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RESEARCH NOTES

A CLASS OF COMPLETELY REGULAR SPACES

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ABSTRACT. In this paper we investigate topologies with ultrafilters having bases of open sets. It is shown that these topologies are completely regular. All results are obtained by using Richardson's compactification of convergence spaces. We also prove a non-existance of a dense convergence space compactification with lifting property.

KEYWORDS AND PHRASES. *Convergence space, compactification.*
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1. PRELIMINARIES

A *convergence space* (Limesraum [1]) is a set X along with a map τ from X to the power set $P(F(X))$ of the set $F(X)$ of filters on X, such that for each $x \in X$ the following conditions are satisfied:

(CI) $\dot{x} \in \tau x$

(CII) $F \in \tau x\ ,\ G \geq F \Rightarrow G \in \tau x$

(CIII) $F, G \in \tau x \Rightarrow F \wedge G \in \tau x$.

Filters in τx are said to be *convergent* or *converging to* x . We call a convergence space *compact* if ultrafilters are convergent. A convergence space is a *pretopology* if each τx contains a smallest filter. This filter will be denoted B_x . We shall write (X,τ) for the convergence space, or simply X if no confusion is possible.

Let (X,τ) and (Y,σ) be convergence spaces. A map $f : X \to Y$ is *continuous* if $F \in \tau x$ implies $f(F) \in \sigma f(x)$.

Throughout the paper we assume all spaces to be *Hausdorff*, i.e. if $\tau x \cap \tau y \neq \emptyset$ then $x = y$. The category of (Hausdorff) convergence spaces with continuous maps as morphisms is denoted HCON . Similarly the full subcategory of compact spaces is denoted by HCCON .

A subset A of X is called *open* if $x \in A$ implies that $A \in F$ for all $F \in \tau x$. A pretopology is a topology if each B_x has a base of open sets. The class of open sets determines a topology called the *topological modification of* X .

The *closure* $\mathrm{cl}[B]$ of a set $B \subseteq X$ is the set of points $x \in X$ such that some $F \in \tau x$ satisfies $F \cap B \neq \emptyset$ for each $F \in F$. The closure $\mathrm{cl} : P(X) \to P(X)$ is in general not idempotent. A pretopology with an idempotent closure is always a topology. If $f : X \to Y$ is continuous we will have $f(\mathrm{cl}[B]) \subseteq \mathrm{cl}[f(B)]$. For a filter F on X the sets $\mathrm{cl}[F]$, $F \in F$, constitute a base for a filter on X, denoted $\mathrm{cl}[F]$. The reader is referred to [1] for more properties of convergence spaces.

In this paper we intend to examine a subclass by using the famous Richardson compactification for convergence spaces. For notational convention and easier understanding of proofs the construction of the compactification is given in the following.

Let (X,τ) be a convergence space and write N_X for the class of non-convergent ultrafilters on X. For $A \subseteq X$ define

$$A^* = A \cup \{ U \in N_X : A \in U \} .$$

We have in particular $X^* = X \cup N_X$. If F is a filter on X we write F^* for the filter on X^* generated by sets F^* where $F \in F$. Now define $\tau^* : X^* \to P(F(X^*))$ through

$$\tau^* x = \{ G \in F(X^*) : \exists F \in \tau x, G \geq F^* \} , \quad x \in X$$
$$\tau^* U = \{ G \in F(X^*) : G \geq U^* \} , \quad U \in N_X .$$

Then (X^*,τ^*) will become a Hausdorff compact convergence space including X as a (dense) subspace. For properties of the Richardson compactification see [2-4].

REMARK. It is not difficult to see that (X^*,τ^*) is isomorphic to the *Kowalsky completion* of the *Cauchy space* (X,C) where

$$C = \cup_{x\in X} \tau x \cup N_X .$$

Cauchy space terminology is found in [5].

2. STRONG COMPLETE REGULARITY

Note that even if X is a topology X^* need not always be one. We do, however, have the following.

THEOREM 2.1. (1) X is a pretopology iff X^* is a pretopology.
(2) X is a topology iff X^* is a topology and non-convergent ultrafilters have bases of open sets.

PROOF. (1) If X is a pretopology then B_x^* is the smallest filter in $\tau^* x$. Since U^*, $U \in N_X$, is the smallest filter in $\tau^* U$ we have that X^* is a pretopology. If X^* is a pretopology then for $x \in X$ every $\tau^* x$ has a smallest filter $G(x)$. Now some $F \in \tau x$ satisfies $G(x) \geq F^*$ and it is not difficult to see that indeed $F = B_x$.
(2) Let X be a topology with non-convergent ultrafilters having bases of open sets. First note that if A is open then also A^* is open. In fact, for $y \in A^*$ we have either $y \in A$ or $y \in N_X$ with $A \in y$. Choose any $G \in \tau^* y$. For $y \in A$ we have $G \geq B_y^*$. Since $A \in B_y$ we get $A^* \in B_y^*$ and so $A^* \in G$. For $y \in N_X$ with $A \in y$ we have $G \geq y^*$. We get $A^* \in y^*$ and therefore $A^* \in G$.

The sets B^* where $B \in B_x$ is open will be a base for B_x^*. Further each

$\mathcal{U} \in N_X$ has a base of open sets. Hence also $\mathcal{U}^*$ has a base of open sets. This shows that X^* is a topology.

Now assume that X^* is a topology. Choose $V \in \mathcal{B}_x$. Then we have $V^* \in \mathcal{B}_x{}^*$ and consequently there is an open set $C \in \mathcal{B}_x{}^*$ and a set $W \in \mathcal{B}_x$ such that $W^* \subseteq C \subseteq V^*$. If $z \in W$ then $z \in C$ and therefore $C \in \mathcal{B}_z{}^*$, which gives a set $W_z \in \mathcal{B}_z$ with $W_z{}^* \subseteq C$. It follows that $W_z \subseteq V$ and so $V \in \mathcal{B}_z$. Similarly it can be proved that each $\mathcal{U} \in N_X$ has a base of open sets $\square$

DEFINITION 2.2. A topology X is said to be *strong completely regular*, SCR for short, if X^* is a topology, i.e. if non-convergent ultrafilters always have bases of open sets.

REMARK 2.3 Closed subspaces of SCR spaces are SCR. Products of SCR spaces need, however, not be SCR.

PROPOSITION 2.4 Strong complete regularity implies complete regularity.

PROOF. A SCR space X is a subspace of the compact topology X^* $\square$.

The converse of Proposition 2.4 is not true. In fact, the real line R with usual topology is not SCR. This is obvious since ultrafilters $\mathcal{U}$ on R contain either the set Q of rational numbers or its complement $R - Q$. Thus $\mathcal{U}$ cannot have a base of open sets. However, we have the following.

PROPOSITION 2.5. If X is completely regular and $\mathcal{U} = \mathrm{cl}[\mathcal{U}]$ for all non-convergent ultrafilters $\mathcal{U}$ on X, then X is strong completely regular.

PROOF. The assumptions together with Corollary 2 p. 1291 in [6] imply that X^* is isomorphic to the Stone-Čech compactification of X. Hence X^* is a topology $\square$

COROLLARY 2.6. The following conditions are equivalent:

(a) X is strong completely regular

(b) X is completely regular and $\mathcal{U} = \mathrm{cl}[\mathcal{U}]$ for non-convergent ultrafilters $\mathcal{U}$

(c) X^* is isomorphic to the Stone-Čech compactification of X.

PROOF. (a) $\Rightarrow$ (c): Obvious by Theorem 2.1. (c) $\Rightarrow$ (b): See [6]. (b) $\Rightarrow$ (a): Prop. 2.5 $\square$

QUESTION 2.7. Can strong complete regularity be used when investigating connectedness concepts ? E.g. when are totally disconnected spaces strong completely regular ?

We conclude with using results from above to illustrate the problem of existance of compactification reflectors for Hausdorff convergence spaces.

THEOREM 2.8. There is no reflector $R : \mathrm{HCON} \to \mathrm{HCCON}$ such that each $X \in |\mathrm{HCON}|$ has a HCCON-reflection (f_X, RX) satisfying $\mathrm{cl}[f_X(X)] = RX$.

PROOF. [7] Let A be a subset of a compact space X. Suppose there is a reflector $R : \mathrm{HCON} \to \mathrm{HCCON}$ such that the HCCON-reflection (f_A, RA) of A satisfies $\mathrm{cl}[f_A(A)] = RA$. Firstly there exists a continuous function $id^* : RA \to X$ such that $id^* \circ f_A = id : A \to X$. Secondly it follows that

$$\begin{aligned} id^*(RA) &= id^*(\mathrm{cl}[f_A(A)]) \\ &\subseteq \mathrm{cl}[id^*(f_A(A))] \\ &= \mathrm{cl}[A]. \end{aligned}$$

But A is a subset of $id^*(RA)$, which is closed. Hence $\mathrm{cl}[A] \subseteq id^*(RA)$, and consequently $\mathrm{cl}[A]$ is closed. This implies that the closure is idempotent. We can conclude that X is a topology and further, by Theorem 2.1 , that any pretopology is a completely regular topology. Thus we have a contradiction $\square$

COROLLARY 2.9. (1) There are compact pretopologies which are not topologies. (2) There are compact pretopologies the topological modification of which is not Hausdorff.

PROOF. (2) If the topological modification of a compact pretopology is Hausdorff, then by Corollary 3 p. 202 in [8] this compact pretopology is a topology $\square$

QUESTION 2.10. Is HCCON reflective in HCON ?

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