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FIXED POINTS FOR NON-SURJECTIVE EXPANSION MAPPINGS

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ABSTRACT . The contractive conditions of Popa (Demonstr. Math. 1990, 23, 213-218) were further improved for four non-surjective expansion mappings, and some common fixed point theorems under semi-compatible pairs of mappings are proved. Our main findings bring improvements to a number of results in the non-metric setting. Some implications for mathematical physics are raised with respect to physical invariants.

KEY WORDS AND PHRASES: Common fixed points; expansion mappings; semi-compatibility; d-complete topological spaces.
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1. INTRODUCTION .

Some fundamental results on fixed points are proved by Hicks (1992) [1], and Hicks and Rhoades (1992) [2] in d-complete topological spaces (Kasahara, 1975a&b [3,4]).

Recently, Popa (1990) [5] has improved previous results on fixed points for expansion mappings (Kahn et al., 1986 [6]; Popa, 1987 [7]), by using a supplementary condition. Here, the object of this paper is twofold. One - to extend the conditions of Popa (1986) [5], for non-surjective expansion mappings. Two - to prove fixed point results in d-complete topological spaces.

This goal will be reached by using semi-compatible pairs of mappings. Our results improve a number of results, including Kang and Rhoades (1992) [8], Kahn et al. [6], Popa [5,7], Rhoades (1985) [9], Hicks and Rhoades [2], Wang et al. [10], in a non-metric setting under semi-compatible conditions.

2. PRELIMINARIES .

Let (X, t) be a topological space and d: $X \times X \mapsto [0, \infty)$, such that a distance is defined by: $d(x, y) = 0$ iff $x = y$. Space X is said to be d-complete if: $\Sigma_{(n=1\mapsto\infty)}\, d(x_n, x_{n+1}) < \infty$ implies that $\{x_n\}$ is a Cauchy sequence in X .

A mapping T: $X \mapsto X$ is w-continuous at x if $x_n \to x$ implies $T(x_n) \to T(x)$, also written Tx. Recall the topological definition: let $\mathcal{U}(x)$ a neighborhood of x. Then: $\forall$ $a \in \mathcal{U}(x)$, $\exists$ $f(a) \in \mathcal{U}(f(x))$. More detailed studies of d-complete topological spaces can be found in Kasahara [3,4].

DEFINITION 2.1 - A symmetric on a set X is a real-valued function d on X such that:

(i) : $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$

(ii) : $d(x, y) = d(y, x)$

This kind of distance essentially differs from a deviation (δ) in that: $\delta(x, y) = 0 \Rightarrow x = y$.

Let d a symmetric on a set X, and :

$(\forall \varepsilon>0)$, $(x \in X)$: $S(x, \varepsilon) = \{y \in X: d(x, y) < \varepsilon)$

which provides open topological balls. The latter are convex spaces, thus provided with the triangular inequality.

From Hicks and Rhoades [2], we define a topology, t(d) on (X) by $\mathcal{U} \in t(d)$ iff for each $x \in \mathcal{U}$ and each $\varepsilon > 0$, $S(x, \varepsilon)$ is neighborhood of (x) in the topology t(d).

$(\forall x \in \mathcal{U})$, $(\varepsilon > 0)$, $S(x, \varepsilon) \in \mathcal{U}(x)$, $(\mathcal{U}(x) \in t(d))$

A topological space X is said to be symmetrizable if its topology is induced by a symmetric on X. Alternatively, let e a neutral element:

$(\forall x \in X)$, $(\exists\, y, x \perp y = e, y \perp x = e)$

DEFINITION 2.2 - Let F, G two self-maps of a topological space (X,t). These mappings are said to be semi-compatible if the following conditions hold:

(D1) : $(p \in X)$, $(Fp = Gp) \Rightarrow FGp = GFp$

(D2) : the w-continuity of G at some point p in X implies:
$\lim_{n\to\infty} FGx_n = Gp$, whenever $\{x_n\}$ is a sequence in X such that:
$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n = p$, for some p in X.

REMARK 2.1 - The d-complete symmetrizable space forms an important class of examples of d-complete topological spaces [1].

REMARK 2.2 - Compatibility [12] is defined in metric while semi-compatibility [11] is defined in non metric setting. Therefore, axioms D1 and D2 are independent.

3. NOTATIONS AND AUXILIARY RESULT.

Throughout this paper, we will adopt the following notations: M is an arbitrary set with values in a Hausdorff topological space (X, t); $\mathbf{N}$ is the set of all positive integers, $\mathbf{R}^+$ is the set of all non-negative real numbers, and Ψ is the family of all functions $\psi : (\mathbf{R}^+)^3 \mapsto \mathbf{R}^+$, satisfying the following properties:

(ψ-1) : (ψ) is continuous on $(\mathbf{R}^+)^3$.

(ψ-2) : $\psi(1, 1, 1) = h > 1$ where $h \in \mathbf{R}^+$.

(ψ-3) : let $(\alpha, \beta) \in \mathbf{R}^+$, such that (12):

(ψ-3-A) : $\alpha \geq \psi(\beta, \beta, \alpha) = h.\beta$

(ψ-3-B) : $\alpha \geq \psi(\beta, \alpha, \beta) = h.\beta$

(ψ-3-C) : $\forall \alpha \neq 0$, $\psi(\alpha, 0\ 0) > \alpha$

For condition (ψ-1), the theorem of Tychonoff states that the product of a family of compact spaces is compact, that is, Hausdorff separated, and each open cover contains a finite subcover (Heine-Borel-Lebesgue property), if each space of the family is compact.

Condition (ψ-3) is that already used by Popa [5,7] for expansion mappings. Alternative forms will be discussed below.

We also need the following important statement.

PROPOSITION 3.1 - Let A, B, S and T be self-maps of M, such that each of the pairs A, S and B, T are semi-compatible, and that, for all x, y in M, and ($\psi \in \Psi$):

$$(3.1) \qquad d(Sx, Ty) \geq \psi(d(Ax, By), d(Ax, Sx), d(By, Ty)).$$

If there exists u, v, and z in M such that Au = Su = Bv = Tv = z , then Az = Bz = Sz = Tz = z.

PROOF . Since A and S are semi-compatible mappings, and Au = Su = z , by property (D1), we have Az = ASu = SAu = Sz. From **3.1** we also have:

$$\begin{aligned} d(Sz, z) &= d(Sz, Tv) \\ &\geq \psi(d(Sz, z), 0, 0) \\ &> d(Sz, z) \end{aligned}$$

by property (ψ-3-C). From this contradiction, it follows that $Sz = z$. By symmetry, $Bz = Tz = z$.

4 - MAIN RESULTS .

We now state and prove our main two theorems and emphasize some of their corollaries and related theorems.

THEOREM 4.1 - Let A, B, S, T, be self-maps of M such that each of the pairs A, S and B, T are semi-compatible mappings and satisfy both the previous condition **3.1**, and the following **4.2** and **4.3**:

(4.2) : $A(M) \subseteq T(M)$, and $B(M) \subseteq S(M)$

(4.3) : $S(M)$ is d-complete.

Then, A, B, S, T have a unique common fixed point in M.

PROOF.

4.1-a Existence of a common fixed point.

For an arbitrary point x_o in M, by **4.2** we define a sequence $\{x_n\}$ in M such that, for all $n = 0, 1, 2, \dots$, :

$$(4.4)\qquad \begin{cases} Tx_{2n+1} = Ax_{2n} = y_{2n} & \text{(say)}, \\ Sx_{2n+2} = Bx_{2n+1} = y_{2n+1} & \text{(say)}, \end{cases}$$

Define $d_n = d(y_n, y_{n+1})$ for all $n = 0, 1, 2, \dots$ Then, by applying **(4.4)**, we obtain:

$$\begin{aligned} d_{2n} &= d(Sx_{2n+2}, Tx_{2n+1}) \\ &\geq \psi(d(Ax_{2n+2}, Bx_{2n+1}), d(Ax_{2n+2}, Sx_{2n+2}), d(Bx_{2n+1}, Tx_{2n+1})) \\ &= \psi(d_{2n+1}, d_{2n+1}, d_{2n}) \\ &\geq h.d_{2n+1}\ , \quad \text{by property } (\psi\text{-3-A}) \end{aligned}$$

This implies that $d_{2n+1} \leq 1/h.\ d_{2n}$. Similarly, we can get $d_{2n+2} \leq 1/h.\ d_{2n+1}$. In general, we have for $d_o > 0$, $d_n \leq 1/h.\ d_{n-1} \leq \dots, \leq 1/h^n.\ d_o$ for all $n \in \mathbf{N}$. Since $h > 1$, this implies that $\lim_{n\to\infty} d_n = 0$. Since $\{d_n\}$ is nondecreasing, $d_{2n} = 0$ for some n. Consequently, $d_{2n+1} = 0$. Thus, we have clearly $\lim_{(n\to\infty)} d_n = 0$

It follows that $\Sigma_{(n=1\to\infty)}\ d(y_n, y_{n+1})$ is convergent.

Since, in addition, S(M) is d-complete, sequence $\{y_n\}$ converges to some z in S(M) hence, the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$, of $\{y_n\}$, also converge to z.

Let $Su = z$ for some u in M.

Putting $x = u$ and $y = x_{2n+1}$ in relation **3.1**, we obtain:

(4.5): $d(Su, Tx_{n+1}) \geq \psi(d(Au, Bx_{2n+1}), d(Au, Su), d(Bx_{2n+1}, Tx_{2n+1}))$.

Letting $n \to \infty$ in **4.5**, we get:

$0 \geq \psi(d(Au, z), d(Au, z), 0)$

$\geq h.d(Au, z)$ by property (ψ-3-A),

which implies that $Au = z$.

Since $z = Au \in A(M) \subseteq T(M)$, there exists a point v in M such that $Au = Tv$. Again, replacing x by u and y by v in **3.1**, we obtain:

$0 = d(Su, Tv) \geq \psi(d(Au, Bv), d(Au, Su), d(Bv, Tv))$

$= \psi(d(z, Bv), 0, d(Bv, z))$

$\geq h.d(Bv, z)$, by property (ψ-3-B),

which implies that $Bv = z$.

Therefore, we have $Au = Su = Bv = Tv = z$, and hence, by proposition **3.1**, it follows that z is a common fixed point of A, B, S, and T.

4.1-b Uniqueness of the common fixed point.

Let us suppose that there exists a second distinct common fixed point w of A, B, S, and T. Then, from relation **3.1**, we have:

$d(z, w) = d(Sz, Tw)$

$\geq \psi(d(Az, Bw), d(Az, Sz), d(Bw, Tw))$

$= \psi(d(z, w), 0, 0)$

$> d(z, w)$ by property (ψ-3-C),

which is a contradiction.

Hence, z is the unique common fixed point of A, B, S, and T.

This completes the proof.

REMARK 4.1 - Theorem **4.1** improves and generalizes Theorem **1** of Popa [7] and theorem **3** of Khan, Khan and Sessa [6], to d-complete topological spaces, under semi-compatible conditions.

Two corollaries and an infering theorem are worth noting.

COROLLARY 4.1 - Let A, B, S, T be self-mappings of Hausdorff space M, such that pairs A, S and B, T are semi-compatible and satisfy both conditions **4.2**, **4.3**, and the following condition **4.6**:

(4.6) : There exists $a, b, c \in \mathbf{R}^+$, with $a > 1$, $b < 1$, $c < 1$, and $a+b+c > 1$, such that, if:

$d^r(Sx, Ty) \geq a.d^r(Ax, By) + b.d^r(Ax, Sx) + c.d^r(By, Ty)$

for all x, y in M , with r a positive integer.

Then, (A), (B), (S), (T), have a unique common fixed point in M.

PROOF - Let us further define the mapping $\psi : (\mathbf{R}^+)^3 \mapsto \mathbf{R}^+$ as follows:

$\forall (t_1, t_2, t_3) \in \mathbf{R}^+, \exists\, r > 0$ such that

$\psi(t_1, t_2, t_3) = (a.t_1^r + b.t_2^r + c.t_3^r)^{1/r}$

Then, $\psi \in \Psi$, and thus, by Theorem **4.1**, this corollary follows.

REMARK 4.2 - It should be noted that corollary **4.1** improves and generalizes Theorem **1** of Popa [5] to this non-metric setting.

If we put $b = c = 0$ in Corollary **4.1**, we obtain the following:

COROLLARY 4.2 - Let A, B, S, T be self-mappings of M, such that pairs A, S and T are semi-compatible and satisfy both conditions **4.2**, **4.3**, and the following condition **4.7**:

(4.7):There exists a constant $k \in (\mathbf{R}^+)$, with $k > 1$, such that :

$$d(Sx, Ty) \geq k.d(Ax, By),$$

for all x, y in (M)

then, A, B, S, and T have a unique common fixed point in M.

If we replace A=B , S by S^2, and T by TS in Theorem **4.1**, then we obtain the following result in which it is important to note that the semi-compatibility condition is no longer necessary:

THEOREM 4.2 - Let S and T be self-mappings of M, such that $S(M) \subseteq TS(M)$ $S(M) \subseteq S^2(M)$ and S(M) is d-complete. Suppose, in addition, that there exists such:

$$(\exists \psi \in \Psi), \ \forall x, y \in M,$$

$$d(S^2x, TSy) \geq \psi\,(d(Sx, Sy), d(Sx, S^2x), d(Sy, TSy)).$$

Then S and T have a unique common fixed point in M.

REMARK 4.3.a - If we define ψ as in the proof of Corollary **4.1**, then the result obtained in this new setting improves and generalizes Theorem **2.4** of Pathak et al., (1996) [13]. The original theorem of this type was proved by these authors in a complete metric space.

Now let Φ denote the family of all functions $\phi : \mathbf{R}^+ \mapsto \mathbf{R}^+$, which are non-decreasing, upper semi-continuous from the right, with:

$\phi(0) = 0$, $\phi(t) < t$ and $\Sigma_{(n=1\to\infty)}\, \phi^n(t) < \infty$, for each $t > 0$,

We finally formulate the following interesting theorem:

THEOREM 4.3 - Let A, B, S, T, self-mappings of M, such that pairs A, S, and B, T are semi-compatible and satisfying both of conditions **4.2**, **4.3** and the following condition **4.8** :

(4.8) : $\phi(d(Sx, Ty)) \geq \max\{d(Ax, By), d(Ax, Sx), d(By, Ty)\}$,

for all x, y in M , where $\phi \in \Phi$).

Then, A, B, S, T, have a unique common fixed point in M.

PROOF - 4-2a. <u>Existence of a common fixed point.</u>

For an arbitrary point $y_o \in M$, define a sequence $\{y_n\}$ as in **4.4**. Also define d_n as :

$$d_n = d(y_n, y_{n+1}), \ \forall\, (n = 0, 1, 2, \dots)$$

Then, by properties **(4.4)** and **(4.8)**, we have:

(4.9) : $\phi(d_{2n}) = \phi(d(Sx_{2n+2}, Tx_{2n+1}))$

$\geq \max\{d(Ax_{2n+2}, Bx_{2n+1}), d(Ax_{2n+2}, Sx_{2n+2}), d(Bx_{2n+1}, Tx_{2n+1})\}$

$= \max\{d_{2n+1}, d_{2n}\}$.

Now, suppose that $\max\{d_{2n+1}, d_{2n}\} = d_{2n}$, for some n. Then from **4.9** we have $\phi(d_{2n}) \geq d_{2n}$ which is a contradiction. Hence, $\max\{d_{2n+1}, d_{2n}\} = d_{2n+1}$ for each n. From **4.9** we have $d_{2n+1} \leq \phi(d_{2n})$, and similarly $d_{2n+2} \leq \phi(d_{2n+1})$.

In general, for $d_o > 0$, and $n \in \mathbf{N}$ $d_n \leq \phi(d_{n-1}) \leq \dots, \leq \phi^i(d_{n-i}), \dots \leq \phi^n(d_o)$.

Since $\Sigma_{(n=1\to\infty)}\, \phi^n(t)$ is convergent for each $t > 0$, it follows that $\Sigma_{(n=1\to\infty))}$ $d(y_n, y_{n+1})$ is convergent. In addition, since S(M) is d-complete, the sequence $\{y_n\}$ converges to some z in S(M), and hence this states that the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$, and $\{Tx_{2n+1}\}$, of $\{y_n\}$ also converge to z.

Let Su = z for some u in X. Putting x = u and $y = x_{2n+1}$ in inequality **4.8** and then, letting limits as $n \to \infty$, we obtain:

$\phi(0) = 0 \geq d(Au, z)$

which implies that Au = z. Since $A(M) \subseteq T(M)$, there is a point v in M such that Au = Tv = z. Again, replacing x by u and y by v in inequality **4.8**, we obtain:

$\phi(0) = 0 = d(Su, Tv) \geq d(Bv, z)$

which means that Bv = z.

Therefore, Au = Su = Bv = Tv = z.

However, since A and S are semi-compatible mappings and Au = Su = z, then, by property **D1**, in definition **2.2**, we have Az = ASu = SAu = Sz.

By property **4.8** we also have:

$\phi(d(Sz, z)) = \phi(d(Sz, Tv))$,

$\geq \max\{d(Az, Bv), d(Az, Sz), d(Bv, Tv\}$

$= d(Sz, z)$

which is a contradiction, since for each $t > 0$:

$\phi(t) < t \Leftrightarrow$ (for: $t = d(Sz, z) > 0$), $\phi(d(Sz, z)) < d(Sz, z)$.

Therefore, Sz = z, and by symmetry, Bz = Tz = z, which demonstrates the existence of z as a common fixed point of A, B, S, and T.

4-2b . Uniqueness of the common fixed point.

This property readily follows from property **4.8**, which completes the proof.

Now, the last two theorems infer from replacing the expression: $\max\{d(Ax, By), d(Ax, Sx), d(By, Ty)\}$ by d(Ax, By), and other terms.

THEOREM 4.4 - Let A, B, S, T, be mappings from M into itself, such that the pairs A, S, and B, T are semi-compatible and fulfill on one hand the previous properties **4.2** , **4.3**, and on the other, at least one of the additional following conditions:

$\forall x, y \in (M)$, $(\phi \in \Phi)$,

(4.10) : $\phi(d(Sx, Ty)) \geq d(Ax, By)$

(4.11) : $\phi(d(Sx, Ty)) \geq 1/2\ [d(Ax, By) + d(Ax, Sx)]$

(4.12) : $\phi(d(Sx, Ty)) \geq 1/2\ [d(Ax, By) + d(By, Ty)]$

(4.13) : $\phi(d(Sx, Ty)) \geq 1/3\ [d(Ax, By) + d(Ax, Sx) + d(By, Ty)]$

Then, A, B, S, T have a unique common fixed point in M.

If lastly we replace A = B by S, S by S^2, and T by TS, then we obtain the following result in which, again, the semi-compatibility concept is not necessary.

THEOREM 4.5 - Let S and T, be mappings from M into itself, such that $S(M) \subseteq TS(M)$, $S(M) \subseteq S^2(M)$, and S(M) is d-complete. Suppose that in addition:

$(\exists \phi \in \Phi)$, $\forall (x, y) \in (M)$, at least one of the following conditions holds:

(4.14) : $\phi(d(S^2x, TSy)) \geq d(Sx, Sy)$

(4.15) : $\phi(d(S^2x, TSy)) \geq 1/2\ [d(Sx, Sy) + d(Sx, S^2x)]$

(4.16) : $\phi(d(S^2x, TSy)) \geq 1/2\ [d(Sx, Sy) + d(Sy, TSy)]$

(4.17) : $\phi(d(S^2x, TSy)) \geq 1/3\ [d(Sx, Sy) + d(Sx, S^2x) + d(Sy, TSy)]$

Then, S and T have a unique common fixed point in M.

Some subsequent remarks will be raised in discussion.

5 - DISCUSSION .

The above results can be examined both from a pure mathematical viewpoint and also with respect to some implications in mathematical physics.

5-1 - Some mathematical remarks.

REMARK 4.1. - An alternative form of condition (ψ -3)$'$ was used first by Delbosco (1981) [17] and then by Fisher and Sessa (1986) [18], for contraction maps in metric spaces. This would be: if $\alpha, \beta \in \mathbf{R}^+$ are such that : $\alpha \geq \psi(\beta, \beta, \alpha)$, or $\alpha \geq \psi(\alpha, \beta, \beta)$, or $\alpha \geq \psi(\beta, \alpha, \beta)$, then $\alpha \geq h.\beta$. In such case, proposition **3.1** of this paper cannot be proved. The question of whether theorem **4.1** could be proved with the latter definition remains open.

REMARK 4.2 - It should be noted that if condition **4.8** was changed into: $\phi(d(Sx, Ty)) \geq \sup\{d(Ax, By), d(Ax, Sx), d(By, Ty)\}$, with $d(Ax, Sx)=0$, $d(By, Ty)=0$, then one could meet the following condition: $\phi(d(Sx, Ty)) \geq d(Ax, By)$. Only in this case would Theorem **4.2** fall into the category of a special case in metric space of the contractive condition of Theorem **2.3** of Kang and Rhoades [8]. Since the authors have proved that surjectivity condition is necessary for this theorem to hold, then Theorem **4.3** of the present paper would not be true in this particular setting, which differs from the one considered here. While the condition of surjectivity is necessary to prove theorem **4.3** in complete metric space, it is not required for this purpose if one considers S(M) d-complete, i.e. a nonmetric setting.

REMARK 4.3.b - In Theorem **4.1** , the results could not be extended to a deviation instead of a symmetric, since a null distance between two points would not necessarily infer from the identity of these two points. The same remark holds for the demonstration of Theorem **4.3**, with:

$$d(Au, z) \leq 0 \Rightarrow Au = z,$$

while in contrast:

$$\delta(Au, z) \leq 0 \neg\Rightarrow Au = z.$$

REMARK 4.4 - If we take X = M, and S and T surjective, in Theorem **4.4**, with condition **4.10**, then we obtain a result in this new setting. It is worthwhile mentioning that the original theorem of this type was proved in 1992 by Kang and Rhoades [8], under the condition of compatibility, in a complete metric space.

REMARK 4.5 - If we put M = X, we obtain the original theorem of this type, namely Theorem **2.1** proved by Pathak, Kang and Ryu [13] for a complete metric space, in this new setting.

REMARK 4.6 - If for $\phi \in \Phi$, we define $\phi : \mathbf{R}^+ \mapsto \mathbf{R}^+$ by $\phi(t) = 1/k.t$, where $k > 1$ then from Theorem **4.4** we obtain Corollary **4.2**, which improves Theorem **2.6** of Kang and Rhoades [8] in this non-distance metric setting.

REMARK 4.7 - If in Theorem **4.2** we replace max $\{\cdot, \cdot, \cdot\}$ by min $\{\cdot, \cdot, \cdot\}$ and take X = M, and if S and T are un-equal surjective, then the statement is false even if A = B = I, the identity mappings.

REMARK 4.8 - Our results improve and generalize several previous results by Daffer and Kaneko (1992) [14], Kang (1993) [15], Rhoades [9], Taniguchi (1989) [16], and Wang et al. [10].

REMARK 5.1 - Theorem **2** transposed to topological aspects suggests that metric distances could surprisingly provide a finer filtering than the symmetric difference previously proposed by some of us (Bounias and Bonaly, 1996) [19] as a non-metric distance between sets. A metric distance between (A) and (B) can be defined by:

$$d(A, B) = \{(x \in A, y \in B), \inf d(x, y)\}$$

while the non-metric distance would be:

$$\Delta(A, B) = \complement_{A \cup B}(A \cap B)$$

Now, this raises an interesting problem. Let sets A, B, S, T, be such that $A \subset S$, $B \subset T$. What are the conditions for: $\Delta(A, B) \subset \Delta(S, T)$, with respect to $d(A, B) \subset d(S, T)$?

The latter result is nearly trivial in the metric space $\mathbf{R}^+$. However, the former involves the following necessary conditions:

$$A \cap B \subseteq (S \cap T)$$

and:

$\mathbf{C}_B(A) \subset \mathbf{C}_S(T)$, $\mathbf{C}_A(B) \subset \mathbf{C}_T(S)$

to be related to conditions **3.1** and **4.8** .

Interestingly, when $A \cap B = (S \cap T)$, the obtained set contains the fixed points of the mappings (F) of $S \times T$ into itself, and when it reduces to one point, it identifies with the fixed points. The role of topological dimensions of the involved sets will be further examined. Let Fix $(S \times T)^{(S \times T)}$ the set of fixed points of the mappings $F: (S \times T \mapsto S \times T)$. Then:

Fix $(S \times T)^{(S \times T)} \subseteq (S \cap T)$

PROPOSITION 5.1 - Let A, B, S, T topological spaces, such that $A \subset S$, $B \subset T$, S and T are complete and having respective topological dimensions $n_S \neq n_T$, if A and B are closed, then the mappings $(S \times T)^{(S \times T)}$ have a common fixed point.

PROOF . If A and B are closed, they contain a Brouwer's type fixed point, denoted a and b. Let $\mathcal{U}(a)$ and $\mathcal{U}(b)$ neighborhoods of these points. The topological continuity in S and T suffices to state that the reciprocal image of $\mathcal{U}(a)$ by any mapping $\mathfrak{f}: A \mapsto S$ is $\mathcal{U}(a)$ and the reciprocal image of $\mathcal{U}(b)$ by any mapping $\mathfrak{f}: B \mapsto T$ is $\mathcal{U}(b)$. Now, provided S and T have topological dimensions n_S and n_T, such that $n_S \neq n_T$, space $(S \cap T)$ is a closed and has a Brouwer's fixed point. Since $A \cap B \subseteq (S \cap T)$, this fixed point is u = a as well as u = b, that is a = b = u, and the proposition is proved in these strictly non-metric conditions.

This brings us now to some last points more closely related to fundamental physics.

5-2 - Some physically relevant remarks.

REMARK 5.2.1 - Conditions (ψ -3-**A**) and (ψ -3-**B**) lead to the same scalar h, and condition (ψ -3-**C**) defines a projection of ($\mathbb{R}^3$) into ($\mathbb{R}$). It should be pointed that the case of a projection of ($\mathbb{R}^4$) into ($\mathbb{R}$) will not be immediate, since major differences lie betwen respective topological properties of 3-spaces and 4-spaces.

The introduction of scalar h makes the case essentially relevant with linear physics. However, later, in corollary **4.1**, exponent r addresses to Euclidean-like norms if it is an integer. In contrast, if it is not integer, the system could be related to fractal scaling. However, it does not match with the alternative non-distance coordinates defined through intersections of sets (Bounias and Bonaly, 1996) [19], since exponents should be a sequence of the following type: {r, r-1, r-2}, with coefficients (b, c) < 0 in relation **4.6**.

REMARK 5.2.2 - In remark **4.6**, we have not called "ϕ (t) = 1/k.t" a metric setting. In fact, it essentially deviates from so-called natural metrics, deriving from Euclidean ones, but it does represent a kind of metric. In contrast, the symmetric difference between sets and its newly defined norm [19], would allow topological generalizations escaping the critical problem of scale inconsistency, in physics. It would then be interesting to re-examine as follows the theory of fixed points with respect to distances defined this way. We thus raise the conjecture that our results on fixed points could further contribute to provide some foundations to the still needed basic justification of the invariance of some physical quantities (see Ashtekar and Magnon-Ashtekar, 1979) [20]. The question of antinomic parity conservation versus parity violation at extreme scales (see Magnon, 1996 [21] for review) could then find some clarification through basic topologies governing the embedded spacetime.

Lastly, we are currently working on purely mathematical aspects of biology [22, 23] in which semi-compatibility condition [11] could provide previously missing basis for the justification of some brain functions.

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