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Research Article

PS-Modules over Ore Extensions and Skew Generalized Power Series Rings

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A right R-module M_R is called a PS-module if its socle, $\mathrm{Soc}(M_R)$, is projective. We investigate PS-modules over Ore extension and skew generalized power series extension. Let R be an associative ring with identity, M_R a unitary right R-module, $O = R[x;\alpha,\delta]$ Ore extension, $M[x]_O$ a right O-module, $(S,\leq)$ a strictly ordered additive monoid, $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism, $A = [[R^{S,\leq},\omega]]$ the skew generalized power series ring, and $B_A = [[M^{S,\leq}]]_{[[R^{S,\leq},\omega]]}$ the skew generalized power series module. Then, under some certain conditions, we prove the following: (1) If M_R is a right PS-module, then $M[x]_O$ is a right PS-module. (2) If M_R is a right PS-module, then B_A is a right PS-module.

This paper is dedicated to Professor M. H. Fahmy on the occasion of his 68th birthday

1. Introduction

Throughout this paper R denotes an associative ring with identity and M_R a unitary right R-module. According to Nicholson and Watters [1], M_R is called a *PS-module* if every simple submodule is projective and equivalently if its socle, $\mathrm{Soc}(M_R)$, is projective. Examples of PS-modules include nonsingular modules, regular modules in the sense of Zelmanowitz [2], and modules with zero socle. The class of PS-modules is closed under direct sums and submodules. In [3], Weimin proved that PS-modules are preserved by Morita equivalences and excellent extensions.

For any subset X of R, denote

$$\mathrm{l}_M(X) = \{m \in M \mid mX = 0\}. \quad (1)$$

Theorem 1 (see [3]). *The following statements are equivalent for a right R-module M_R:*

(1) *M_R is a PS-module.*

(2) *If L is a maximal right ideal of R, then either $l_M(L) = 0$ or $L = eR$, where $e^2 = e \in R$.*

A left PS-module ${}_RM$ is defined analogously. A ring R is said to be a *left PS-ring* if ${}_RR$ is a PS-module. Every semiprime ring is a PS-ring. Every PP-ring is a PS-ring (where a ring R is called PP-ring if every principal left ideal is projective). In particular every Baer ring is a PS-ring (where a ring R is called Baer if every left (or right) annihilator is generated by an idempotent). A ring for which every simple singular module is injective is a PS-ring. If $\mathrm{l}_R(J(R)) = 0$, then R is a PS-ring. In fact $J(R) \subset L$ for every maximal right ideal so $\mathrm{l}_R(L) = 0$.

The notion of PS-rings is not left-right symmetric (cf. [1]). A ring R is *duo* if each one-sided ideal of R is a two-sided ideal. As a generalization of left duo rings, a ring R is called *weakly left duo* if for every $r \in R$ there is a natural number $n(r)$ such that $Rr^{n(r)}$ is a two-sided ideal of R. A ring R is weakly duo if it is weakly right and left duo. In [3], Weimin proved that a duo ring R is a PS-ring if and only if it is a right PS-ring. In [4], Dingguo generalized this result to weakly duo rings as follows: a weakly duo reduced ring R is a PS-ring if and only if R is a right PS-ring.

If R is a PS-ring so also are $R[x]$ and $R[[x]]$. The converse of this result is false by the following example.

Example 2 (see [1], Example 3.2). If $R = \mathbb{Z}_4$, then $R[x]$ and $R[[x]]$ are PS-rings but R is not PS-ring.

The motivation of this paper is to investigate the PS property of Ore extension modules and the skew generalized power series extension modules. These results generalize the corresponding results for polynomial rings, generalized power series rings, and modules [5, 6].

2. PS-Modules over Ore Extension Rings

This section is devoted to study the relationship between the PS property of a right R-module M_R and the PS property of the right Ore extension module $M[x]_O(M[x]_{R[x;\alpha,\delta]})$.

Let α be an endomorphism of R and $\delta : R \to R$ an α-derivation of R, that is, an additive map such that

$$\delta(ab) = \delta(a)b + \alpha(a)\delta(b), \quad \forall a, b \in R. \tag{2}$$

In case α is the identity map, δ is called just a derivation of R.

The Ore extension $O = R[x;\alpha,\delta]$ is the set of all polynomials $\sum_{i=0}^{n} a_i x^i$ with the usual sum and the following multiplication rule:

$$xa = \alpha(a)x + \delta(a). \tag{3}$$

We assume that 1 is the identity element of $O = R[x;\alpha,\delta]$. This means that $\alpha(1) = 1$ and $\delta(1) = 0$. This definition of noncommutative polynomial rings with identity was first introduced by Ore [7]. Ever since the appearance of Ore's fundamental paper [7], Ore extensions have played an important role in noncommutative ring theory and many noncommutative ring theorists have investigated Ore extensions from different points of view such as ideal theory, order theory, Galois theory, and homological algebras.

For integers i, j with $j \geq i \geq 0$, $f_i^j \in \mathrm{End}(R,+)$ will denote the map which is the sum of all possible words in α, δ built with i letters of α and $j-i$ letters of δ. For instance,

$$\begin{aligned} f_0^0 &= \mathrm{Id}_R, \\ f_j^j &= \alpha^j, \\ f_0^j &= \delta^j, \\ f_{j-1}^j &= \alpha^{j-1}\delta + \alpha^{j-2}\delta\alpha + \cdots + \delta\alpha^{j-1}. \end{aligned} \tag{4}$$

For any positive integer n and $r \in R$, we have

$$x^n r = \sum_{i=0}^{n} f_i^n(r)x^i \tag{5}$$

(see [8], Lemma 4.1). This formula uniquely determines a general product of (left) polynomials in O and will be used freely in what follows.

Given a right R-module M_R, $M[x]_O$ is a right O-module with the natural action of O on $M[x]$ applying the above twist whenever necessary. The verification that this defines a valid O-module structure on $M[x]$ is almost identical to the verification that $O = R[x;\alpha,\delta]$ is a ring and it is straightforward (see [9]).

Definition 3 (see [9]). Given a module M_R, an endomorphism $\alpha : R \to R$ and an α-derivation $\delta : R \to R$. One says that M_R is α-compatible if for each $m \in M_R$, $r \in R$, one has $mr = 0 \Leftrightarrow m\alpha(r) = 0$. Moreover, One says that M_R is δ-compatible if for each $m \in M_R$, $r \in R$, one has $mr = 0 \Rightarrow m\delta(r) = 0$. If M_R is both α-compatible and δ-compatible, one says that M_R is (α,δ)-compatible.

Note that if M_R is α-compatible (resp., δ-compatible), then M_R is α^i-compatible (resp., δ^i-compatible) for all $i \geq 1$. It is clear that M_R is α-compatible (resp., δ-compatible), then so is any submodule of M_R. A ring R is (α,δ)-compatible if and only if R_R is an (α,δ)-compatible module.

As an immediate consequence of Definition 3, we obtain the following.

Lemma 4. *Let M_R be an (α,δ)-compatible module. For each $m \in M$ and $a \in R$, one has the following:*

(1) *$ma = 0$ if and only if $m\alpha^n(a) = 0$ for any positive integer n.*

(2) *If $ma = 0$, then $mf_i^j(a) = 0$ for all $j \geq i \geq 0$.*

Lemma 5 (see [10], Lemma 2.5). *Let M_R be an (α,δ)-compatible module, $m(x) = m_0 + \cdots + m_k x^k \in M[x]$, and $r \in R$. If $m(x)r = 0$, then $m_i r = 0$ for each i.*

Definition 6 (see [10]). Given a module M_R, an endomorphism $\alpha : R \to R$ and an α-derivation $\delta : R \to R$. One says that M_R is (α,δ)-Armendariz if whenever $m(x) = \sum_{i=0}^{k} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{n} a_j x^j \in R[x;\alpha,\delta]$ satisfy $m(x)f(x) = 0$, one has $m_i x^i a_j x^j = 0$ for all i, j.

A ring R is called (α,δ)-Armendariz if R_R is an (α,δ)-Armendariz module.

Using Lemma 5 it is easy to deduce that if M_R is (α,δ)-compatible and (α,δ)-Armendariz, then for any $m(x) = \sum_{i=0}^{k} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{n} a_j x^j \in R[x;\alpha,\delta]$, $m(x)f(x) = 0$ if and only if $m_i a_j = 0$ for all i, j.

Theorem 7. *Let M_R be an (α,δ)-compatible and (α,δ)-Armendariz module. If M_R is a PS-module, then $M[x]_O$ is a PS-module.*

Proof. Let L be a maximal right ideal of O. We will show that either $\mathrm{l}_{M[x]}(L) = 0$ or $L = hO$, where $h^2 = h \in O$. Let I be the set of all coefficients of all polynomials in L and let J be the right ideal of R generated by I. If $J = R$, then there exist $a_1, \ldots, a_n \in I$ and $r_1, \ldots, r_n \in R$ such that

$$1 = a_1 r_1 + \cdots + a_n r_n. \tag{6}$$

Suppose that $\varphi(x)=\sum_{i=0}^{k} m_i x^i \in l_{M[x]}(L)$ and $\varphi \neq 0$, then for every $g(x)=\sum_{j=0}^{n} a_j x^j \in L$, we have

$$\varphi(x) g(x)=\left(\sum_{i=0}^{k} m_i x^i\right)\left(\sum_{j=0}^{n} a_j x^j\right)=0. \quad (7)$$

Since M_R is (α,δ)-compatible and (α,δ)-Armendariz, it follows that

$$m_i a_j=0, \quad \forall 0 \leq i \leq k,\ 0 \leq j \leq n. \quad (8)$$

Consequently, for every $a \in I$, $m_i a=0$, $0 \leq i \leq k$. Hence we get

$$\begin{aligned} m_i &= m_i 1=m_i\left(a_1 r_1+\cdots+a_n r_n\right) \\ &= \left(m_i a_1\right) r_1+\cdots+\left(m_i a_n\right) r_n=0, \end{aligned} \quad (9)$$

a contradiction. Then $l_{M[x]}(L)=0$. Suppose that $J \neq R$. We will show that J is a maximal right ideal of R. Let $r \in R-J$. If $r \in L$, then $r \in I$ and so $r \in J$, a contradiction. Thus $r \notin L$. Since L is a maximal right ideal of O,

$$O=L+rO. \quad (10)$$

It follows that there exist $g(x)=\sum_{i=0}^{n} a_i x^i \in L$ and $h(x)=\sum_{j=0}^{m} b_j x^j \in O$ such that $1=a_0+rb_0$. If $a_0=0$, then $1=rb_0 \in rR$ and so $R=J+rR$. If $a_0 \neq 0$, then $a_0 \in I \subseteq J$ which implies that $R=J+rR$. Hence J is a maximal right ideal of R. Since M_R is a PS-module, it follows that either $l_M(J)=0$ or $J=eR$, where $e^2=e \in R$. According to that we have the following two cases.

Case 1. Suppose that $l_M(J)=0$. We will show that $l_{M[x]}(L)=0$. Let $\varphi(x)=\sum_{i=0}^{k} m_i x^i \in l_{M[x]}(L)$ and $\varphi \neq 0$; then for every $g(x)=\sum_{j=0}^{n} a_j x^j \in L$, we have

$$\varphi(x) g(x)=\left(\sum_{i=0}^{k} m_i x^i\right)\left(\sum_{j=0}^{n} a_j x^j\right)=0. \quad (11)$$

Since M_R is (α,δ)-compatible and (α,δ)-Armendariz, it follows that $m_i a_j=0$ for all $0 \leq i \leq k$ and $0 \leq j \leq n$. Consequently, for every $a \in I$, $m_i a=0$, $0 \leq i \leq k$. For any $r \in J$, there exist $a_1,\ldots,a_n \in I$ and $r_1,\ldots,r_n \in R$ such that $r=a_1 r_1+\cdots+a_n r_n$. Hence

$$\begin{aligned} m_i r &= m_i\left(a_1 r_1+\cdots+a_n r_n\right) \\ &= \left(m_i a_1\right) r_1+\cdots+\left(m_i a_n\right) r_n=0, \end{aligned} \quad (12)$$

which implies that $m_i \in l_M(J)=0$. Thus $\varphi(x)=0$, a contradiction. Hence $l_{M[x]}(L)=0$.

Case 2. Suppose that $J=eR$, where $e^2=e \in R$. We will show that $L=eO$, where $e^2=e \in O$. To show that $eO \subseteq L$, we need to prove that $e \in L$. If $e \notin L$, then $O=L+eO$. Thus there exist $g(x)=\sum_{i=0}^{n} a_i x^i \in L$ and $h(x)=\sum_{j=0}^{m} b_j x^j \in O$ such that $1=a_0+eb_0$. If $a_0=0$, then $1=eb_0 \in eR=J$, a contradiction. If $a_0 \neq 0$, then $a_0 \in I \subseteq J$ which implies that $1=a_0+eb_0 \in J+eR=J$, a contradiction. Therefore $e \in L$ which implies that $eO \subseteq L$. Now we show that $L \subseteq eO$. Suppose that $g(x)=\sum_{i=0}^{n} a_i x^i \in L$; then, for all $0 \leq i \leq n$, $a_i \in I \subseteq J=eR$ and so $a_i=ea_i$. We have

$$eg(x)=e\sum_{i=0}^{n} a_i x^i=\sum_{i=0}^{n}\left(ea_i\right) x^i=\sum_{i=0}^{n} a_i x^i=g(x); \quad (13)$$

it follows that $g(x) \in eO$. Thus $L \subseteq eO$. □

If $M_R=R_R$, we get the following.

Corollary 8. *Let R be an (α,δ)-compatible and (α,δ)-Armendariz ring. If R is a right PS-ring, then $R[x;\alpha,\delta]$ is a right PS-ring.*

3. PS-Modules over Skew Generalized Power Series Rings

Let $(S,\leq)$ be an ordered commutative monoid. Unless stated otherwise, the operation of S will be denoted additively, and the identity by 0. Recall that $(S,\leq)$ is artinian if every strictly decreasing sequence of elements of S is finite and that $(S,\leq)$ is narrow if every subset of pairwise order-incomparable elements of S is finite. The following construction is due to Zhongkui [11].

Let $(S,\leq)$ be a strictly ordered monoid (i.e., if $s,s',t \in S$ and $s<s'$, then $s+t<s'+t$), R a ring, and $\omega: S \rightarrow \operatorname{End}(R)$ a monoid homomorphism. Consider the set $A=[[R^{S,\leq},\omega]]$ of all maps $f: S \rightarrow R$ whose support ($\operatorname{supp}(f)=\{s \in S \mid f(s) \neq 0\}$) is artinian and narrow.

For every $s \in S$ and $f,g \in A$, let

$$\begin{aligned} & X_s(f,g) \\ & \quad=\{(u,v) \in S \times S \mid u+v=s;\ f(u) \neq 0,\ g(v) \neq 0\}. \end{aligned} \quad (14)$$

It follows from ([12], 4.1) that $X_s(f,g)$ is a finite set.

This fact allows defining the operation of multiplication (*convolution*) as follows:

$$(fg)(s)=\sum_{(u,v) \in X_s(f,g)} f(u)\,\omega_u(g(v)), \quad (15)$$

and $(fg)(s)=0$ if $X_s(f,g)=\phi$. With this operation and pointwise addition $A=[[R^{S,\leq},\omega]]$ becomes a ring, which is called the *ring of skew generalized power series* with coefficients in R and exponents in S.

In [13], Zhao and Jiao generalized this construction to obtain the skew generalized power series modules over skew generalized power series rings as follows.

Let M_R be a right R-module; let B be the set of all maps $\varphi: S \rightarrow M$ such that $\operatorname{supp}(\varphi)=\{s \in S \mid \varphi(s) \neq 0\}$ is artinian and narrow. With pointwise addition, $B=[[M^{S,\leq}]]$ is an abelian additive group. For each $f \in A=[[R^{S,\leq},\omega]]$ and $\varphi \in B$, the set

$$\begin{aligned} & X_s(\varphi,f) \\ & \quad=\{(u,v) \in S \times S \mid u+v=s;\ \varphi(u) \neq 0,\ f(v) \neq 0\} \end{aligned} \quad (16)$$

is finite (see [14], Lemma 1). This allows defining the scalar multiplication of the elements of B by scalars from A as follows:

$$(\varphi f)(s) = \sum_{(u,v)\in X_s(\varphi,f)} \varphi(u)\,\omega_u(f(v)), \tag{17}$$

and $(\varphi f)(s) = 0$ if $X_s(\varphi, f) = \phi$. With this operation and pointwise addition, one can easily show that B is a right A-module, which is called the *module of skew generalized power series* with coefficients in M and exponents in S.

For every $s \in S$ if we set $\omega(s) = \mathrm{Id}_R \in \mathrm{Aut}(R) \subset \mathrm{End}(R)$, the identity map of R, then $A = [[R^{S,\leq}, \omega]] = [[R^{S,\leq}]]$ is the ring of generalized power series in the sense of Ribenboim [12] and $B = [[M^{S,\leq}]]$ is the untwisted module of generalized power series in the sense of [15].

For any $r \in R$ we associated the map $c_r \in A$ defined by

$$c_r(x) = \begin{cases} r, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0. \end{cases} \tag{18}$$

For any $m \in M$ and $s \in S$, we define a map $d_m^s \in B$ by

$$d_m^s(x) = \begin{cases} m, & \text{if } x = s, \\ 0, & \text{if } x \neq s. \end{cases} \tag{19}$$

Definition 9 (see [13]). A right R-module M_R is called ω-compatible whenever $ma = 0$ if and only if $m\omega_s(a) = 0$ for any $s \in S$, $m \in M$, and $a \in R$.

Clearly, R is an ω-compatible ring if and only if R_R is an ω-compatible R-module.

Theorem 10. *Let $(S, \leq)$ be a strictly totally ordered monoid which satisfies the condition $0 \leq s$ for every $s \in S$ and let M_R be an ω-compatible module. If M_R is a PS-module, then B_A is a PS-module.*

Proof. Let L be a maximal right ideal of A. We will show that either $\mathrm{l}_B(L) = 0$ or $L = hA$, where $h^2 = h \in A$. Since $(S, \leq)$ is a strictly totally ordered monoid, $\mathrm{supp}(f)$ is a nonempty well-ordered subset of S, for every $0 \neq f \in A$. We denote by $\pi(f)$ the smallest element of support f.

For any $s \in S$, set

$$\begin{aligned} I_s &= \{f(s) \mid f \in L, \pi(f) = s\} \subset R, \\ I &= \bigcup_{s\in S} I_s. \end{aligned} \tag{20}$$

Let J be the right ideal of R generated by I. If $J = R$, then there exist $s_1, \ldots, s_n \in S$, $f_1, \ldots, f_n \in L$, and $r_1, \ldots, r_n \in R$ such that

$$1 = f_1(s_1) r_1 + \cdots + f_n(s_n) r_n, \tag{21}$$

where $f_i(s_i) \in I_{s_i}$ and $\pi(f_i) = s_i$, for every $1 \leq i \leq n$. We will show that $\mathrm{l}_B(L) = 0$. Suppose that $\varphi \in \mathrm{l}_B(L)$ and $\varphi \neq 0$. Then $\mathrm{supp}(\varphi)$ is a nonempty well-ordered subset of S. Let $t = \pi(\varphi)$; if

$$\varphi(t)\,\omega_t(f_i(s_i)) \neq 0, \quad \text{for some } 1 \leq i \leq n, \tag{22}$$

then

$$\begin{aligned} (\varphi f_i)(t + s_i) &= \sum_{(u,v)\in X_{t+s_i}(\varphi,f_i)} \varphi(u)\,\omega_u(f_i(v)) \\ &= \varphi(t)\,\omega_t(f_i(s_i)) \neq 0. \end{aligned} \tag{23}$$

This means that $\varphi f_i \neq 0$ for some $1 \leq i \leq n$, a contradiction. Thus

$$\varphi(t)\,\omega_t(f_i(s_i)) = 0, \quad \forall 1 \leq i \leq n. \tag{24}$$

Since M_R is an ω-compatible module, we get

$$\varphi(t) f_i(s_i) = 0, \quad \forall 1 \leq i \leq n. \tag{25}$$

Consequently

$$\begin{aligned} \varphi(t) &= \varphi(t)\,1 = \varphi(t)\,(f_1(s_1) r_1 + \cdots + f_n(s_n) r_n) \\ &= (\varphi(t) f_1(s_1)) r_1 + \cdots + (\varphi(t) f_n(s_n)) r_n = 0, \end{aligned} \tag{26}$$

a contradiction. Thus $\mathrm{l}_B(L) = 0$. Suppose that $J \neq R$. We will show that J is a maximal right ideal of R. Let $r \in R - J$. If $c_r \in L$, then $r = c_r(0) \in I_0 \subset I$ and so $r \in J$, a contradiction. Therefore $c_r \notin L$. Since L is a maximal right ideal of A,

$$A = L + c_r A. \tag{27}$$

It follows that there exist $f \in L$ and $g \in A$ such that $c_1 = f + c_r g$. Thus

$$\begin{aligned} 1 &= c_1(0) = f(0) + (c_r g)(0) = f(0) + r\omega_0(g(0)) \\ &= f(0) + rg(0). \end{aligned} \tag{28}$$

If $f(0) = 0$, then $1 = rg(0) \in rR$. So, $R = J + rR$.

If $f(0) \neq 0$, then $0 \in \mathrm{supp}(f)$. Since $0 \leq s$ for every $s \in S$, $\pi(f) = 0$. Thus $f(0) \in I_0 \subset I \subset J$, which implies that $R = J + rR$.

Hence J is a maximal right ideal of R. Since M_R is a PS-module, it follows that either $\mathrm{l}_M(J) = 0$ or $J = eR$, where $e^2 = e \in R$. According to that we have the following two cases.

Case 1. Suppose that $\mathrm{l}_M(J) = 0$. We will show that $\mathrm{l}_B(L) = 0$. Let $\varphi \in \mathrm{l}_B(L)$ and $\varphi \neq 0$. Then $\mathrm{supp}(\varphi)$ is a nonempty well-ordered subset of S. Let $s = \pi(\varphi)$. For any $r \in J$, there exist $s_1, \ldots, s_n \in S$, $f_1, \ldots, f_n \in L$, and $r_1, \ldots, r_n \in R$ such that

$$r = f_1(s_1) r_1 + \cdots + f_n(s_n) r_n, \tag{29}$$

where $f_i(s_i) \in I_{s_i}$ and $\pi(f_i) = s_i$, for every $1 \leq i \leq n$. Since $\varphi \in \mathrm{l}_B(L)$, $f_1, \ldots, f_n \in L$, we get $\varphi f_i = 0$ for every $1 \leq i \leq n$. If

$$\varphi(s)\,\omega_s(f_i(s_i)) \neq 0, \quad \text{for some } 1 \leq i \leq n, \tag{30}$$

then

$$(\varphi f_i)(s+s_i) = \sum_{(u,v)\in X_{s+s_i}(\varphi,f_i)} \varphi(u)\,\omega_u(f_i(v)) = \varphi(s)\,\omega_s(f_i(s_i)) \neq 0. \quad (31)$$

This means that $\varphi f_i \neq 0$ for some $1 \le i \le n$, a contradiction. Thus

$$\varphi(s)\,\omega_s(f_i(s_i)) = 0, \quad \forall 1 \le i \le n. \quad (32)$$

Since M_R is an ω-compatible module, we get

$$\varphi(s)\,f_i(s_i) = 0, \quad \forall 1 \le i \le n. \quad (33)$$

Consequently

$$\begin{aligned}\varphi(s)\,r &= \varphi(s)\,(f_1(s_1)\,r_1 + \cdots + f_n(s_n)\,r_n)\\ &= (\varphi(s)\,f_1(s_1))\,r_1 + \cdots + (\varphi(s)\,f_n(s_n))\,r_n\\ &= 0.\end{aligned} \quad (34)$$

Therefore, $\varphi(s) \in \mathrm{l}_M(J) = 0$ and $\pi(\varphi) = s$. Thus $\varphi = 0$, a contradiction. Hence $\mathrm{l}_B(L) = 0$.

Case 2. Suppose that $J = eR$, where $e^2 = e \in R$. We will show that $L = c_eA$, where $(c_e)^2 = c_e \in A$. To show that $c_eA \subseteq L$, we need to prove that $c_e \in L$. If $c_e \notin L$, then $A = L + c_eA$. Thus there exist $f \in L$ and $g \in A$ such that $c_1 = f + c_eg$. Thus

$$\begin{aligned}1 = c_1(0) &= f(0) + (c_eg)(0) = f(0) + e\omega_0(g(0))\\ &= f(0) + eg(0).\end{aligned} \quad (35)$$

If $f(0) = 0$, then $1 = eg(0) \in eR = J$, a contradiction.

If $f(0) \neq 0$, then $0 \in \mathrm{supp}(f)$. Since $0 \le s$ for every $s \in S$, $\pi(f) = 0$. Thus $f(0) \in I_0 \subset I \subset J$, which implies that $f(0) \in J$ and $J = eR$. Hence $1 = f(0) + eg(0) \in J + eR = J$, a contradiction. Therefore $c_e \in L$ which implies that $c_eA \subseteq L$.

Conversely, suppose that $f \in L$ and $\pi(f) = s$; then $f(s) \in I_s \subset I \subset J = eR$ and so $f(s) = ef(s)$. We claim that $f(u) = ef(u)$ for any $u \in \mathrm{supp}(f)$.

Suppose that $f(v) = ef(v)$ for each $v < u$. Consider the following element $f_u \in A$ defined by

$$f_u(x) = \begin{cases} f(x), & x < u,\\ 0, & x \ge u.\end{cases} \quad (36)$$

Thus $\pi(f - f_u) = u$. By hypothesis it is easy to see that $f_u = c_ef_u \in c_eA \subset L$. Thus $f - f_u \in L$. By analogy with the proof above, it follows that

$$(f - f_u)(u) = e(f - f_u)(u), \quad (37)$$

which implies that $f(u) = ef(u)$. Thus our claim holds. Therefore

$$(c_ef)(t) = \sum_{(u,v)\in X_t(c_e,f)} c_e(u)\,\omega_u(f(v)) = c_e(0)\,\omega_0(f(t)) = ef(t) = f(t). \quad (38)$$

Hence $f = c_ef \in c_eA$. Thus $L = c_eA$ and the result follows since c_e is an idempotent of A. □

As a special case of the last result if we set $M_R = R_R$ we get the following.

Corollary 11. *Let $(S,\le)$ be a strictly totally ordered monoid which satisfies the condition that $0 \le s$ for every $s \in S$ and let R be an ω-compatible ring. If R is a right PS-ring, then $A = [[R^{S,\le},\omega]]$ is a right PS-ring.*

If we set $\omega(s) = \mathrm{Id}_R$, for every $s \in S$, we get the following as a corollary.

Corollary 12 (see [6], Theorem 1). *Let $(S,\le)$ be a strictly totally ordered monoid which satisfies the condition that $0 \le s$ for every $s \in S$. If M_R is a PS-module, then $[[M^{S,\le}]]_{[[R^{S,\le}]]}$ is a PS-module.*

If $M_R = R_R$, we get the following as a corollary.

Corollary 13 (see [5], Theorem 4). *Let $(S,\le)$ be a strictly totally ordered monoid which satisfies the condition that $0 \le s$ for every $s \in S$. If R is a right PS-ring, then $[[R^{S,\le}]]$ is a right PS-ring.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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