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# ON THE STRONG LAW FOR ARRAYS AND FOR THE BOOTSTRAP MEAN AND VARIANCE

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**ABSTRACT.** Chung type strong laws of large numbers are obtained for arrays of rowwise independent random variables under various moment conditions. An interesting application of these results is the consistency of the bootstrap mean and variance.

**KEY WORDS AND PHRASES.** *Strong law of large numbers, rowwise independent, bootstrap mean and variance.*
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## 1. INTRODUCTION

Let $\{X_n : n = 1, 2, ...\}$ be a sequence of independent and identically distributed (i.i.d.) random variables (r.v.'s). The Marcinkiewicz-Zygmund [1] strong law of large numbers (SLLN) provides that

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} (X_i - EX_i) \to 0 \quad \text{a.s. for} \quad 1 \le \alpha < 2 \tag{1.1}$$

and

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} X_i \to 0 \quad \text{a.s. for} \quad 0 < \alpha < 1 \tag{1.2}$$

if and only if $E|X_1|^\alpha < \infty$. The case $\alpha = 1$ is due to Kolmogorov [2]. If the sequence $\{X_n\}$ is independent (but not necessarily identically distributed), Chung's [3] SLLN yields (1.1) with $\alpha = 1$ if $\psi(t)$ is a positive, even, continuous function such that either

$$\frac{\psi(t)}{|t|} \downarrow \quad \text{as} \quad |t| \uparrow \infty, \tag{1.3}$$

and

$$\sum_{n=1}^{\infty} \frac{E\psi(X_n)}{\psi(n)} < \infty \tag{1.4}$$

hold, or,

$$\frac{\psi(t)}{|t|} \uparrow \quad \text{and} \quad \frac{\psi(t)}{t^2} \downarrow \quad \text{as} \quad |t| \uparrow \infty, \tag{1.5}$$

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$EX_n = 0$ and (1.4) hold. Hu, Moricz and Taylor [4] considered SLLN's for arrays $\{X_{ni}\}$ of i.i.d. r.v.'s, and in particular, they showed that, for $1 \le \alpha < 2$,

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} (X_{ni} - EX_{ni}) \to 0 \quad \text{a.s. if and only if} \quad E|X_{11}|^{2\alpha} < \infty.$$

In Section 2, SLLN's for triangular arrays $\{X_{ni}\}$ of rowwise independent (but neither necessarily identically distributed nor independent between rows) r.v.'s are established under conditions similar to those of Chung. In Section 3, these results are related to verifying consistency of the bootstrap mean and bootstrap variance.

## 2. STRONG LAWS OF LARGE NUMBERS

In this section let $\{X_{ni} : 1 \le i \le n, n \ge 1\}$ be a triangular array of rowwise independent random variables. Let $\{a_n\}$ be a sequence of positive real numbers such that $a_{n+1} > a_n$ and $\lim a_n = \infty$. Let $\psi(t)$ be a positive, even function such that $\frac{\psi(|t|)}{|t|^p}$ is an increasing function of $|t|$ and $\frac{\psi(|t|)}{|t|^{p+1}}$ is a decreasing function of $|t|$ respectively, that is,

$$\frac{\psi(|t|)}{|t|^p} \uparrow \quad \text{and} \quad \frac{\psi(|t|)}{|t|^{p+1}} \downarrow \quad \text{as} \quad |t| \uparrow \tag{2.1}$$

for some nonnegative integer p. Conditions (2.2) - (2.4) are given as

$$EX_{ni} = 0, \tag{2.2}$$

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} E \frac{\psi(X_{ni})}{\psi(a_n)} < \infty, \tag{2.3}$$

$$\sum_{n=1}^{\infty} \left( \sum_{i=1}^{n} E \left( \frac{X_{ni}}{a_n} \right)^2 \right)^{2k} < \infty \tag{2.4}$$

where $k$ is a positive integer. Condition (2.3) is a necessary condition for the SLLN (Condition (2.5) below) to hold in some sense (see Chung [3]). Different SLLN's are obtained for $p \ge 2$, $p = 1$, and $p = 0$, and are explicitly stated in Theorem 2.1, 2.2 and 2.3.

**Theorem 2.1.** Let $\{X_{ni} : 1 \le i \le n, n \ge 1\}$ be an array of rowwise independent random variables and let $\psi(t)$ satisfy (2.1) for some integer $p \ge 2$. Then Conditions (2.2), (2.3), and (2.4) imply

$$\frac{1}{a_n} \sum_{i=1}^{n} X_{ni} \to 0 \quad \text{a.s.} \tag{2.5}$$

**Proof.** Let $Y_{ni} = X_{ni} I_{[|X_{ni}| \le a_n]}$ and $Z_{ni} = X_{ni} I_{[|X_{ni}| > a_n]}$ where $I[.]$ is the indicator function. Since $\psi(|t|)$ is an increasing function of $|t|$,

$$\begin{aligned} \sum_{n=1}^{\infty} P\left[ \sum_{i=1}^{n} \frac{X_{ni}}{a_n} \ne \sum_{i=1}^{n} \frac{Y_{ni}}{a_n} \right] &\le \sum_{n=1}^{\infty} \sum_{i=1}^{n} P\left[ |X_{ni}| > a_n \right] \\ &\le \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E\psi(X_{ni})}{\psi(a_n)} < \infty \end{aligned} \tag{2.6}$$

by Markov's inequality and Condition (2.3). Thus by (2.6), the two sequences $\{\sum_{i=1}^{n} \frac{X_{ni}}{a_n}\}$ and $\{\sum_{i=1}^{n} \frac{Y_{ni}}{a_n}\}$ are equivalent. Next, note that $EX_{ni} = 0$ implies that

$$\left| \sum_{n=1}^{\infty} \sum_{i=1}^{n} E\left( \frac{Y_{ni}}{a_n} \right) \right| = \left| \sum_{n=1}^{\infty} \sum_{i=1}^{n} E\left( \frac{Z_{ni}}{a_n} \right) \right| \le \sum_{n=1}^{\infty} \sum_{i=1}^{n} E\left( \frac{\psi(X_{ni})}{\psi(a_n)} \right) < \infty \tag{2.7}$$

since $\frac{\psi(|t|)}{|t|}$ is an increasing function of $|t|$ and Condition (2.3) holds. Therefore from the above equivalence and (2.7), proving (2.5) is equivalent to showing

$$\sum_{i=1}^{n} \frac{Y_{ni} - EY_{ni}}{a_n} \to 0 \quad \text{a.s.} \tag{2.8}$$

as $n \to \infty$. To prove (2.8), first define

$$W_{ni} = \frac{Y_{ni}}{a_n} - E\frac{Y_{ni}}{a_n} \quad \text{for all} \quad n \quad \text{and} \quad i.$$

It is easy to see that $W_{ni}$ is bounded by 2 and that for $1 \leq u < \nu$,

$$E(W_{ni})^{\nu} \leq 2^{\nu} E|\frac{Y_{ni}}{a_n}|^{\nu} \leq 2^{\nu} E|\frac{Y_{ni}}{a_n}|^{u}. \tag{2.9}$$

Next, similar to the technique in Pruitt [5] express

$$E\left(\sum_{i=1}^{n} W_{ni}\right)^{2pk} = \sum_{k_1,\dots,k_{2pk}} E\left(\prod_{j=1}^{2pk} W_{nk_j}\right),$$

where the sum is extended over all $2pk$-tuples $(k_1, \dots, k_{2pk})$ with $k_j = 1, 2, \dots, n$ for each $j$. There is no contribution to the sum on the right so long as there is a $j$ with $k_i \neq k_j$ for all $i \neq j$ since $W_{ni}$ are independent and $EW_{ni} = 0$. For each $n$, the general term to be considered then will have

$$q_1 \text{of the k's} = \xi_1, \quad \dots, \quad q_m \text{of k's} = \xi_m$$

$$r_1 \text{of the k's} = \eta_1, \quad \dots, \quad r_l \text{of k's} = \eta_l$$

where $q_i$ and $r_j$ have the following relation:

$$2 \leq q_i \leq p, \quad p < r_j \quad \text{and} \quad \sum_{i=1}^{m} q_i + \sum_{j=1}^{l} r_j = 2pk. \tag{2.10}$$

Clearly $0 \leq m \leq pk$. Then from (2.9),

$$EW_{ni}^{q_i} \leq 2^{q_i} E|\frac{Y_{ni}}{a_n}|^2$$

$$EW_{ni}^{r_j} \leq 2^{r_j} E|\frac{Y_{ni}}{a_n}|^{p+1}. \tag{2.11}$$

Combining all possible terms yields

$$\begin{aligned} &E\left(\sum_{i=1}^{n} W_{ni}\right)^{2pk} \\ &\leq C \sum^{*}_{q_1,\dots,q_m;r_1,\dots,r_l} \sum^{**}_{\xi_1,\dots,\xi_m;\eta_1,\dots,\eta_l} E\left(\prod_{i=1}^{m} W_{n\xi_i}^{q_i}\right)\left(\sum_{j=1}^{l} W_{n\eta_j}^{r_j}\right) \\ &= C \sum^{*}_{q_1,\dots,q_m;r_1,\dots,r_l} S_{q_1,\dots,q_m;r_1,\dots,r_m}, \quad \text{say,} \end{aligned}$$

where $\Sigma^*$ is extended over all m-tuples $(q_1, \dots, q_m)$ and $l$-tuples $(r_1, \dots, r_l)$ such that condition (2.10) holds (the case $m = 0$ or $l = 0$ may occur) while $\Sigma^{**}$ is extended over all $(m + l)$-tuples $(\xi_1, \dots, \xi_m; \eta_1, \dots, \eta_l)$ of different integers between 1 and $n$ and $C$ is a constant which is independent of $n$, for example, we may take $C = (2pk)!$. Two cases are distinguished according to $l = 0$ and $l \geq 1$.

Case $l = 0$: Note $m \geq 2k$. Thus, for $n$ large,

$$S_{q_1,\ldots,q_m;0} = \sum_{\xi_1,\ldots,\xi_m}^{**} \prod_{i=1}^{m} EW_{n\xi_i}^{q_i} \leq \sum_{\xi_1,\ldots,\xi_m}^{**} \prod_{i=1}^{m} 2^{q_i} E|\frac{Y_{n\xi_i}}{a_n}|^2$$
$$\leq 2^{2pk}\left(\sum_{i=1}^{n} E|\frac{Y_{ni}}{a_n}|^2\right)^m \leq 2^{2pk}\left(\sum_{i=1}^{n} E|\frac{Y_{ni}}{a_n}|^2\right)^{2k}$$

by (2.6) and the fact $\sum_{i=1}^{n} E|\frac{Y_{ni}}{a_n}|^2 < 1$.

Case $l \geq 1$: From (2.10) and (2.11), for large $n$,

$$S_{q_1,\ldots,q_m;r_1,\ldots,r_l} = \sum_{\xi_1,\ldots,\xi_m;\eta_1,\ldots,\eta_l}^{**} \prod_{i=1}^{m} EW_{n\xi_i}^{q_i} \prod_{j=1}^{l} EW_{n\eta_j}^{r_j}$$
$$\leq \sum_{\xi_1,\ldots,\xi_m;\eta_1,\ldots,\eta_l}^{**} \prod_{i=1}^{m} 2^{q_i} E|\frac{Y_{n\xi_i}}{a_n}|^2 \prod_{j=1}^{l} 2^{r_j} E|\frac{Y_{n\xi_i}}{a_n}|^{p+1}$$
$$\leq 2^{2pk}\left(\sum_{\xi_1,\ldots,\xi_m}^{**} \prod_{i=1}^{m} E|\frac{Y_{n\xi_i}}{a_n}|^2\right) \cdot \left(\sum_{\eta_1,\ldots,\eta_l}^{**} \prod_{j=1}^{l} E|\frac{Y_{n\eta_i}}{a_n}|^{p+1}\right)$$
$$\leq 2^{2pk}\left(\sum_{i=1}^{n} E|\frac{Y_{ni}}{a_{ni}}|^2\right)^m \left(\sum_{i=1}^{n} E|\frac{Y_{ni}}{a_n}|^{p+1}\right)^l \leq 2^{2pk}\left(\sum_{i=1}^{n} E|\frac{Y_{ni}}{a_n}|^{p+1}\right)$$
$$\leq 2^{2pk}\left(\sum_{i=1}^{n} E\frac{\psi(Y_{ni})}{\psi(a_n)}\right)$$

by the fact that $\sum_{i=1}^{n} E|\frac{Y_{ni}}{a_n}|^2 < 1$ and that $\frac{\psi(|t|)}{|t|^{p+1}}$ is an decreasing function of $|t|$.

Therefore, from Conditions (2.3) and (2.4),

$$\sum_{n=1}^{\infty} E\left(\sum_{i=1}^{n} W_{ni}\right)^{2pk} < \infty.$$

Finally, by Markov's inequality (2.8) is obtained, and the proof of Theorem 2.1 is completed.

If $a_n = n^{2/p}$ and $\psi(t) = |t|^p(\log^+|t|)^{1+\delta}$ for some $2 \leq p < 4$, and $\delta > 0$, then

$$\sup E|X_{ni}|^p(\log^+|X_{ni}|)^{1+\delta} \leq M \tag{2.12}$$

for some $M > 0$, implies Condition (2.3) holds and Condition (2.4) holds for $2k > \frac{p}{4-p}$. Thus, the following corollary for Theorem 2.1 is established.

**Corollary.** Let $\{X_{ni} : 1 \leq i \leq n; n \geq 1\}$ be an array of rowwise independent random variables. Then (2.2) and (2.12) imply that

$$\frac{1}{n^{2/p}} \sum_{i=1}^{n} X_{ni} \to 0.$$

**Theorem 2.2.** Let $\{X_{ni} : 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise independent random variables. and let $\psi(t)$ satisfy (2.1) for $p = 1$. Then Conditions (2.2) and (2.3) imply (2.5).

**Proof.** Condition (2.3) holding for $\{X_{ni}\}$ and $p = 1$ implies that Condition (2.3) holds for $\{Y_{ni}\}$ and $p = 1$. Since $|Y_{ni}| \leq a_n$, Condition (2.4) for $\{Y_{ni}\}$ follows and, the proof of Theorem 2.2 follows from the proof of Theorem 2.1.

When Condition (2.1) holds for $p = 0$, Conditions (2.2) and (2.4) are no longer needed.

**Theorem 2.3.** Let $\{X_{ni} : 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise independent random variables and let $\psi(t)$ satisfy (2.1) for $p = 0$. Then Condition (2.3) implies (2.5).

**Proof.** If Condition (2.3) holds for $p = 0$, then $|Y_{ni}| \leq a_n$ implies $\frac{\psi(Y_{ni})}{|Y_{ni}|} \geq \frac{\psi(a_n)}{a_n}$. Hence, for arbitrary $\epsilon > 0$,

$$P\left[|\frac{1}{a_n}\sum_{i=1}^{n} Y_{ni}| > \epsilon\right] \leq \frac{1}{\epsilon}\sum_{i=1}^{n} E|\frac{Y_{ni}}{a_n}| \leq \frac{1}{\epsilon}\sum_{i=1}^{n} E\frac{\psi(Y_{ni})}{\psi(a_n)}$$

and the proof follows from (2.3).

To close this section, an example is given to show that Condition (2.4) is a necessary condition for the SLLN. For each $n$, let $\{X_{ni} : 1 \leq i \leq n\}$ be independent, identically distributed normal random variables with mean zero and variance $n$. For $\psi(t) = t^6$ and $a_n = n$,

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} E\frac{\psi(X_{ni})}{\psi(n)} = \sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{15}{n^3} < \infty$$

and

$$\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} E\left(\frac{X_{ni}}{n}\right)^2\right)^{2k} = \sum_{n=1}^{\infty} 1 = \infty.$$

But, $\frac{1}{n}\sum_{i=1}^{n} X_{ni}$ has the standard normal distribution for each $n$, and hence can not converge to zero.

## 3. CONVERGENCE OF THE BOOTSTRAP MEAN AND VARIANCE

In this section the strong consistency of bootstrap estimators is considered. Let $X_1, X_2, \ldots$ be independent random variables with common distribution function $F$. Assume that $F$ has finite mean $\mu$ and variance $\sigma^2$, both unknown. The conventional estimators for $\mu$ and $\sigma^2$ are the sample average and sample variance, respectively. By the strong law of large numbers,

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \to \mu \quad \text{a.s.}$$

and

$$S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \to \sigma^2 \quad \text{a.s.}$$

In this situation, the consistencies of these estimators have been well-known. However, there has been considerable theoretical and practical interest in examining how the "bootstrap estimator" performs.

Let $F_n$ be the empirical distribution function of $X_1, X_2, \ldots, X_n$, putting mass $1/n$ on each $X_i$. The next step in the bootstrap method is to resample the data. Given $(X_1, X_2, \ldots, X_n)$, let $X_{n1}^*, X_{n2}^*, \ldots, X_{nm}^*$ be conditionally independent random variables with common distribution $F_n$. We denote the "bootstrap mean" by $\bar{X}_{nm}^*$ and the "bootstrap variance" by $S_{nm}^{*2}$, where

$$\bar{X}_{nm}^* = \frac{1}{m}\sum_{i=1}^{m} X_{ni}^* \quad \text{and} \quad S_{nm}^{*2} = \frac{1}{m}\sum_{i=1}^{m}(X_{ni}^* - \bar{X}_{nm}^*)^2. \tag{3.1}$$

Bickel and Freedman [6] showed the following result:

Suppose $X_1, X_2, \ldots$ are independent, identically distributed random variables which have finite positive variance $\sigma^2$. Along almost all sample sequences $X_1, X_2, \ldots$ given $(X_1, \ldots, X_n)$, if $n$ and $m$ tend to $\infty$, then $S^*_{nm} \to \sigma$ in conditional probability, that is, for all positive $\epsilon$,

$$P\left[|S^*_{nm} - \sigma| > \epsilon | X_1, \ldots, X_n\right] \to 0 \quad \text{a.s.}$$

Athreya, Ghosh, Low and Sen [7] obtained

$$P\left[\sup_{n \geq N} \sup_{m \geq dn} |\bar{X}^*_{nm} - \bar{X}_n| > \epsilon\right] \to 0 \tag{3.2}$$

as $N \to \infty$ for all $\epsilon > 0$ and $d > 0$ when $E|X_1|^{1+\delta} < \infty$ for some $\delta > 0$. Clearly, (3.2) yields the strong consistency of the bootstrap mean $\bar{X}_{nm}$. Athreya [8] and Csörgö [9] also provided proofs of the consistency of $\bar{X}^*_{nm}$ under slightly different settings. It will now be shown that Theorem 2.1 can be used to obtain the strong consistency of the bootstrap mean and variance in a very natural formulation in a fairly different approach from previously cited references.

**Theorem 3.1.** Let r.v.'s $X_1, X_2, \ldots$ be i.i.d. with mean $\mu$ and $E|X_1|^{1+\delta} < \infty$ for some $\delta > 0$. Along almost all sample sequences $X_1, X_2, \ldots$ given $(X_1, \ldots, X_n)$, if $n$ tends to $\infty$, then $\bar{X}^*_{nn} \to \mu$ with conditional probability 1.

**Proof.** The approach in this proof will be to start with a sample path realization $X_1, X_2, X_3, \ldots$ from which the array $\{X^*_{ni}\}$ is computerized generated and then show that the hypotheses of Theorem 2.1 are satisfied. It is understood that null sets from the domain of the observations $\{X_n\}$ must be excluded, and for the proof of Theorem 3.1 the specific null sets to be excluded are indicated by (3.3), (3.4) and (3.5). This approach is more natural for the experimenter and provides a short proof for the consistency of the bootstrap mean as an application of Theorem 2.1.

Let $E|X_1|^{1+\delta} < \infty$ from some $\delta, 0 < \delta < 1$. Let $k$ be an integer such that $k > 2$ and $k(\delta/(1+\delta)) > 1$. Applying the Marcinkiewicz-Zygmund SLLN to $\{|X_i|^{k-j}, i \geq 1\}$ with $\alpha = \frac{1+\delta}{k-j}$ in (1.2) and $\alpha = 1$ in (1.1), we have

$$\frac{1}{n^{(k-j)/(1+\delta)}} \sum_{i=1}^{n} |X_i|^{k-j} \to 0 \quad \text{a.s. for} \quad j = 0, \ldots, k-2 \tag{3.3}$$

$$\frac{1}{n} \sum_{i=1}^{n} |X_i| \to E|X_1| \quad \text{a.s.} \tag{3.4}$$

and

$$\frac{1}{n} \sum_{i=1}^{n} X_i \to \mu \quad \text{a.s.} \tag{3.5}$$

Given $(X_1, \ldots, X_n), n \geq 1$, the resampled data $\{X^*_{ni}, 1 \leq i \leq n\}, n \geq 1$, form a triangular array of r.v.'s which are rowwise i.i.d. with distribution $F_n$ and mean

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

If we choose $\psi(t) = |t|^k$, then

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{i=1}^{n} E(\psi(X_{ni}^* - \bar{X}_n)|X_1, ..., X_n)/\psi(n) \\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{n} E(|X_{ni}^* - \bar{X}_n|^k|X_1, ..., X_n)/n^k \\
&\le \sum_{n=1}^{\infty} \frac{1}{n^{k\delta/(1+\delta)}} \sum_{i=1}^{n} \frac{|X_i|^k}{n^{k/(1+\delta)}} + ... + \sum_{n=1}^{\infty} \frac{\binom{k}{j}|\bar{X}_n|^j}{n^{(k\delta+j)/(1+\delta)}} \sum_{i=1}^{n} \frac{|X_j|^{k-j}}{n^{(k-j)/(1+\delta)}} + ... \\
&+ \sum_{n=1}^{\infty} \frac{\binom{k}{k-1}|\bar{X}_n|^{k-1}}{n^{k-1}} \left(\sum_{i=1}^{n} \frac{|X_i|}{n}\right) + \sum_{n=1}^{\infty} \frac{|\bar{X}_n|^k}{n^{k-1}} < \infty \quad \text{a.s.}
\end{aligned} \tag{3.6}$$

by (3.3), (3.4) and the choice of $k$. Next,

$$\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} \frac{E((X_{ni}^* - \bar{X}_n)^2|X_1, ..., X_n)}{n^2}\right)^{2k} = \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\right)^{2k} / n^{4k}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{4k\delta/(1+\delta)}} \left(\sum_{i=1}^{n} \frac{X_i^2}{n^{2/(1+\delta)}} - \frac{\bar{X}_n^2}{n^{(1-\delta)/(1+\delta)}}\right)^{2k} < \infty \quad \text{a.s.} \tag{3.7}$$

by (3.3) and (3.5). Therefore, Theorem 2.1 implies

$$\frac{1}{n}\sum_{i=1}^{n}(X_{ni}^* - \bar{X}_n) \to 0 \quad \text{a.s.}$$

for almost all sample paths $X_1, X_2, ...$, and

$$\frac{1}{n}\sum_{i=1}^{n} X_{ni}^* \to \mu$$

with probability one when considering the total probability space.

The bootstrap variance can be written as $\frac{1}{n}\sum_{i=1}^{n} X_{ni}^{*2} - (\bar{X}_{nn}^*)^2$. Thus, with $X_{ni} = X_{ni}^{*2}$, in Theorem 3.1, we have $\frac{1}{n}\sum_{i=1}^{n} X_{ni}^{*2} \to EX_1^2$ with conditional probability 1. Therefore, the consistency of the bootstrap variance is established and is formally stated in Theorem 3.2, and the proof is similar to the proof of Theorem 3.1.

**Theorem 3.2.** Let r.v.'s $X_1, X_2, ...$ be i.i.d. with mean $\mu$, variance $\sigma^2$ and $E|X_1|^{2+\delta} < \infty$ for some $\delta > 0$. Along almost all sample sequences $X_1, X_2, ...$ given $(X_1, ..., X_n)$, if $n$ tends to $\infty$, then $S_{nn}^{*2} \to \sigma^2$ with conditional probability one.

In practice there is considerable interest in allowing the resampled implications $m$ (in 3.1) to be different from the number of original data points $n$. Consistency of these bootstrap estimators was established by Athreya [8] and Csörgö [9] when $m$ and $n$ had certain relationships. Their results are available in Theorems 3.1 and 3.2 by a different choice of $k$ (depending on the relationship) in the proof of Theorem 3.1.

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