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EXPANSION OF A CLASS OF FUNCTIONS INTO AN INTEGRAL INVOLVING ASSOCIATED LEGENDRE FUNCTIONS

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ABSTRACT. A theorem for expansion of a class of functions into an integral involving associated Legendre functions is obtained in this paper. This is a somewhat general integral expansion formula for a function $f(x)$ defined in (x_1, x_2) where $-1 < x_1 < x_2 < 1$, which is perhaps useful in solving certain boundary value problems of mathematical physics and of elasticity involving conical boundaries.

KEY WORDS AND PHRASES. Integral expansion of a function, associated Legendre function, Mehler-Fok integral transform.
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1. INTRODUCTION.

Integral transforms are often used to solve the problems of mathematical physics involving linear partial differential equations and also other problems. Integral expansions involving spherical functions of a class of functions are known as Mehler-Fok type transforms. In these transform formulae, the subscript of the Legendre functions appear as the integration variable while its superscript is either zero or a fixed integer (see Sneddon [10]). There is another class of integral transforms involving associated Legendre functions somewhat related to the Mehler-Fok transforms, in which the superscript of the associated Legendre function appears in the integration formula while the subscript (complex) is kept fixed. Felsen [2] first developed this type of transform formulae involving $P^{-\mu}_{-1/2+i\tau}(\cos\theta)$ as kernel where $0 < \theta < \pi$ from a unique δ-function representation. Later Mandal ([6], [7]) obtained somewhat similar types of two transform formulae from the solution of two appropriately designed boundary value problems. In the first type, the argument x of $P^{-\mu}_{-1/2+i\tau}(x)$ ranges from -1 to 1 while in the second, the argument z of $P^{\mu}_{-1/2+i\tau}(z)$ ranges from 1 to ∞. Recently Mandal and Guha Roy [8] used a similar technique to establish another Mehler-Fok type integral transform formula involving $P^{-\mu}_{-1/2+i\tau}(\cos\theta)$ as kernel $(0 < \theta < \alpha)$.

In the present paper, an integral expansion of a class of functions defined in (x_1, x_2) where $-1 < x_1 < x_2 < 1$, involving associated Legendre functions is obtained. Based on direct investigation of the properties of spherical functions, sufficient conditions which would establish the validity of this expansion formula for a wide class of functions are obtained in a manner similar to the ideas used in ([3]-[5]). The main result is given in section 2 in the form of a theorem. Recently, we have used a similar technique to establish another type of integral representation [9] involving $P^{-\mu}_{-1/2+i\tau}(\cosh\alpha)$ as kernel where $0<\alpha<\alpha_0$.

2. INTEGRAL EXPANSION OF A FUNCTION IN (x_1,x_2) WHERE $-1<x_1<x_2<1$.

We present the main result of this paper in the form of the following theorem.

THEOREM. Let $f(x)$ be a given function defined on the interval (x_1,x_2) where $-1<x_1<x_2<1$ and satisfies the following conditions:

(1) The function $f(x)$ is piecewise continuous and has a bounded variation in the open interval (x_1,x_2).

(2) The function $f(x)(1-x^2)^{-1}\ell n(1-x^2)^{-1}\in L(x_1,x_2)$, $-1<x_1<x_2<1$.

Then we have

$$f(x)=\sum_k \sigma_k\,\Gamma\left(\tfrac{1}{2}+i\tau-i\sigma_k\right)\Gamma\left(\tfrac{1}{2}-i\tau-i\sigma_k\right)\frac{M(x,x_2;i\sigma_k)}{(\partial/\partial\sigma_k)M(x_2,x_1;i\sigma_{\hat{k}})}\,F(\sigma_{\hat{k}})$$
$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\sigma\,\Gamma\left(\tfrac{1}{2}+i\tau-i\sigma\right)\Gamma\left(\tfrac{1}{2}-i\tau-i\sigma\right)\frac{M(x,x_2;i\sigma)}{M(x_2,x_1;i\sigma)}\,F(\sigma)\,d\sigma \tag{2.1}$$

where

$$F(\sigma)=\int_{x_1}^{x_2}\frac{f(x)}{1-x^2}\,M(x,x_1;i\sigma)\,d\sigma, \tag{2.2}$$

$$-1<x_1<x_2<1,\ M(x,y;i\sigma)=P^{i\sigma}_{-1/2+i\tau}(x)P^{i\sigma}_{-1/2+i\tau}(-y)-P^{i\sigma}_{-1/2+i\tau}(-x)P^{i\sigma}_{-1/2+i\tau}(y)$$

and σ_k's$,\sigma,\tau$ are real. The equation (2.2) may be regarded as an integral transform of the function $f(x)$ defined in (x_1,x_2) and (2.1) is its inverse. (2.1) and (2.2) together give the integral expansion of the function $f(x)$.

PROOF OF THE EXPANSION THEOREM. To prove this expansion theorem, we first note that the representation (cf. Erdélyi [1])

$$P^{i\sigma}_{-1/2+i\tau}(x)=\left(\frac{1+x}{1-x}\right)^{i\sigma/2}F\left(\tfrac{1}{2}+i\tau,\ \tfrac{1}{2}-i\tau;\ 1-i\sigma;\ \tfrac{1-x}{2}\right)/\,\Gamma(1-i\sigma),$$

$-1<x_1<x<x_2<1$, where $F(a,b;c;x)$ denotes the hypergeometric series, implies $P^{i\sigma}_{-1/2+i\tau}(x)$ is continuous in the region defined by $-1<x_1<x<x_2<1$, $-\infty<\sigma<\infty$ and satisfies the inequality

$$\left|P^{i\sigma}_{-1/2+i\tau}(x)\right|\le\sqrt{sh\pi\sigma/\pi\sigma}\;P_{-1/2+i\tau}(x), \tag{2.3}$$

where the Legendre function $P_{-1/2+i\tau}(x)$ is positive.

Using (2.3) it follows from (2.2) that

$$\int_{x_1}^{x_2}\left|\frac{f(x)}{1-x^2}\left[P^{i\sigma}_{-1/2+i\tau}(x)\,P^{i\sigma}_{-1/2+i\tau}(-x_1)-P^{i\sigma}_{-1/2+i\tau}(-x)\,P^{i\sigma}_{-1/2+i\tau}(x_1)\right]\right|dx$$

$$\le\sqrt{sh\pi\sigma/\pi\sigma}\int_{x_1}^{x_2}\frac{|f(x)|}{1-x^2}\left\{P_{-1/2+i\tau}(x)\,P_{-1/2+i\tau}(-x_1)-P_{-1/2+i\tau}(-x)\,P_{-1/2+i\tau}(x_1)\right\}dx,$$

and this shows that the conditions imposed on $f(x)$ imply that the integral $F(\sigma)$ is absolutely and uniformly convergent for $\sigma\in[-T,T]$ where T is a positive large number. Hence $F(\sigma)$ is continuous on $[-T,T]$ and the repeated integral

$$J(x,T)=\frac{1}{2\pi i}\int_{-T}^{T}\sigma\Gamma\left(\tfrac{1}{2}+i\tau-i\sigma\right)\Gamma\left(\tfrac{1}{2}-i\tau-i\sigma\right)\frac{M(x,x_2;i\sigma)}{M(x_2;x_1;i\sigma)}\,d\sigma\cdot\int_{x_1}^{x_2}\frac{f(y)}{1-y^2}\,M(y,x_1;i\sigma)\,dy$$

is meaningful. Also, uniform convergence allows us to change the order of integration and write $J(x,T)$ as

$$J(x,T) = \int_{x_1}^{x_2} \frac{f(y)}{1-y^2} K(x,y,T)\, dy, \tag{2.4}$$

where

$$K(x,y,T) = \frac{1}{2\pi i} \int_{-T}^{T} \sigma\, \Gamma\left(\tfrac{1}{2}+i\tau-i\sigma\right)\Gamma\left(\tfrac{1}{2}-i\tau-i\sigma\right)\frac{M(x,x_2;i\sigma)M(y,x_1;i\sigma)}{M(x_2,x_1;i\sigma)}\, d\sigma. \tag{2.5}$$

Now we shall show that the kernel $K(x,y,T)$ is symmetric in the variables x and y. By definition, we have

$$K(x,y,T) - K(y,x,T) = \frac{1}{2\pi i} \int_{-T}^{T} \sigma\, \Gamma\left(\tfrac{1}{2}+i\tau-i\sigma\right)\Gamma\left(\tfrac{1}{2}-i\tau-i\sigma\right)\frac{1}{M(x_2,x_1;i\sigma)}$$

$$\cdot\left[M(x,x_2;i\sigma)M(y,x_1;i\sigma) - M(y,x_2;i\sigma)M(x,x_1;i\sigma)\right] d\sigma.$$

It follows from the properties of associated Legendre functions (cf. Erdélyi [1]) that the integrand in the above integral is an odd function of σ, hence the integral vanishes. Thus

$$K(y,x,T) = K(x,y,T). \tag{2.6}$$

To investigate the behavior of $K(x,y,T)$ as $T\rightarrow\infty$, by writing $\mu = -i\sigma$, we write (2.5) as

$$K(x,y,T) = \frac{1}{2\pi i} \int_{-iT}^{iT} \mu\, \Gamma\left(\tfrac{1}{2}+i\tau+\mu\right)\Gamma\left(\tfrac{1}{2}-i\tau+\mu\right)\frac{M(x,x_2;-\mu)M(y,x_1;-\mu)}{M(x_2,x_1;-\mu)}\, d\mu. \tag{2.7}$$

Expression under the integral sign in (2.7) is analytic function fo the complex variable μ and it has no singularity in the semi-plane $Re\mu \geq 0$, except for simple poles at $\mu = -i\sigma_k$ (k is positive integer) (cf. Felsen [2]), where

$$M(x_2,x_1;i\sigma_k) = 0,\ \sigma_k > 0. \tag{2.8}$$

Completing the contour of integration on (2.7) with the arc Γ_T of radius T situated in the semi-plane $Re\mu \geq 0$ and applying the residue theorem, we obtain

$$K(x,y,T) = K_1(x,y,T) - \sum_k \sigma_k\, \Gamma\left(\tfrac{1}{2}+i\tau-i\sigma_k\right)\Gamma\left(\tfrac{1}{2}-i\tau-i\sigma_k\right)\cdot\frac{M(x,x_2;i\sigma_k)M(y,x_1;i\sigma_k)}{(\partial/\partial\sigma_k)M(x_2;x_1;i\sigma_k)}, \tag{2.9}$$

where

$$K_1(x,y,T) = \frac{1}{2\pi i} \int_{\Gamma_T} \mu\, \Gamma\left(\tfrac{1}{2}+i\tau+\mu\right)\Gamma\left(\tfrac{1}{2}-i\tau+\mu\right)\frac{M(x,x_2;-\mu)M(y,x_1;-\mu)}{M(x_2,x_1;-\mu)}\, d\mu. \tag{2.10}$$

Suppose that $y \leq x$. By virtue of the definition

$$P^{-\mu}_{-1/2+i\tau}(x) = \left(\frac{1+x}{1-x}\right)^{-\mu/2}\frac{1}{\Gamma(1+\mu)}\left[1+0(|\mu|^{-1})\right],$$

$$P^{-\mu}_{-1/2+i\tau}(-x) = \left(\frac{1-x}{1+x}\right)^{-\mu/2}\frac{1}{\Gamma(1+\mu)}\left[1+0(|\mu|^{-1})\right] \tag{2.11}$$

Using (2.11) and asymptotic properties of the gamma function for large μ, we conclude that

$$\mu\, \Gamma\left(\tfrac{1}{2}+i\tau+\mu\right)\Gamma\left(\tfrac{1}{2}-i\tau+\mu\right)\frac{M(x,x_2;-\mu)M(y,x_1;-\mu)}{M(x_2,x_1;-\mu)}$$

$$=\frac{\left[\left(\frac{1+x}{1-x}\,\frac{1-x_2}{1+x_2}\right)^{-\mu/2}-\left(\frac{1-x}{1+x}\,\frac{1+x_2}{1-x_2}\right)^{-\mu/2}\right]\left[\left(\frac{1+y}{1-y}\,\frac{1-x_1}{1+x_1}\right)^{-\mu/2}-\left(\frac{1-y}{1+y}\,\frac{1+x_1}{1-x_1}\right)^{-\mu/2}\right]}{\left[\left(\frac{1+x_2}{1-x_2}\,\frac{1-x_1}{1+x_1}\right)^{-\mu/2}-\left(\frac{1-x_2}{1+x_2}\,\frac{1+x_1}{1-x_1}\right)^{-\mu/2}\right]}$$

$$\cdot\left[1+0(|\mu|^{-1})\right]. \tag{2.12}$$

Now introduce the new variables

$$\xi=\frac{1}{2}\,\ell n\,\frac{1+x}{1-x},\ \eta=\frac{1}{2}\,\ell n\,\frac{1+y}{1-y},\ \alpha=\frac{1}{2}\,\ell n\,\frac{1+x_1}{1-x_1}\ \text{and}\ \beta=\frac{1}{2}\,\ell n\,\frac{1+x_2}{1-x_2}.$$

Then, for large μ, from (2.10) - (2.12) we obtain for $y\le x$

$$K_1(x,y,T)=\frac{1}{2\pi i}\int_{\Gamma_T}\left[exp\{-\mu(\xi-\eta)\}+exp\{-\mu(2\beta-2a-\xi+\eta)\}\right.$$

$$\left.-exp\{-\mu(\xi+\eta-2\alpha)\}-exp\{-\mu(2\beta-\xi-\eta)\}\right]d\mu$$

$$+O(1)\int_0^{\pi/2}exp\{-\mu(\xi-\eta)\cos\varphi\}+exp\{-\mu(2\beta-2\alpha-\xi+\eta)\cos\varphi\}]$$

$$-exp\{-\mu(\xi+\eta-2\alpha)\cos\varphi\}-exp\{-\mu(2\beta-\xi-\eta)\cos\varphi\}]\,d\varphi,$$

$$\text{for } \alpha<\eta\le\xi<\beta.$$

Using the identity

$$\frac{2}{\pi}\int_0^{\pi/2}exp\{-\lambda T\cos\varphi\}\,d\varphi\le\frac{1-exp(-\lambda T)}{\lambda T},\ \lambda\ge 0,$$

we obtain for $y\le x$,

$$K_1(x,y,T)=\frac{1}{\pi}\left[\frac{\sin T(\xi-\eta)}{\xi-\eta}+\frac{\sin T(2\beta-2\alpha-\xi+\eta)}{2\beta-2\alpha-\xi+\eta}-\frac{\sin T(\xi+\eta-2a)}{\xi+\eta-2a}\right.$$

$$\left.-\frac{\sin T(2\beta-\xi-\eta)}{2\beta-\xi-\eta}\right]+O(1)\left[\frac{1-exp\{-T(\xi-\eta)\}}{T(\xi-\eta)}+\frac{1-exp\{-T(2\beta-2\alpha-\xi+\eta)\}}{T(2\beta-2\alpha-\xi+\eta)}\right.$$

$$\left.-\frac{1-exp\{-T(\xi+\eta-2\alpha)\}}{T(\xi+\eta-2\alpha)}-\frac{1-exp\{-T(2\beta-\xi-\eta)\}}{T(2\beta-\xi-\eta)}\right],\qquad \alpha<\eta\le\xi<\beta, \tag{2.13}$$

where the factor $O(1)$ is independent of y.

Again for $y\ge x$, we use the symmetry property (2.6) and the representation (2.10) of $K_1(x,y,T)$ with the variables x,y replaced by y,x.

Now we write (2.4) as

$$J(x,T)=\int_{x_1}^{x}\frac{f(y)}{1-y^2}\,K_1(x,y,T)\,dy+\int_{x}^{x_2}\frac{f(y)}{1-y^2}\,K_1(x,y,T)\,dy$$

$$-\sum_k\sigma_k\,\Gamma\left(\frac{1}{2}+i\tau-i\sigma_k\right)\Gamma\left(\frac{1}{2}-i\tau-i\sigma_k\right)\frac{M(x,x_2;i\sigma_k)}{(\partial/\partial\sigma_k)M(x_2,x_1;i\sigma_k)}\int_{x_1}^{x_2}\frac{f(y)}{1-y^2}\,M(y,x_1;i\sigma_k)\,dy$$

$$=J_1(x,T)+J_2(x,T)-\sum_k\sigma_k\,\Gamma\left(\frac{1}{2}+i\tau-i\sigma_k\right)\Gamma\left(\frac{1}{2}-i\tau-i\sigma_k\right)\frac{M(x,x_2;i\sigma_k)}{(\partial/\partial\sigma_k)M(x_2,x_1;i\sigma_k)}\times$$

$$\times\cdot\int_{x_1}^{x_2}\frac{f(y)}{1-y^2}M(y,x_1;i\sigma_k)dy. \tag{2.14}$$

Using (2.13) in J_1, we obtain

$$J_1(x,T)=\frac{1}{\pi}\left[\int_\alpha^\xi f(\tanh\eta)\frac{\sin T(\xi-n)}{\xi-\eta}\,d\eta+\int_\alpha^\xi f(\tanh\eta)\frac{\sin T(2\beta-2\alpha-\xi+\eta)}{2\beta-2\alpha-\xi+\eta}\,d\eta\right.$$

$$\left.-\int_\alpha^\xi f(\tanh\eta)\frac{\sin T(\xi+\eta-2\alpha)}{\xi+\eta-2\alpha}\,d\eta-\int_\alpha^\xi f(\tanh\eta)\frac{\sin T(2\beta-\xi-\eta)}{2\beta-\xi-\eta}\,d\eta\right]$$

$$+O(1)\left[\int_\alpha^\xi |f(\tanh\eta)|\frac{1-exp\{-T(\xi-\eta)\}}{T(\xi-\eta)}\,d\eta\right.$$

$$+\int_\alpha^\xi |f(\tanh\eta)|\frac{1-exp\{-T(2\beta-2\alpha-\xi+\eta)\}}{T(2\beta-2\alpha-\xi+\eta)}\,d\eta$$

$$-\int_\alpha^\xi |f(\tanh\eta)|\frac{1-exp\{-T(\xi+\eta-2\alpha)\}}{T(\xi+\eta-2\alpha)}\,d\eta$$

$$\left.-\int_\alpha^\xi |f(\tanh\eta)|\frac{1-exp\{-T(2\beta-\xi-\eta)\}}{T(2\beta-\xi-\eta)}\,d\eta\right]. \tag{2.15}$$

The conditions satisfied by $f(x)$ imply that $f(\tanh\eta)\in L(\alpha,\beta)$; hence, by virtue of Dirichlet's theorem, for $T\to\infty$

$$\frac{1}{\pi}\int_\alpha^\xi f(\tanh\eta)\frac{\sin T(\xi-\eta)}{\xi-\eta}\,d\eta=\frac{1}{2}f(\tanh\xi-o)+o(1)$$

$$=\frac{1}{2}f(x-o)+o(1),$$

$$\frac{1}{\pi}\int_\alpha^\xi f(\tanh\eta)\frac{\sin T(2\beta-2a-\xi+\eta)}{2\beta-2\alpha-\xi+\eta}\,d\eta=o(1),$$

$$\frac{1}{\pi}\int_\alpha^\xi f(\tanh\eta)\frac{\sin T(\xi+\eta-2\alpha)}{\xi+\eta-2a}\,d\eta=o(1),$$

and

$$\frac{1}{\pi}\int_\alpha^\xi f(\tanh\eta)\frac{\sin T(2\beta-\xi-\eta)}{2\beta-\xi-\eta}\,d\eta=o(1).$$

Moreover, if the integral of integration is divided into the subintervals $(\xi-\delta,\xi)$ and $(\alpha,\xi-\delta)$ and if a sufficiently small positive δ (implying a sufficiently large T) is chosen, then we have

$$\int_\alpha^\xi |f(\tanh\eta)|\frac{1-exp\{-T(\xi-\eta)\}}{T(\xi-\eta)}\,d\eta$$

$$\le\frac{1}{\delta T}\int_\alpha^{\xi-\delta}|f(\tanh\eta)|\,d\eta+\int_{\xi-\delta}^\xi|f(\tanh\eta)|\,d\eta$$

$$=O(T^{-1})+o(1)=o(1)\text{ for }T\to\infty,$$

$$\int_\alpha^\xi |f(\tanh\eta)|\frac{1-exp\{-T(2\beta-2\alpha-\xi+\eta)\}}{T(2\beta-2\alpha-\xi+\eta)}\,d\eta\le\frac{1}{\xi T}\int_\alpha^\xi|f(\tanh\eta)|\,d\eta$$

$$= O(T^{-1}) = o(1) \text{ for } T \to \infty,$$

$$\int_{\alpha}^{\xi} |f(tanh\ \eta)| \frac{1 - exp\{-T(\xi + \eta - 2\alpha)\}}{T(\xi + \eta - 2\alpha)} d\eta \le \frac{1}{\xi T} \int_{\alpha}^{\xi} |f(tanh\ \eta)|\, d\eta$$

$$= O(T^{-1}) = o(1) \text{ for } T \to \infty,$$

and

$$\int_{\alpha}^{\xi} |f(tanh\ \eta)| \frac{1 - exp\{-T(2\beta - \xi - \eta)\}}{T(2\beta - \xi - \eta)} d\eta \le \frac{1}{\xi T} \int_{\alpha}^{\xi} |f(tanh\ \eta)|\, d\eta$$

$$= O(T^{-1}) = o(1) \text{ for } T \to \infty. \quad (2.17)$$

Thus (2.15) to (2.17) leads to

$$\lim_{T \to \infty} J_1\, (tanh\ \xi, T) = \frac{1}{2} f(tanh\ \xi - o) = \frac{1}{2} f(x - o). \quad (2.18)$$

Similarly,

$$\lim_{T \to \infty} J_2(tanh\ \xi, T) = \frac{1}{2} f(tanh\ \xi + o) = \frac{1}{2} f(x + o). \quad (2.19)$$

Hence,

$$\lim_{T \to \infty} J(x,T) = \frac{1}{2}[f(x+o) + f(x-o)] - \sum_k \sigma_k\, \Gamma\left(\frac{1}{2} + i\tau - i\sigma_k\right) \Gamma\left(\frac{1}{2} - i\tau - i\sigma_k\right)$$

$$\cdot \frac{M(x, x_2; i\sigma_k)}{(\partial/\partial\sigma_k) M(x_2, x_1; i\sigma_k)} F(\sigma_k). \quad (2.20)$$

Thus, at the points of continuity of $f(x)$ we obtain (2.1). We note that (2.1) becomes a result in [5] when $x_1 = -1$ and $x_2 = 1$.

It follows from the foregoing theorem that, at points of continuity of $f(x)$, we have

$$f(x) = \sum_k \sigma_k\, \Gamma\left(\frac{1}{2} + i\tau - i\sigma_k\right) \Gamma\left(\frac{1}{2} - i\tau - i\sigma_k\right) \frac{R(x, x_2; i\sigma_k)}{(\partial^2/\partial x_2 \partial\sigma_k) R(x_2, x_1; i\sigma_k)} F(\sigma_k)$$

$$+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \sigma\, \Gamma\left(\frac{1}{2} + i\tau - i\sigma\right) \Gamma\left(\frac{1}{2} - i\tau - i\sigma\right) \frac{R(x, x_2; i\sigma)}{(\partial/\partial x_2) R(x_2, x_1; i\sigma)} F(\sigma) d\sigma, \quad (2.21)$$

where

$$F(\sigma) = \int_{x_1}^{x_2} \frac{f(x)}{1 - x^2} R(x, x_1; i\sigma)\, dx,\ -1 < x_1 < x_2 < 1, \quad (2.22)$$

$$R(x, y; i\sigma) = P^{i\sigma}_{-1/2 + i\tau}(x) \frac{\partial}{\partial y} P^{i\sigma}_{-1/2 + i\tau}(-y) - P^{i\sigma}_{-1/2 + i\tau}(-x) \frac{\partial}{\partial y} P^{i\sigma}_{-1/2 + i\tau}(y)$$

and σ_k's, σ, τ are real.

The integrand in (2.21) has singularities at $\sigma = \sigma_k$(k is positive integers) which are simple poles along the positive σ-axis, where

$$\frac{\partial}{\partial x_2} R(x, x_1; i\sigma_k) = 0,\ (\sigma_k > 0). \quad (2.23)$$

To prove (2.21) we use the following asymptotic formulas for large μ:

$$\frac{\partial}{\partial x} P^{-\mu}_{-1/2 + i\tau}(x) = -\frac{\mu}{\Gamma(1+\mu)} \frac{1}{(1-x)(1+x)} \left(\frac{1+x}{1-x}\right)^{-\mu/2} [1 + O(|\mu|^{-1})],$$

$$\frac{\partial}{\partial x} P^{-\mu}_{-1/2 + i\tau}(-x) = -\frac{\mu}{\Gamma(1+\mu)} \frac{1}{(1+x)(1-x)} \left(\frac{1-x}{1+x}\right)^{-\mu/2} [1 + O(|\mu|^{-1})], \quad (2.24)$$

The proof of (2.21) is similar to the proof in the section 2, and we do not reproduce it. We note that (2.21) becomes a result in [5] when $x_1 = -1$ and $x_2 = 1$.

3. EXAMPLES.

We now give examples of expansions of some functions.

$$(1)\quad (1-x^2)^{\nu/2} = \sum_k \sigma_k \Gamma\left(\tfrac{1}{2}+i\tau-i\sigma_k\right)\Gamma\left(\tfrac{1}{2}-i\tau-i\sigma_k\right)\frac{M(x,x_2;i\sigma_k)}{(\partial/\partial\sigma_k)M(x_2,x_1;i\sigma_k)}$$

$$\cdot\frac{2^\nu\Gamma(1+\nu)}{(\nu^2+\sigma_k^2)}(\nu+i\sigma_k)\left[P_\nu^{-\nu}(x_1)M_1(x_1,x_1;i\sigma_k)-P_\nu^{-\nu}(x_2)\, M_1(x_2,x_1;i\sigma_k)\right]$$

$$+\frac{2^\nu\Gamma(1+\nu)}{2\pi i}\int_{-\infty}^{\infty}\frac{\sigma(\nu+i\sigma)}{\nu^2+\sigma^2}\Gamma\left(\tfrac{1}{2}+i\tau-i\sigma\right)\Gamma\left(\tfrac{1}{2}-i\tau-i\sigma\right)\frac{M(x,x_2;i\sigma)}{M(x_2,x_1;i\sigma)}$$

$$\cdot\left[P_\nu^{-\nu}(x_1)M_1(x_1,x_1;i\sigma)-P_\nu^{-\nu}(x_2)\, M_1(x_2,x_1;i\sigma)\right]d\sigma,$$

$$(-1<x_1<x<x_2<1)$$

where

$$M(x,y;i\sigma)=P_\nu^{i\sigma}(x)\,P_\nu^{i\sigma}(-y)-P_\nu^{i\sigma}(-x)\,P_\nu^{i\sigma}(y),$$

$$M_1(x,y;i\sigma)=P_{\nu-1}^{i\sigma}(x)\,P_\nu^{i\sigma}(-y)-P_{\nu-1}^{i\sigma}(-x)\,P_\nu^{i\sigma}(y) \text{ and } \nu=-1/2+i\tau.$$

$$(2)\quad P_\nu^\mu(x)=\sum_k\sigma_k\Gamma\left(\tfrac{1}{2}+i\tau-i\sigma_k\right)\Gamma\left(\tfrac{1}{2}-i\tau-i\sigma_k\right)\frac{M(x,x_2;i\sigma_k)}{(\partial/\partial\sigma_k)M(x_2,x_1;i\sigma_k)}\;\frac{1}{(\mu^2+\sigma_k^2)}$$

$$\cdot\Big[(\nu+\mu)\,P_{\nu-1}^\mu(x_2)M(x_2,x_1;i\sigma_k)+(\nu+i\sigma_k)\,\{P_\nu^\mu(x_1)M_1(x_1,x_2;i\sigma_k)$$

$$-P_\nu^\mu(x_2)M_1(x_2,x_2;i\sigma_k)\}\Big]+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\sigma}{\mu^2+\sigma^2}\Gamma\left(\tfrac{1}{2}+i\tau-i\sigma\right)\Gamma\left(\tfrac{1}{2}-i\tau-i\sigma\right)$$

$$\cdot\frac{M(x,x_2;i\sigma)}{M(x_2,x_1;i\sigma)}\Big[(\nu+\mu)\,P_{\nu-1}^\mu(x_2)M(x_2,x_1;i\sigma)+(\nu+i\sigma)\,\{P_\nu^\mu(x_1)$$

$$\cdot M_1(x_1,x_2;i\sigma)-P_\nu^\mu(x_2)M_1(x_2,x_1;i\sigma)\}\Big]\,d\sigma.$$

In all these results the conditions under which the expansion theorem hold are satisfied.

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REFERENCES

1. ERDÉLYI, A.; MAGNUS, W.; OBERHITTINGER, F. & TRICOMI, F.G., *Higher Transcendental Functions, Vol. 1*, McGraw Hill Co., 1953.

2. FELSEN, L.B., Some new transform theorems involving Legendre functions, *J. Math. Phys.* **37** (1958), 188-191.

3. LEBEDEV, N.N. & SKAL'SKAYA, I.P., Integral expansion of an arbitrary function in terms of spherical functions, *PMM* **30** (1966), 252-258.

4. LEBEDEV, N.N. & SKAL'SKAYA, I.P., Expansion of an arbitrary function into an integral in terms of associated spherical functions, *PMM* **32** (1968), 421-427.

5. LEBEDEV, N.N. & SKAL'SKAYA, I.P., Integral representations related to Mehler-Fok transformations, *Differential Equations* **22** (1986), 1050-1056.

6. MANDAL, B.N., An integral transform associated with degree of Legendre functions, *Bull. Cal. Math. Soc.* **63** (1971), 1-6.

7. MANDAL, B.N., Note on an integral transform, *Bull. Math. de la Soc. Math. de la R.S. de Roumanie* **63** (1971), 87-93.

8. MANDAL, B.N. & GUHA ROY, P., On a Mehler-Fok type integral transform, *Appl. Math. Lett.* **4** (1991), 29-32.

9. MANDAL, N. & MANDAL, B.N., Integral representation of a function in terms of associated Legendre functions, *Bull. Math. de la Soc. Math. de la R.S. de Roumanie* **4** (1991), to appear.

10. SNEDDON, I.N., *The Use of Integral Transforms*, McGraw Hill Co., 1972.


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