = |S|$, then the only solution would be $b_{i,m} = 0$ for all $(i,m) \in S$, contradicting the hypothesis that R_α is nonzero. Unfortunately, for a given polynomial P, one does not know a priori what the set S of nonzero $b_{i,m}$ is or how large it is. Nevertheless, we may summarize the results obtained so far in a corollary to the powersum satisfaction theorem.

Corollary 4.3 (the powersum formula). *Let $R_\alpha \equiv \sum_{(i,m)\in S} b_{i,m}\cdot\alpha^i D^m y$ be an α-resolvent of P, where $S \subset \mathbb{N}_0\times\mathbb{N}_0$ is a finite set. If there exists a set of $|S|-1$ integers $\Gamma\subset\mathbb{N}$ such that not all the $F_{i,m}$ given by the powersum formula $F_{i,m} \equiv (-1)^{\mathrm{sgn}(i,m)}\cdot |q^{i'}D^{m'}p_q|_{(i',m')\neq(i,m)\ q\in\Gamma}$ are zero, then the linear ordinary differential equation (ODE), $\mathfrak{R}_\alpha \equiv \sum_{(i,m)\in S} F_{i,m}\cdot\alpha^i D^m y$, is an integral α-resolvent of P. If no such set of integers Γ exists, then the powersum formula yields all zeroes for $F_{i,m}$.*

The author believes that the resolvent $\mathfrak{R}_\alpha$ given by the powersum formula will be a $\mathbb{Q}\langle e\rangle$-multiple, not just a $\mathbb{Q}\langle e\rangle(\alpha)$-multiple of R_α, but this requires proof. For example, let $\alpha^M\cdot D^H y$ denote the highest power of α on the highest derivative of y in R_α. Even though $F_{M,L}/b_{M,L}\cdot R_\alpha$ and $\mathfrak{R}_\alpha$ are both resolvents (provided that $F_{M,L}\neq 0$) with the same coefficient function of $\alpha^M\cdot D^H y$, one must eliminate the possibility that their other terms may differ due to the possibility that P has resolvents of lower order.

5. Example. We will now apply the powersum formula to compute a particular α-resolvent of a particular trinomial. It has not yet been proved that this formula yields a nonzero differential equation for every polynomial. However, in every polynomial the author has tested, it has been possible to set up an α-resolvent, itself a polynomial in the power α, and choose the proper set of powersums such that the powersum formula yields a nonzero answer. If the powersum formula yields a nonzero answer, then it is guaranteed by Corollary 4.3 that the answer is a (nonzero) resolvent of the polynomial. By a very long and difficult proof in [4, Theorem 41, page 74] and [5, Theorem 4.1, page 726], it has been shown that in case the distinct roots of the polynomial are differentially independent over constants (i.e., they satisfy no polynomial differential equations over $\mathbb{Q}$), then the powersum formula yields a nonzero resolvent.

The powersum formula has the advantage of giving a resolvent in an integral form. In the next example, this means the powersum formula gives a resolvent all of whose terms lie in the ring $\mathbb{Z}[x,\alpha]$.

Example 5.1 (Sir James Cockle's resolvent of a trinomial). Cockle [1] gave a formula for a linear differential α-resolvent (although he did not call it that) for any trinomial of the form $t^n + x\cdot t^p - 1$, where $Dx \equiv 1$. Consider the particular trinomial $P(t)\equiv t^3+x\cdot t^2-1$, where $n=3$ and $p=2$. Then, Cockle's resolvent specializes to $27\cdot D^3 y = 4\cdot(x\cdot D+\alpha/2)(x\cdot D+3/2+\alpha/2)(x\cdot D-\alpha)y$. This expands to $27\cdot D^3 y = (4\cdot(x\cdot D)^3 + 6\cdot(x\cdot D)^2 - 3\cdot\alpha\cdot(1+\alpha)\cdot(x\cdot D) - \alpha^2\cdot(3+\alpha))y$. Replacing $(x\cdot D)^3$ with $x^3\cdot D^3+3\cdot x^2\cdot D^2+x\cdot D$ and $(x\cdot D)^2$ with $x^2\cdot D^2+x\cdot D$ yields $(4x^3-27)\cdot D^3 y+18\cdot x^2\cdot D^2 y+(10-3\cdot\alpha-3\cdot\alpha^2)\cdot x\cdot Dy-\alpha^2\cdot(3+\alpha)\cdot y=0$, which has the form $f_1\cdot D^3 y+f_2\cdot D^2 y+(f_3+f_4\cdot\alpha+f_5\cdot\alpha^2)\cdot Dy+(f_6\cdot\alpha^2+f_7\cdot\alpha^3)\cdot y=0$. The powersum formula requires one to know a priori the various powers of α appearing in a resolvent. Specialize α to one of the six integers $q\in\{1,2,3,4,5,6\}$, then sum the resulting equation over each of the three roots. Doing this for each $q \in \{1,2,3,4,5,6\}$, one gets a system of six linear equations in the undetermined coefficient functions $\{f_k\}_{k=1}^{7}$ of the form $\mathfrak{M} \cdot \vec{f} = \vec{0}$, where $\mathfrak{M}$ is the 6×7 matrix defined by

$$\mathfrak{M} \equiv \begin{bmatrix} D^3p_1 & D^2p_1 & Dp_1 & 1\cdot Dp_1 & 1^2\cdot Dp_1 & 1^2\cdot p_1 & 1^3\cdot p_1 \\ D^3p_2 & D^2p_2 & Dp_2 & 2\cdot Dp_2 & 2^2\cdot Dp_2 & 2^2\cdot p_2 & 2^3\cdot p_2 \\ D^3p_3 & D^2p_3 & Dp_3 & 3\cdot Dp_3 & 3^2\cdot Dp_3 & 3^2\cdot p_3 & 3^3\cdot p_3 \\ D^3p_4 & D^2p_4 & Dp_4 & 4\cdot Dp_4 & 4^2\cdot Dp_4 & 4^2\cdot p_4 & 4^3\cdot p_4 \\ D^3p_5 & D^2p_5 & Dp_5 & 5\cdot Dp_5 & 5^2\cdot Dp_5 & 5^2\cdot p_5 & 5^3\cdot p_5 \\ D^3p_6 & D^2p_6 & Dp_6 & 6\cdot Dp_6 & 6^2\cdot Dp_6 & 6^2\cdot p_6 & 6^3\cdot p_6 \end{bmatrix}, \tag{5.1}$$

$\vec{f}$ is the 7×1 column vector defined by

$$\vec{f} \equiv \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}, \tag{5.2}$$

and $\vec{0}$ is the 6×1 column vector defined by

$$\vec{0} \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{5.3}$$

The following program, written in Mathematica 4.0 for Students and run on a Dell Dimension XPS R400 computer using Windows 98 operating system, computes the seven terms $\{f_k\}_{k=1}^{7}$ by setting each f_k to the appropriate cofactor of $\mathfrak{M}$. This matrix is denoted by T in the program. The symbol $s[k]$ stands for the kth powersum p_k of the roots of P. The output is denoted by f, which is defined as the transpose of $\vec{f}$. The result is

$$\begin{bmatrix} -27+4x^3 & 18x^2 & 10x & -3x & -3x & -3 & -1 \end{bmatrix}, \tag{5.4}$$

which is the Cockle resolvent. The computation time is less than 5 seconds.

```
x=.; s[0]=3; s[1]=-x; s[2]=x^2;
Table[s[k+3]=Expand[-x*s[k+2]+s[k]],{k,0,3}];
T=Table[{D[s[k],{x,3}],D[s[k],{x,2}],D[s[k],x],
k*D[s[k],x],k^2*D[s[k],x],k^2*s[k],k^3*s[k]},{k,1,6}];
M=Minors[t,6];
f=Table[Simplify[m[[1,k]]*(-1)^(7-k)/(466560*x)],{k,1,7}].
```

To see the output in Mathematica for other variables, remove the semicolon after its formula. For the record, the first six powersums are (written in the form Mathematica gives) $p_1 = -x$, $p_2 = x^2$, $p_3 = 3-x^3$, $p_4 = -4x+x^4$, $p_5 = 5x^2-x^5$, and $p_6 = 3-6x^3+x^6$. The cofactors of the matrix $\mathfrak{M}$ had to be divided by $466560 \cdot x = 2^7 \cdot 3^6 \cdot 5^1 \cdot x$ to get the resolvent in *Cohnian* form, that is, such that the only divisors in $\mathbb{Z}[x,\alpha]$ among all the terms of the resolvent are ± 1.

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