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ON SOME CLASSES OF BCH-ALGEBRAS

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ABSTRACT. The concept of a BCH-algebra is a generalization of the concept of a BCI-algebra. It is shown that weakly commutative BCH-algebras are weakly commutative BCI-algebras. Moreover, the concepts of weakly positive implicative and weakly implicative BCH-algebras are defined and it is shown that every weakly implicative BCH-algebra is a weakly positive implicative BCH-algebra. The weakly positive implicative BCH-algebras are characterized with the help of their self maps. Two open problems are posed.

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1. Introduction. In 1966, Imai and Iséki introduced two classes of abstract algebras, BCK-algebras and BCI-algebras [6, 7]. BCI-algebras are a generalization of BCK-algebras. These algebras have been extensively studied since their introduction. In 1983, Hu and Li [4, 5] introduced the notion of a BCH-algebra, which is a generalization of the notions of BCK- and BCI-algebras. They have studied a few properties of these algebras. Certain other properties have been studied by Chaudhry [2] and Dudek and Thomys [3]. It has been shown [3, 4, 5] that there are no proper associative and medial BCH-algebras, that is, associative and medial BCH-algebras are associative and medial BCI-algebras, respectively.

The purpose of this paper is to investigate the existence of certain classes of proper BCH-algebras and study their properties. It is shown that proper weakly commutative BCH-algebras do not exist. However, proper weakly positive implicative and proper weakly implicative BCH-algebras exist and every weakly implicative BCH-algebra is a weakly positive implicative BCH-algebra but not conversely. Weakly positive implicative BCH-algebras have been characterized in terms of their self maps. The results proved in this paper are general in the sense that corresponding results for BCK-algebras and BCI-algebras become special cases.

2. Preliminaries. In this section, we describe certain definitions, known results, and examples that will be used in the sequel.

DEFINITION 2.1 (see [9]). A BCI-algebra is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following conditions:

(1) $(x * y) * (x * z) \leq z * y$,
(2) $x * (x * y) \leq y$,
(3) $x \leq x$,
(4) $x \leq y$ and $y \leq x$ imply $x = y$,
(5) $x \leq 0$ implies $x = 0$, where $x \leq y$ is defined by $x * y = 0$.

If (5) is replaced by $0 \le x$, then the algebra is called a BCK-algebra. It is known that every BCK-algebra is a BCI-algebra but not conversely. Further, in a BCI-algebra the identity $(x * y) * z = (x * z) * y$ holds [9].

Definition 2.2 (see [4]). A BCH-algebra is an algebra $(X, *, 0)$ of type (2, 0) satisfying the following conditions:

(3) $x \le x$,
(4) $x \le y$, $y \le x$ imply $x = y$,
(6) $(x * y) * z = (x * z) * y$, where $x \le y$ if and only if $x * y = 0$.

In any BCH-algebra, the following hold:

(2) $x * (x * y) \le y$ [4],
(5) $x * 0 = 0$ implies $x = 0$ [4],
(7) $0 * (x * y) = (0 * x) * (0 * y)$ [3],
(8) $x * 0 = x$ [3],
(9) $(x * y) * x = 0 * y$ [4],
(10) $x \le y$ implies $0 * x = 0 * y$ [2].

It is known that every BCI-algebra is a BCH-algebra but the following example shows that the converse is not true.

Example 2.3 (see [4]). Let $X = \{0, 1, 2, 3\}$ in which $*$ is defined by:

*	0	1	2	3
0	0	0	0	0
1	1	0	3	3
2	2	0	0	2
3	3	0	0	0

Then $(X, *, 0)$ is a BCH-algebra but it is not a BCI-algebra because

$$(2 * 3) * (2 * 1) = 2 * 0 = 2 \not\le 1 * 3 = 3. \tag{2.1}$$

Example 2.4 (see [2]). Let $X = \{0, 1, 2, 3, 4\}$ in which $*$ is defined by:

*	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	1	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

Routine calculations give that $(X, *, 0)$ is a BCH-algebra but it is not a BCI-algebra because

$$(1 * 3) * (1 * 2) = 1 * 0 = 1 \not\le 2 * 3 = 0. \tag{2.2}$$

In the sequel a BCH-algebra will be simply denoted by X.

Definition 2.5 (see [5]). A BCH-algebra X is called proper if it is not a BCI-algebra.

We note that BCH-algebras of Examples 2.3 and 2.4 are proper BCH-algebras.

Definition 2.6 (see [4]). A BCH/BCI-algebra X is called associative if $(x*y)*z = x*(y*z)$.

Definition 2.7 (see [3]). A BCH/BCI-algebra X is called medial if $(x*y)*(z*\mu) = (x*z)*(y*\mu)$.

In the sequel, we shall need the following result.

(11) A BCH-algebra X is proper if and only if it does not satisfy (1) (see [4]).

3. Classification of BCH-algebras.

It is known that associative BCH-algebras are associative BCI-algebras and medial BCH-algebras are medial BCI-algebras [3, 4]. Thus a natural question arises whether there exist some interesting classes of proper BCH-algebras or not? We show that there exist proper weakly positive implicative BCH-algebras as well as weakly implicative BCH-algebras. Moreover, the class of weakly implicative BCH-algebras is a proper subclass of the class of weakly positive implicative BCH-algebras. However, weakly commutative BCH-algebras are weakly commutative BCI-algebras.

Definition 3.1 (see [8]). A BCK-algebra X is called positive implicative if $(x*y)*z = (x*z)*(y*z)$. It is called implicative if $x*(y*x) = x$. It is commutative if $x*(x*y) = y*(y*x)$.

It is well known that positive implicative BCI-algebras, implicative BCI-algebras and commutative BCI-algebras are positive implicative BCK-algebras, implicative BCK-algebras and commutative BCK-algebras, respectively [9].

In [1], Chaudhry defined three classes of proper BCI-algebras, namely, weakly positive implicative BCI-algebras, weakly implicative BCI-algebras, and weakly commutative BCI-algebras. He also investigated a few properties of these algebras. We recall these definitions and the following result.

Definition 3.2 (see [1]). A BCI-algebra X is called weakly positive implicative if

(12) $(x*y)*z = ((x*z)*z)*(y*z)$.

It is called weakly implicative if

(13) $(x*(y*x))*(0*(y*x)) = x$.

It is called weakly commutative if

(14) $(x*(x*y))*(0*(x*y)) = y*(y*x)$.

Theorem 3.3 (see [1]). *A BCI-algebra X is weakly positive implicative if and only if*

(15) $x*y = ((x*y)*y)*(0*y)$.

We note that Theorem 3.3 tells us that (12) and (15) are equivalent in a BCI-algebra. However, they are not equivalent in a BCH-algebra. We consider the BCH-algebra X of Example 2.4. Then easy calculations give that (15) is satisfied but (12) is not satisfied because $(1*2)*3 = 0 \neq 1 = ((1*3)*3)*(2*3)$. Further the following theorem tells us that BCH-algebras satisfying (12) are BCI-algebras.

Theorem 3.4. *A BCH-algebra satisfying $(x*y)*z = ((x*z)*z)*(y*z)$ is a BCI-algebra.*

Proof. In view of (11) it is sufficient to prove that (1) holds.
Consider

$$\begin{aligned}((x*y)*(x*z))*(z*y) &= ((x*(x*z))*y)*(z*y)\\ &= (((x*y)*y)*((x*z)*y))*(z*y) \quad \text{(by (12))}\\ &= (((x*y)*y)*(z*y))*((x*z)*y)\\ &= ((x*z)*y)*((x*z)*y) \quad \text{(by (12))}\\ &= 0.\end{aligned} \tag{3.1}$$

This completes the proof. □

In view of Theorems 3.3 and 3.4 and the comments made between them, we adopt the following definitions for BCH-algebras.

Definition 3.5. A BCH-algebra X is weakly positive implicative if

$$x*y = ((x*y)*y)*(0*y) \quad \forall x,y \in X. \tag{3.2}$$

We note that the BCH-algebra of Example 2.4 satisfies (3.2). Thus there exist proper weakly positive implicative BCH-algebras.

Definition 3.6. A BCH-algebra X is weakly implicative if

$$(x*(y*x))*(0*(y*x)) = x \quad \forall x,y \in X. \tag{3.3}$$

Definition 3.7. A BCH-algebra X is weakly commutative if

$$(x*(x*y))*(0*(x*y)) = y*(y*x). \tag{3.4}$$

Theorem 3.8. *Every weakly implicative BCH-algebra X is a weakly positive implicative BCH-algebra.*

Proof. Let X be weakly implicative. Then

$$(x*(z*x))*(0*(z*x)) = x. \tag{3.5}$$

Putting $x = z*x$ in (3.5), we get

$$((z*x)*(z*(z*x)))*(0*(z*(z*x))) = z*x. \tag{3.6}$$

Since $z*(z*x) \le x$, therefore (10) gives $0*(z*(z*x)) = 0*x$. Thus

$$z*x = ((z*x)*(z*(z*x)))*(0*x) = ((z*(z*(z*x)))*x)*(0*x). \tag{3.7}$$

Now

$$\begin{aligned}&(z*x)*(z*(z*(z*x)))\\ &\quad= ((z*(z*(z*x))*x)*(0*x))*(z*(z*(z*x)))\\ &\quad= ((z*(z*(z*x))*x)*(z*(z*(z*x))))*(0*x)\\ &\quad= (0*x)*(0*x) = 0.\end{aligned} \tag{3.8}$$

Hence $z * x \le z * (z * (z * x)) \le z * x$. Thus

$$z * (z * (z * x)) = z * x \tag{3.9}$$

holds in a weakly implicative BCH-algebra. Putting (3.9) in (3.7) we get $z * x = ((z * x) * x) * (0 * x)$. Hence, X is weakly positive implicative. This completes the proof. □

Remark 3.9. It is known that $0 * x = 0 * (0 * (0 * x))$ holds in a BCH-algebra [3], but it is still not known that in a BCH-algebra the identity $x * y = x * (x * (x * y))$ holds or not, although it holds in BCI-algebras and weakly implicative BCH-algebras (as shown in (3.9)).

Remark 3.10. Since every BCI-algebra is a BCH-algebra and weak positive implicativeness and weak implicativeness coincide with positive implicativeness and implicativeness, respectively, in BCK-algebras [1], therefore the following results of Chaudhry and Iséki follow as corollaries of Theorem 3.4.

Corollary 3.11 (see [1]). *Every weakly implicative BCI-algebra is a weakly positive implicative BCI-algebra.*

Corollary 3.12 (see [8]). *Every implicative BCK-algebra is a positive implicative BCK-algebra.*

Theorem 3.13. *A BCH-algebra X satisfying $(x*(x*y))*(0*(x*y)) = y*(y*x)$ is a BCI-algebra.*

Proof. It is sufficient to show that (1) holds. We consider

$$\begin{aligned}
&((x * y) * (x * z)) * (z * y) \\
&\quad = ((x * (x * z)) * y) * (z * y) \\
&\quad = (((z * (z * x)) * (0 * (z * x))) * y) * (z * y) \quad \text{(by given condition)} \\
&\quad = (((z * (z * x)) * y) * (0 * (z * x))) * (z * y) \\
&\quad = (((z * y) * (z * x)) * (0 * (z * x))) * (z * y) \\
&\quad = (((z * y) * (z * y)) * (z * x)) * (0 * (z * x)) \\
&\quad = (0 * (z * x)) * (0 * (z * x)) = 0.
\end{aligned} \tag{3.10}$$

This completes the proof. □

We now pose the following open problem.

Open problem 1. Do there exist classes of proper BCH-algebras other than the classes of weakly positive implicative and weakly implicative BCH-algebras, which are generalizations of the known classes of BCI- as well as BCK-algebras.

4. Characterization of weakly positive implicative BCH-algebras.

In this section, we characterize weakly positive implicative BCH-algebras by their self maps.

Definition 4.1. Let X be a BCH-algebra. For a fixed x in X, the map $R_x : X \to X$ given by $R_x(t) = t * x$ for all $t \in X$ is called a right self map.

Definition 4.2. Let X be a BCH-algebra. For a fixed x in X, the map $R'_x : X \to X$ given by $R'_x(t) = (t * x) * (0 * x)$ for all $t \in X$ is called a weak right self map.

The following theorem gives us a characterization of a weakly positive implicative BCH-algebra with the help of its right and weak right self maps.

Theorem 4.3. *A BCH-algebra X is weakly positive implicative if and only if $R_z = R'_z \circ R_z$ for all $z \in X$, where "$\circ$" is composition of functions.*

Proof. Let X be a BCH-algebra and $R_z = R'_z \circ R_z$. Then $R_z(y) = R'_z \circ R_z(y)$ for all $y \in X$. Thus $y * z = R'_z(y * z) = ((y * z) * z) * (0 * z)$ for all $y, z \in X$. Hence X is a weakly positive implicative BCH-algebra. Conversely, if X is a weakly positive implicative BCH-algebra, then $y * z = ((y * z) * z) * (0 * z)$. Thus $R_z(y) = (R_z(y) * z) * (0 * z) = R'_z(R_z(y)) = R'_z \circ R_z(y)$ for all $y, z \in X$. Hence $R_z = R'_z \circ R_z$. This completes the proof. □

Theorem 4.4. *Let X be a weakly positive implicative BCH-algebra. Then $R'_y = R'_y \circ R'_y = (R'_y)^2$.*

Proof. Since X is weakly positive implicative, therefore $x * y = ((x * y) * y) * (0 * y)$. Thus

$$\begin{aligned}(x * y) * (0 * y) &= (((x * y) * y) * (0 * y)) * (0 * y) \\ &= (((x * y) * (0 * y)) * y) * (0 * y).\end{aligned} \tag{4.1}$$

Hence

$$R'_y(x) = R'_y((x * y) * (0 * y)) = R'_y(R'_y(x)) = R'_y \circ R'_y(x) = (R'_y)^2(x) \tag{4.2}$$

for all $x, y \in X$. This completes the proof. □

The following example shows that the converse of the above theorem is not true.

Example 4.5. Let $X = \{0, a, b, c\}$ in which $*$ is defined by:

$*$	0	a	b	c
0	0	0	b	b
a	a	0	b	b
b	b	b	0	0
c	c	b	a	0

Then X is a BCI-algebra. Further X is not weakly positive implicative because $a = c * b \neq ((c * b) * b) * (0 * b) = (a * b) * (0 * b) = b * b = 0$. Moreover, easy calculations give that

$$R'_0 = (R'_0)^2, \qquad R'_a = (R'_a)^2, \qquad R'_b = (R'_b)^2, \qquad R'_c = (R'_c)^2. \tag{4.3}$$

This shows that the converse of Theorem 4.4 does not hold for the class of BCH-algebras, because it does not hold for BCI-algebras.

We now pose another open problem.

Open problem 2. What are the characterizations of weakly positive implicative BCH-algebras and weakly implicative BCH-algebras in terms of their ideals.

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